Name: ____Solutions____________ Student Number: ______________________	CISC 203 Discrete Mathematics for Computing Science Test 4, Fall 2011 Professor Mary McCollam

Please write in pen and only in the box marked "Answer".
This is a closed-book exam. No computers or calculators are allowed.

Question 1: [10 marks]

Suppose a computer worm replicates itself once an hour, so that the total number of copies of the worm in a system doubles every hour (assume its growth is unchecked).

Answer:

i) [3 marks] Set up a recurrence relation, w_n, for the number of copies of the worm after n hours have elapsed.

$$w_n = 2 \cdot w_{n-1}$$

ii) [5 marks] Find an explicit formula for the number of copies of the worm after n hours have elapsed if 10 copies of the worm are released initially.

$$w_0 = 10 = 2^0 \cdot 10$$

$$w_1 = 20 = 2^1 \cdot 10$$

$$w_2 = 40 = 2^2 \cdot 10$$

$$w_4 = 80 = 2^3 \cdot 10$$

…

$$w_n = 2^n \cdot 10$$

iii) [2 marks] If 10 copies of the worm are released initially, how many copies of the worm will there be after 10 hours?

$$W_{10} = 2^{10} \cdot 10 = 10{,}240$$

Question 2: [10 marks]

a) [6 marks]   Determine whether the relation ***R*** on the set of all real numbers is symmetric, antisymmetric, or transitive where $(x,y) \in \boldsymbol{R}$ if and only if $x = 1$ or $y = 1$.
Justify each answer with a brief explanation.

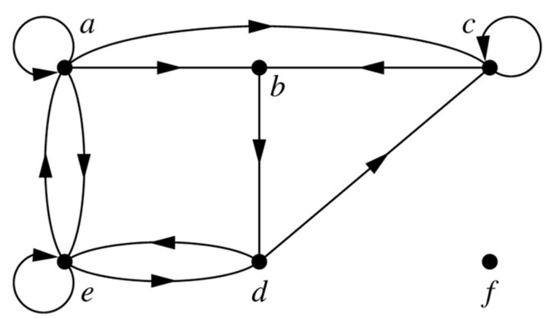
Answer:

R is symmetric since if x = 1 or y = 1 then y = 1 or x = 1

Not antisymmetric, since, for example, both (1,5) and (5,1) are in *R*

Not transitive, since for example (8,1) and (1,8) are in *R*, but (8,8) is not in *R*

b) [4 marks] Determine whether the relation *R* represented by this directed graph is reflexive, symmetric, antisymmetric and/or transitive. Justify each answer with a brief explanation.



Answer:

Not reflexive since there is not a loop at every vertex, for example vertices b and d

Not symmetric since, for example, there is an edge from d to c, but no edge from c to d

Not antisymmetric since, for example, there is an edge from a to e and one from e to a

Not transitive since, for example, there is an edge from e to d and from d to c, but there is no edge from e to c

Question 3: [10 marks}

a) [5 marks] Suppose that ***R*** and ***S*** are reflexive relations on a set $\mathcal{A}$. Prove that ***S*** ∘ ***R*** is reflexive. Recall that ***S*** ∘ ***R*** is the composite of ***R*** and ***S***.

Answer:

By the definition of ***S*** ∘ ***R***, if $(a,b) \in$ R and $(b,c) \in$ S, then $(a,c) \in$ ***S*** ∘ ***R***

Therefore, since it is true that for all $a \in \mathcal{A}$, $(a,a) \in R$ and $(a,a) \in S$, since *R* and *S* are reflexive, then it is true that for all $a \in \mathcal{A}$, $(a,a) \in$ ***S*** ∘ ***R***

Therefore ***S*** ∘ ***R*** is reflexive.

b) [5 marks] Let ***S*** be the set of subroutines of a computer program (i.e., functions in Python, methods in Java). Define the relation ***R*** by P ***R*** Q if subroutine P contains a call to subroutine Q, so Q might be called during the execution of P.

Answer:

i) [1 mark] For which type of subroutine P does (P,P) belong to ***R***?

P is recursive

ii) [2 marks] Describe the relation $\boldsymbol{R}^2$

(P,Q) is in $\boldsymbol{R}^2$ if subroutine P calls a subroutine that calls subroutine Q, where “calls” means “contains a call to”

ii) [2 marks] Describe the transitive closure of ***R***

(P,Q) is in $\boldsymbol{R}^*$ if there is some sequence of calls such that P calls a subroutine that calls a subroutine that calls ... that calls Q, where “calls” means “contains a call to”

Question 4: [10 marks]

Let ***R*** be the following relation on the set **{ 3,4,5 }**:

{ (3,3), (3,4), (4,5), (5,4) }

Use the **0-1 matrix** representation for relations to find the transitive closure of ***R***. Show the formula used to find the transitive closure of ***R*** from its 0-1 matrix representation and show the matrices in the intermediate steps in the algorithm, as well as the result matrix.

Answer:

Since M_R is a 3 × 3 matrix, the formula for its transitive closure is: $M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]}$

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad M_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

So, $M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$