CISC 204 Class 3

Proof Rules for Implication Elimination

Text Correspondence: pp. 9–11

Main Concepts:

- *$\rightarrow$e: implication elimination, or Modus Ponens*
- *MT: derived rule of Modus Tollens*

In natural deduction – as in most of symbolic logic – the concept of "if ... then" has a specific meaning. The English word "implies" is less than perfect in logic because it might carry a meaning of a causal relationship between the two propositions under discussion. We will follow common logical usage and call this *material implication*, understanding that when we use the word "implication" we always intend the logical meaning and not the common English meaning.

3.1 Implication Elimination

A very important proof rule – in many axiomatic systems this is the only proof rule – is *implication*. This is a formal version of English "if-then" reasoning.

Proof Rule: **implication**-*elimination*: $\rightarrow$e
(also known as Modus Ponens, MP)

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow \mathrm{e}$$

Formal terminology for implication is, for the formula $\phi \rightarrow \psi$, the proposition ϕ is the *antecedent* and the proposition ψ is the *consequent*. The rule Modus Ponens is also known as "affirming the antecedent", because we are affirming that ϕ is true.

The rule $\rightarrow$e can be applied to a complex sequent, such as $(p \wedge q) \rightarrow (q \vee r), p \wedge q \vdash q \vee r$ which can be proved as

1	$(p \wedge q) \rightarrow (q \vee r)$	premise
2	$p \wedge q$	premise
3	$q \vee r$	$\rightarrow$e 2, 1

3.2 Modus Tollens, or Denying the Consequent

A proof rule that is closely related to the rule of implication elimination is *derived* from the rule $\rightarrow$e:

Proof Rule, derived: **denying the consequent**, or *Modus Tollens*: MT

$$\frac{\phi \rightarrow \psi \quad \neg\,\psi}{\neg\,\phi}\,\text{MT}$$

This rule asserts that, for a given implication, from the falsity of the consequent we can infer the falsity of the antecedent.

A simple English example might be:

> *If the instructor is a super-hero then the instructor could jump over Grant Hall; the instructor cannot jump over Grant Hall; therefore the instructor is no super-hero*

As with Modus Ponens, this rule can be used for complex propositions. A caution to the reader: some proof software, such as JAPE, may not have MT "built in" to the logical theory.

A simple problem is to prove that the following sequent is valid:

$p \rightarrow \neg\,q, q \vdash \neg\,p$

This is an application of the MT rule that also benefits from the use of double negation.