CISC 204 Class 20

Commuting Universal Quantifiers

Text Correspondence: pp. 117–119

Main Concepts:

- *Universal quantifiers commute*

The next sequent that we will prove is from the textbook, Theorem 2.13, Part 4(a). In English, the theorem states that universal quantifiers commute; if we can prove the first "direction" then the second "direction" will be true by re-naming symbols. In symbols, the sequent is

$$\forall x\, \forall y\, \phi \vdash \forall y\, \forall x\, \phi$$

We will prove an instance of the formula that is

$$\forall x\, \forall y\, R(x, y) \vdash \forall y\, \forall x\, R(x, y) \tag{20.1}$$

After writing the premise as Line 1, we look at the conclusion. It is a universal quantification so the rule $\forall x$ i might be useful. N.B. The textbook calls the rule $\forall x$ i, but this does not mean that the variable x is always quantified; we can use another one and, indeed, for this example we must use the variable y. Some texts call the rule UI, for universal introduction, which is neutral with respect to the variable name.

Using "backward" reasoning, we can deduce the conclusion by using the rule $\forall x$ i. This requires using a "fresh" variable so we will use z, which does not appear in the premise or in the conclusion. Our strategy so far is

1		$\forall x\, \forall y\, R(x, y)$	premise
2	z		
$\vdots$		$\vdots$	
		$\forall x\, R(x, z)$	
?		$\forall y\, \forall x\, R(x, y)$	$\forall x$ i 2–...

The formula in the last line of the assumption box in general is ψ, which in this proof is $\forall x\, R(x, z)$. This formula ψ is also universally quantified, so a "nested" application of $\forall x$ i might be useful. The form of this nested deduction could be

?	w	
$\vdots$	$\vdots$	
	$R(w,z)$	
?	$\forall x\, R(x,z)$	$\forall x$ i ?–. . .

To avoid any potential confusion over the order of substitution, we can introduce the "fresh" variables before we perform the substitution. Putting these observations together, one proof of Sequent 20.1 is

1	$\forall x\, \forall y\, R(x,y)$	premise
2	z	
3	w	
4	$\forall y\, R(w,y)$	$\forall x$ e 1
5	$R(w,z)$	$\forall x$ e 4
6	$\forall x\, R(x,z)$	$\forall x$ i 3–5
7	$\forall y\, \forall x\, R(x,y)$	$\forall x$ i 2–6

In English, using variable names, the proof is

Line 2: introduce fresh z.

Line 3: introduce fresh w.

Line 4: replace w for x in Line 1.

Line 5: replace z for y in Line 4.

Line 6: from Lines 3–5, generalize over all x. (This is the step where the commutation of quantifiers takes place; we substituted for y and are now generalizing over x.)

Line 7: from Lines 2–6, generalize over all y.

At this stage of the course, students should understand the textbook proofs for parts of Theorem 2.13 and be able to prove any of its sequents. On tests, we will work with specific predicates (such s $P(x)$ or $R(x,y)$ rather than with a general formula (such as ϕ).