CISC 204 Class 5

Proof Rules for Negation Introduction and Proof By Contradiction

Text Correspondence: pp. 22–26, 16–17

Main Concepts:

- *Proof using contradiction: a rule that uses an assumption box and a contradiction*
- $\neg$ *i: negation introduction*
- *Proof By Contradiction (PBC): derived proof rule*
- $\vee$ *i: disjunction introduction*

This class explores another proof rule that uses the concept of contradiction. In plain English, the idea for this rule is: suppose that a proposition is true; show that the assumption leads to a contradiction; deduce that the assumed proposition is false.

That is, by assuming that a proposition is true "for the sake of argument", we can introduce the negation of the proposition into a valid proof.

5.1 Negation Introduction

The rule of negation introduction $\neg$ i says that, to prove a formula that is a negation which is of the form $\neg\phi$, we assume the "positive" formula ϕ. If we can deduce a contradiction $\perp$ then our assumption ϕ must be false, so $\neg\phi$ must be true.

Proof Rule: **negation**-*introduction,* $\neg\, i$

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \perp\end{array}}}{\neg\,\phi}\ \neg\,\mathrm{i}$$

Self-Study Sequent 5.1: $p \rightarrow q, p \rightarrow \neg q \vdash \neg p$ *H&R, p. 21*

An example proof strategy and proof is on the next page of these notes.

Proof Strategy: After writing down the conclusion, we see that it is a negation *and* that the negation is the antecedent of at least one premise. This suggests that a proof by contradiction is useful, so we begin by assuming p. This does indeed lead to a contradiction, which means that the assumption was false, which in turn means that the conclusion is true.

This strategy is used in the textbook proof, but without this plain English explanation.

Self-Study Sequent 5.1: $p \to q, p \to \neg q \vdash \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to$e 3, 1
5	$\neg q$	$\to$e 3, 2
6	$\bot$	$\neg$e 4, 5
7	$\neg p$	$\neg$i 3–6

5.2 Proof By Contradiction (PBC)

A *proof by contradiction*, abbreviated as PBC, is a derived rule that is based on the rule of negation introduction $\neg$ i. To prove a conclusion $\vdash \phi$, the negation $\neg \phi$ is assumed; if this assumption leads to a contradiction then the assumption was false. The form of this rule is

Proof Rule, derived: *proof by contradiction*, PBC

$$\frac{\boxed{\begin{array}{c}\neg \phi \\ \vdots \\ \bot\end{array}}}{\phi}\ \text{PBC}$$

Example:

"Suppose that: the truth of p and the falsity of q implies r; that r is false; and that p is true. We can conclude that q is true."

$p \wedge \neg q \rightarrow r, \neg r, p \vdash q$ *H&R, p. 23*

Proof Strategy: The proof can be performed by writing down the premises of the sequent, then using a single instance of proof by contradiction. *Using PBC removes line 8 from the proof and directly deduces the conclusion q.*

The core of our alteration of the proof that the textbook provides, in English, is:

- Assume $\neg q$
- Form the conjunction $p \wedge \neg q$ using $\wedge \mathrm{i}_1$
- Apply Modus Ponens, $\rightarrow$ e, to deduce r
- Apply negation elimination, $\neg$ e, to deduce $\bot$
- Apply PBC to deduce q

There is a proof rule in natural deduction that allows us to conduct a simple mode of reasoning about "either…or". Suppose that we take a proposition to be true, such as the atomic proposition p. We can reason that, for any other proposition such as the atomic proposition q, that either p or q is true, and perhaps both are true. Using this mode of reasoning, we can deduce $p \vee q$; we can just as well deduce $q \vee p$.

5.3 Disjunction Introduction

There are, technically, two proof rules in natural deduction for the introduction of a disjunction in a valid proof.

Proof Rule: **disjunction**-*introduction*: $\vee\mathrm{i}$

$$\frac{\phi}{\phi \vee \psi}\ \vee\mathrm{i}_1$$

$$\frac{\phi}{\psi \vee \phi}\ \vee\mathrm{i}_2$$

We can now prove that disjunction is commutative:

$$p \vee q \vdash q \vee p$$ *H&R, p. 17*

We can also prove that disjunction can be introduced *into* an implication:

$$q \to r \vdash p \vee q \to p \vee r$$ *H&R, p. 18*

5.4 Strategies For Propositional Assumptions

In natural deduction, we have used "assumption boxes" in our exploration of propositional logic. Let us review some of these uses so that we are sure that we understand them.

5.4.1 Strategies For Implication Introduction

This is used when we want to deduce a formula $\phi \to \psi$. We assume ϕ and, using deduction rules, assert ψ; from this we can conclude that ϕ implies ψ.

Example: "Suppose that the falsity of q implies the falsity of p; we can conclude that p implies q."

$\neg q \to \neg p \vdash p \to q$ *H&R, p. 13*

Proof Strategy: We can prove this using double negation and MT, Modus Tollens. To begin, we write the premise and the conclusion; we leave space for the intermediate lines of the proof.

1	$\neg q \to \neg p$	premise
	$\vdots$	
	$p \to q$	

The conclusion is an implication so we "open" an $\to$i box. *Important*: we put the antecedent at the top and the consequent at the bottom so that we know exactly what we are assuming and what we wish to deduce.

1	$\neg q \to \neg p$	premise
2	p	assumption
	$\vdots$	
?	q	
6	$p \to q$	$\to$ i 2–?

We can now fill in the rest of the proof. From p we can deduce $\neg\neg p$; applying the rule MT we can deduce $\neg\neg q$; and from this we can deduce q. The complete proof looks like

1	$\neg q \to \neg p$	premise
2	p	assumption
3	$\neg\neg p$	$\neg\neg$i, 2
4	$\neg\neg q$	MT 1,3
5	q	$\neg\neg$i, 4
6	$p \to q$	$\to$ i 2–7

Key Concept: The $\to$i rule uses "backward" logical reasoning. When we open the $\to$i "box", we put the antecedent at the top – as an assumption – and the consequent at the bottom – as the desired conclusion. This constrains our reasoning so that the $\to$i rule is applied correctly.

5.4.2 Strategies For Negation Introduction

The rule of negation introduction $\neg$ i says that, to prove a formula that is a negation which is of the form $\neg\phi$, we assume the "positive" formula ϕ. If we can deduce a contradiction $\perp$ then our assumption ϕ must be false, so $\neg\phi$ must be true.

Example: "Suppose that: p implies q is true, and that p implies q is false. We can conclude that p is false."

$$p \to q, p \to \neg q \vdash \neg p$$ *H&R, p. 22*

Proof Strategy: We can reason that, if the opposite of the conclusion is assumed, then we can deduce both q and $\neg q$ which are contradictory.

We write the premises and conclusion, leaving space for the rest of the proof.

1	$p \to q$	premise
2	$p \to \neg q$	premise
	$\vdots$	
?	$\neg p$	

Our strategy is to assume the opposite of our desired conclusion. We do this by opening an assumption box. The simplest "opposite" of $\neg p$ is p; to correctly apply the ¬i rule, we need to end with a contradiction $\bot$. We know two additional things: the ¬i rule is used to deduce the conclusion and the ¬e rule will be used to deduce the contradiction $\bot$. We note these in the "rules" column; the question marks are in these notes but in a proof we would temporarily leave them blank.

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
	$\vdots$	
	$\bot$	¬e ?,?
?	$\neg p$	¬i ?–?

We can now finish the proof. From the assumption p we can deduce q, and from p we can also deduce $\neg q$. From these we can deduce that a contradiction is present and we have solved the problem.

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	→e 1, 3
5	$\neg q$	→e 2, 3
6	$\bot$	¬e 4, 5
7	$\neg p$	¬i 3–6

Key Concept: The ¬i rule uses "backward" logical reasoning. When we open the ¬i "box", we put the "positive" of the conclusion at the top of the box – as an assumption – and the contradiction symbol $\bot$ at the bottom – as the desired conclusion. This constrains our reasoning so that the ¬i rule is applied correctly.

5.4.3 Strategies For Assumption Boxes

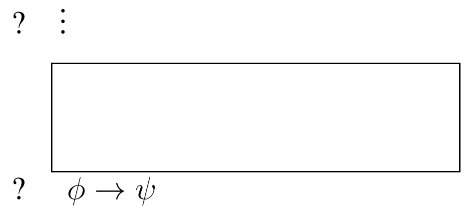
We can summarize a strategy for using assumption boxes as a series of 4 steps. We *always* describe these with "backward" reasoning, although a student may prefer to use "forward" reasoning in proving a particular given sequent.

We will describe the strategy for the rule on implication introduction, $\rightarrow$i, understanding that the strategy works for all of the assumption-box rules in natural deduction. These are the steps we will use to prove that formula ϕ implies formula ψ, which is $\phi \rightarrow \psi$, using the rule $\rightarrow$i.

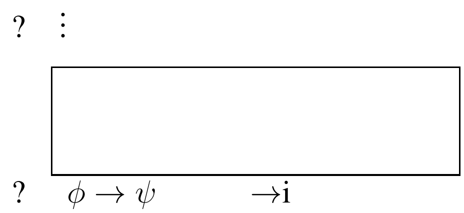
Prior to using the rule, we have a proof so far that is indicated by vertical dots, and we have the conclusion $\phi \rightarrow \psi$ that we are trying to prove. This is

$$\begin{array}{ll} ? & \vdots \\ ? & \phi \rightarrow \psi \end{array}$$

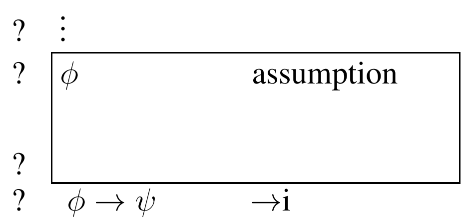
Step 1: Draw the assumption box. This indicates that we are planning to use one of the more powerful rules in natural deduction.



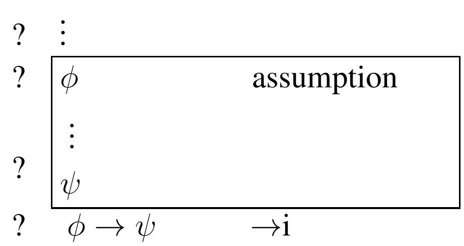
Step 2: State the rule beside the conclusion, outside of the box. This specifies the kind of reasoning that we are intending to employ in this part of the proof.



Step 3: Write the assumption as the first line in each assumption box, and state that this is an assumption. We will be able to use this line, within the scope of the assumption box, as we proceed in the proof.



Step 4: Write the necessary proposition as the last line in each assumption box. We now know what proposition we are aiming to prove.



Using these steps is a useful strategy for correctly employing assumption boxes in a proof that uses natural deduction.