CISC 204 Class 13

Motivation for Predicate Logic

Text Correspondence: pp. 93–95

Main Concepts:

- *Need to quantify logical statements*
- *Parse Tree: graphical represnetation of a parsing of a formula*

13.1 Predicate Logic: the Need for a Richer Language

Some logical arguments cannot be expressed using propositional logic. A classical syllogism of Aristotle is

$$\begin{array}{l} \text{All humans are mortal} \\ \text{Socrates is human} \\ \text{Therefore, Socrates is mortal} \end{array} \tag{13.1}$$

If we try to express this argument using propositional logic, we can see that propositions do not entirely capture our intent. For example, we might try rephrasing the syllogism to be

$$\begin{array}{l} \text{A human is mortal} \\ \text{Socrates is a human} \\ \text{Therefore, Socrates is mortal} \end{array} \tag{13.2}$$

The logic of Argument 13.2 could be written as

$$\begin{array}{l} H \to M \\ S \to H \\ \hline S \to M \end{array} \tag{13.3}$$

Although Argument 13.3 properly encodes Argument 13.2, it does not capture the sense of "all" that is in Argument 13.1[1].

To express such arguments we must be able to separate individuals from their properties. This is achieved by using *predicates* to describe properties or relationships. We will also introduce *functions* into the language to express mathematical relationships.

Roughly speaking, predicate logic is propositional logic augmented by *quantifiers*, predicates and functions. Many authoritative texts prefer to call this *first-order logic*, because only variables can be quantified; we will simply call it predicate logic.

We can begin to understand how predicate logic works by introducing *variables* into propositional logic. We might use the symbol x to represent a variable; for now we will not be concerned with values that a variable can have, leaving that until we discuss semantics.

One of the easier ways to represent the concept "A human is mortal" is to say that, if a variable satisfies the predicate "is human", then it satisfies the predicate "is mortal. We can re-write the antecedent of the first line of Argument 13.2 as

$$H(x) \to M(x)$$

Introducing the predicate "is Socrates", and writing that a variable satisfies this predicate as $S(x)$, we can translate Argument 13.2 into a form that uses predicates:

$$\begin{array}{l} H(x) \to M(x) \\ S(x) \to H(x) \\ \hline S(x) \to M(x) \end{array} \tag{13.4}$$

This expresses the ideas in Argument 13.2 but it does not yet capture the concepts in the original syllogism of Argument 13.1. To do this, we need to *quantify* the variable x. For the first line of the original syllogism, we could write

For all x, if x is a human then x is mortal

This is typically done by using the *universal quantifier* symbol $\forall$. The first line can be written, in predicate logic, as

$$\forall x \, (H(x) \to M(x))$$

[1] This lack of expressiveness and rigor was observed by the philosopher Gottlob Frege in 1879, later made understandable and applied to number theory by Giuseppe Peano. We owe the systematic study of the foundations of mathematics to these and other intellectual pioneers.

The second line of Argument 13.1 is asserting that some individual is Socrates and that this individual is also human. There are at least two ways to write this, using implication and using conjunction. Choosing implication as better capturing the sense of the second line, we can write

For all x, if x is Socrates, then x is human

In predicate logic, this can be expressed using the *existential quantifier* $\exists$ as the formula

$$\exists x(S(x) \wedge H(x))$$

To go further, we will formally define the language of predicate logic [2].

[2] The quantifier $\exists$ was introduced by Peano in 1897; the quantifier $\forall$ was introduced in 1935 by Gerhard Gentzen, who formalized natural deduction as presented in this course.