

CISC 204 Class 17

Proof Rules For Existential Quantifiers

Text Correspondence: pp. 112–117

Main Concepts:

- *Existential Introduction: replacing a variable with an existential quantifier*
- *Existential Elimination: deducing a formula by replacing an existential quantifier with an assumed “fresh” variable*

Existential quantification is the assertion that, for the universe of values that a variable may have, there is at least one value that makes a logical formula true. If the formula is ϕ , and the variable x is free in ϕ , then we assert this idea using the formula $\exists x \phi$.

Consider a simple situation, where we know that the value a has a property P . We write this assertion as $P(a)$. From this, we intuitively understand that we can conclude that *some* x has the property P ; we want to be able to perform the reasoning

$$\frac{P(a)}{\exists x P(x)}$$

Next, consider a situation where the value a has property P and property Q . We want to conclude that there is some x that also has both properties. This means that we want to be able to argue

$$\frac{P(a) \wedge Q(a)}{\exists x (P(x) \wedge Q(x))}$$

Can we make this more general? What we would much rather be able to do is to work with a formula ϕ and a variable x that is free in ϕ . If the formula ϕ is true when we substitute some specific term t for x , then we know that there is *some* value of x for which the formula ϕ is valid. The substitution is written as $\phi[t/x]$ and we already know how to write the existential quantification.

17.1 Existential Introduction

The above mode of intuitive reasoning can be added to predicate logic as a rule of deduction.

Proof Rule: Existential Introduction, $\exists x i$

$$\frac{\phi[t/x]}{\exists x \phi} \exists x i$$

Example

Consider the sequent

$$\forall x P(x) \vdash \exists x P(x)$$

One strategy for proving the validity of the sequent is to examine the conclusion, which is $\exists x \phi$. We observe that this could be deduced if we know that some value a has the property P .

Examining the single premise, we see that *any* value of x has the property P . We can apply universal elimination to the formula and deduce that a value a has P ; from this we can deduce the conclusion. One proof would look like

1	$\forall x P(x)$	premise
2	$P(a)$	$\forall x e 1$
3	$\exists x P(x)$	$\exists i 2$

This strategy can be used to prove a much more general theorem, which is an example in the textbook.

Example

Consider the sequent

$$\forall x \phi \vdash \exists x \phi$$

One strategy for proving the validity of the sequent is to examine the conclusion, which is $\exists x \phi$. We observe that this could be deduced if we know that some value a satisfied ϕ .

Examining the single premise, we see that ϕ is satisfied by *any* value of x . We can apply universal elimination to the formula and deduce that a value a satisfies ϕ ; from this we can deduce the conclusion. One proof would look like

1	$\forall x \phi$	premise
2	$\phi[a/x]$	$\forall x e 1$
3	$\exists x \phi$	$\exists i 2$

What we have done in Line 2 of the proof is to assert that, because ϕ is satisfied by every value, it is satisfied by a specific value. The line can be confusing because, up to now, we have mainly used substitution to justify the use of a rule of deduction. In this case, we are using substitution to make an assertion. This is another reason why we must be careful about the scope of variables: substitution is a highly effective tool in predicate logic so it must be used carefully.

17.2 Existential Elimination

Intuitively, we can see that if there is some value of x that has a property P , then we can give that value at least one name. This name is often called a *dummy variable*¹ or a dummy value. The textbook also uses the wording “fresh” variable, which is the wording preferred in this course.

We might, for instance, use the value a to be the “fresh” value so that we can assert $P(a)$. This would allow us to reason that

$$\frac{\exists x P(x)}{P(a)}$$

This way of reasoning is, superficially, not too useful but we must consider it carefully. Suppose that we want to perform this line of reasoning:

$$\begin{array}{l} \text{All humans are mortal} \\ \text{Something is human} \\ \text{Something is mortal} \end{array} \quad (17.1)$$

We can translate the premises into predicate logic as

$$\forall x (H(x) \rightarrow M(x)), \exists x H(x)$$

and we can translate the conclusion as

$$\exists x M(x)$$

We know that we can apply universal elimination to the first premise, so we can assert that any particular value satisfies the implication. What we need to be able to do is to give a “fresh” name to a value that satisfies the second premise, so that we can use Modus Ponens and eventually get to the conclusion, without making an incorrect assertion along the way.

Another way to write this is: suppose a is a “fresh” value; assume that a has property H ; then a has property M ; then there is some x that has property M .

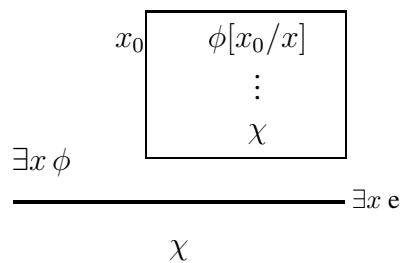
¹Here, the sense of “dummy” is as a copy or a substitute and is not meant to be pejorative.

In English, we want to be able to argue that: given an existentially quantified formula, if we introduce a “fresh” variable *into that formula* and can arrive at a conclusion that does not mention the “fresh” variable, then we can assert the conclusion.

To formalize this as a new rule of deduction, we need to clarify some terminology.

Let x_0 be a “fresh” value and let ϕ be a formula such that x_0 is free in ϕ for x . Let χ be a formula such that x_0 is not “mentioned” in χ , that is, x_0 is not in any term in χ . The rule of deduction is called existential elimination, because we will use the rule to eliminate the existential quantifier of the first premise.

Proof Rule: Existential Elimination, $\exists x e$



With this rule, we can now prove Syllogism 17.1.

Example

Consider the sequent

$$\forall x (H(x) \rightarrow M(x)), \exists x H(x) \vdash \exists x M(x)$$

The second premise is of the form $\exists x \phi$. Can we find a “fresh” variable that leads us to the conclusion? We can use z as such a variable.

Substituting z into ϕ , we get $\phi[z/x]$ which is $H(z)$. From this we can use existential introduction and deduce that there is some x that has property M , which in symbols is $\exists x M(x)$. This latter formula does not use the “fresh” variable z , so it meets the requirement of the rule of existential elimination. Our proof is:

1	$\forall x(H(x) \rightarrow M(x))$	premise
2	$\exists x H(x)$	premise
3	$z \quad H(z)$	assumption
4	$H(z) \rightarrow M(z)$	$\forall x e 1$
5	$M(z)$	$\rightarrow e 4, 3$
6	$\exists x M(x)$	$\exists x i, 5$
7	$\exists x M(x)$	$\exists x e 2, 3-6$ where ϕ is $H(x)$ and χ is $\exists x M(x)$

We were careful, in this proof, to use the same formula ϕ in the scope of the quantifier of Line 2 and in the assumption of Line 3, where Line 3 is a substitution. In Line 6, the formula χ does not use the “fresh” variable z so the rule of deduction has been correctly applied. (See the textbook for further discussion, including an invalid variant of this proof.)