Name:____________SOLUTIONS_____________ Student Number:_____________________________	CISC 203 Discrete Mathematics for Computing Science Test 5 Fall 2011 Professor Mary McCollam

Please write in pen and only in the box marked "Answer".
This is a closed-book exam. No computers or calculators are allowed.

Question 1: [10 marks]

(a) Let R be the relation on the set of ordered pairs of positive integers such that $((a,b), (c,d)) \in R$ if and only if $ad = bc$. Show that R is an ***equivalence relation***.

Answer:
R is reflexive since $((a,b), (a,b)) \in R$ because ab = ba

R is symmetric since if $((a,b), (c,d)) \in R$ then ad = bc, which also means that cb = da, so $((c,d), (a,b)) \in R$

R is also transitive. If $((a,b), (c,d)) \in R$ and $((c,d), (e,f)) \in R$, then ad = bc and cf = de.
Multiplying these equations gives acdf = bcde, which also means that af = be, so $((a,b), (e,f)) \in R$

Since R is reflexive, symmetric, and transitive, R is an equivalence relation.

(b) Describe the ***equivalence class*** of **(1,3)** with respect to that equivalence relation, in other words**,** describe $[\,(1,3)\,]_R$.

Answer:
ad = bc if and only if a/b = c/d

Therefore, the equivalence class of (1,3) is the set of all pairs (a,b) such that the fraction a/b equals 1/3

Another way of expressing this is that the equivalence class of (1,3) is the set of all pairs (a,b) such that b = 3a

Question 2: [10 marks]

(a) Determine whether the relation represented by the following zero-one matrix is a ***partial order*** over the set {*a,b,c,d*}. Justify your answer.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Answer:

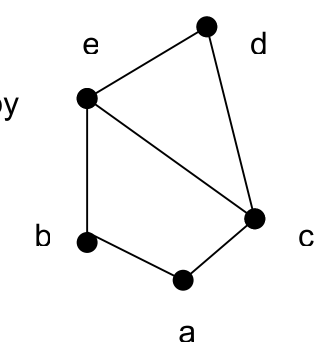
The relation must be reflexive, antisymmetric, and transitive to be a partial order.

It is reflexive, since all vertices on the main diagonal are 1s.

However, it is not antisymmetric, since cRb and bRc

Therefore the relation is not a poset.

(b) List all ordered pairs in the ***partial ordering*** represented by the following Hasse diagram.



Answer:

{ (a,a), (a,b), (a,c), (a,d), (a,e),
(b,b), (b,e), (b,d),
(c,c) (c, d), (c,e),
(d,d)
(e,e), (e,d) }

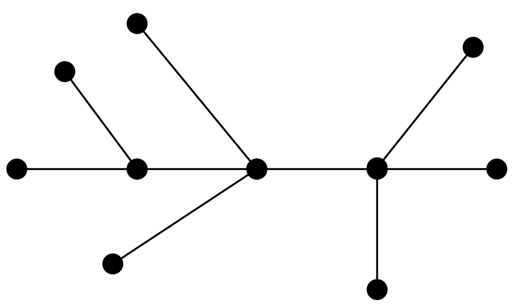
Question 3: [10 marks]

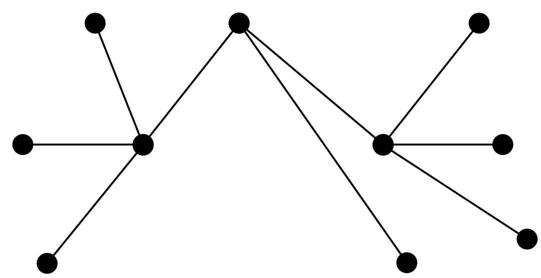
(a) Describe what the adjacency matrix for the graph K_n looks like. Recall that K_n is the ***complete graph*** with n vertices.

Answer:

It is an n x n matrix that has zeroes on the main diagonal and ones everywhere else.

(b) Determine whether the following pair of graphs are ***isomorphic***. Exhibit an isomorphism or provide a rigorous argument that none exists. You may label the vertices in order to refer to them.





Answer:

These two graphs are not isomorphic.

They have the same number of vertices and the same number of edges.

They also both have 7 vertices of degree 1, 1 vertex of degree 3, and 2 vertices of degree 4.

However, the two vertices of degree 4 are adjacent to each other in the graph on the left, whereas they are not adjacent in the graph to the right.

So the sub-graph of the vertices of degree 4 for the graph on the left consists of the two vertices with an edge connecting them, whereas the sub-graph of the vertices of degree 4 for the graph on the right consists of the two vertices with no connecting edge. Thus the sub-graphs are not isomorphic and therefore the original graphs are not isomorphic.

Question 4: [10 marks]

Use ***Huffman coding*** to encode these symbols with given frequencies: A: 0.23, B: 0.37, C: 0.06, D: 0.14, E: 0.16, F: 0.01, G: 0.03. What is the resulting code for each letter? Compute the average number of bits required to encode a symbol.

Answer:

In what follows, the heavier tree is always to the left (edges going left are labeled with a 0, while edges going right are labeled with a 1).

Combine G and F into a tree T_1 of weight 0.04

Combine C and T_1 into a tree T_2 of weight 0.10

Combine D and T_2 into a tree T_3 of weight 0.24

Combine A and E into a tree T_4 of weight 0.39

Combine B and T_3 into a tree T_5 of weight 0.61

Finally, combine T_5 and T_4

This gives the following code: A: 10, B: 00, C: 0110, D: 010, E: 11, F: 01111, G: 01110

The average number of bits required is given by:

$$2(0.23) + 2(0.37) + 4(0.06) + 3(0.14) + 2(0.16) + 5(0.01) + 5(0.03) = 2.38$$

which means that on average 2.38 bits are needed to encode a character.