CISC 204 Class 26

Models of Formulas in Predicate Logic

Text Correspondence: pp. 128–129

Main Concepts:

- $\forall x$ *semantics: must hold for all possibl assignments of the bound variable to a value*
- $\exists x$ *semantics: must hold for at least one possible assignment of the bound variable to a value*

26.1 Models of Formulas

As described in the textbook, quantifiers have the semantics that we might expect from our intuitions about logic:

- $\forall x\, \psi$ holds for all possible substitutions of x with a value $a \in A$
- $\exists x\, \psi$ holds for some possible substitution of x with a value $a \in A$

It is useful to examine some assertions that are stated in symbolic logic, to see whether the assertions hold in this model. The textbook considers five formulas:

1. $\forall x\, ((x \leq x \cdot e) \land (x \cdot e \leq x))$
2. $\exists y\, \forall x\, (y \leq x)$
3. $\forall x\, \exists y\, (y \leq x)$
4. $\forall x\, \forall y\, \forall z\, ((x \leq y) \rightarrow (x \cdot z \leq y \cdot z))$
5. $\neg \exists x\, \forall y\, ((x \leq y) \rightarrow (y \leq x))$

First, we must understand why the predicate and functions are not restricted to the model $\mathcal{M}$, that is, why we do not use $\leq^{\mathcal{M}}$ and $\cdot^{\mathcal{M}}$ and $e^{\mathcal{M}}$. This is because the logical statements are general formulas; they do not have "meaning" until we provide a model. Once we have a model, we have a specific way of modeling everything that is stated in a general logical formula.

Looking at the first assertion, there are certainly other models where this does not hold; it holds because the function e in the model $\mathcal{M}$ is the empty string, which does not "change" a string. Students should be able to translate this formula into English; in addition to the textbook translation, we could say "The prefix property is invariant to post-concatenation with the empty string".

Students should be able to translate each of the five formulas into English. Where an existential quantifier is used, a student should be able to provide a concrete example; an instance with specific strings of 0's and 1's will help in learning how to apply this semantic model to predicate logic.

Example: Finite-state machine

Let us consider an example from computer science. A finite-state machine has *states*, often represented as *nodes* in a directed graph; the *edges* of the graph are the state transitions, which are the allowable ways of moving from one state to another. The initial state is often represented as an arrow pointing into the graph, and the final state often is a double circle. There is usually only one initial state, but there can be more than one final state.

The textbook, in Example 2.15, gives a logical model for a finite-state machine. We can draw the model as shown in Figure 26.1, where the arcs are the allowed state transitions. In this machine, the initial state is a and the final states are b and c. From State a we can go to any state; from State b we can go only to State c; and from State c we can go only to State c.

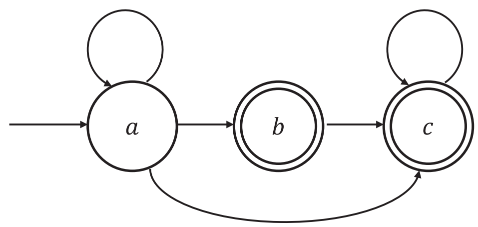

Figure 26.1: A finite-state machine with 3 states. State a is the initial state and either b or c is a final state. Allowed transitions are arcs between states.

The textbook model for logical reasoning about this machine begins with the universe of discourse. The authors decided that they wanted to reason about states in the machine, so they picked the states as the values that a variable can take. This is why they selected

$$A \stackrel{\text{def}}{=} \{a, b, c\}$$

To be able to formulate statements about the initial state, they chose to introduce a constant function. There were no other functions of interest, so they limited the space of functions to

$$\mathcal{F} \stackrel{\text{def}}{=} \{i\}$$

Consulting Figure 26.1, they want the initial function to return State a. We must be careful here: i is a nullary function, so in the absence of a model we can do relatively little with it. However, $i^{\mathcal{M}}$ is a specific function within the model; once it is semantically defined, we can reason about it with specificity. The authors define it as

$$i^{\mathcal{M}} \stackrel{\text{def}}{=} a$$

The authors chose to have just two predicates in the model. From Figure 26.1, the predicates are to determine whether or not a state is a final state, and whether or not there is a transition from one state to another. They picked the symbol F to represent the unary function that lets us determine whether State x is a final state or not. They picked the symbol R to represent the binary function that lets us determine whether State x can transition to State y. This means that the space of predicates is

$$\mathcal{P} \stackrel{\text{def}}{=} \{F, R\}$$

The authors chose to use extensional definitions of the predicates in their model. The final-state predicate $F^{\mathcal{M}}(x)$ is true if and only if x is either b or c. The transition relation $P^{\mathcal{M}}(x, y)$ is true if and only if the diagram in Figure 26.1 has a transition from State x to State y.

The semantic model of the state machine, presented by the authors, is

$$\begin{aligned}
A &\stackrel{\text{def}}{=} \{a, b, c\} \\
\mathcal{P} &\stackrel{\text{def}}{=} \{F, R\} \\
\mathcal{F} &\stackrel{\text{def}}{=} \{i\} \\
F^{\mathcal{M}}(x) &\stackrel{\text{def}}{=} \{b, c\} \\
R^{\mathcal{M}}(x, y) &\stackrel{\text{def}}{=} \{(a, a), (a, b), (a, c), (b, c), (c, c)\} \\
i^{\mathcal{M}} &\stackrel{\text{def}}{=} a
\end{aligned}$$

As examples of reasoning within this semantic model, the textbook gives the formulas:

1.	$\exists y\, R(i, y)$	Consider $R(a, b)$
2.	$\neg\, F(i)$	Consider $F(a)$
3.	$\neg\, \forall x\, \forall y\, \forall z\, ((R(x, y) \wedge R(x, z)) \rightarrow (y = z))$	Consider $R(a, b) \wedge R(a, c)$
4.	$\forall x\, \exists y\, R(x, y)$	Consider $\forall x\, R(x, c)$

Students can use the transition diagram, and the textbook reasoning, to verify these formulas. Students should also be able to translate each of these symbolic formulas into plain English; examples for the first formula include “The initial state transitions to some state” or “Some state has a transition from the initial state”. Many other translations are also reasonable.

End of Extra Notes