Sound and Complete Bidirectional Typechecking for Higher-Rank Polymorphism with Existentials and Indexed Types

JOSHUA DUNFIELD, Queen’s University, Canada
NEELAKANTAN R. KRISHNASWAMI, University of Cambridge, United Kingdom

Bidirectional typechecking, in which terms either synthesize a type or are checked against a known type, has become popular for its applicability to a variety of type systems, its error reporting, and its ease of implementation. Following principles from proof theory, bidirectional typing can be applied to many type constructs. The principles underlying a bidirectional approach to indexed types (generalized algebraic datatypes) are less clear. Building on proof-theoretic treatments of equality, we give a declarative specification of typing based on focalization. This approach permits declarative rules for coverage of pattern matching, as well as support for first-class existential types using a focalized subtyping judgment. We use refinement types to avoid explicitly passing equality proofs in our term syntax, making our calculus similar to languages such as Haskell and OCaml. We also extend the declarative specification with an explicit rules for deducing when a type is principal, permitting us to give a complete declarative specification for a rich type system with significant type inference. We also give a set of algorithmic typing rules, and prove that it is sound and complete with respect to the declarative system. The proof requires a number of technical innovations, including proving soundness and completeness in a mutually recursive fashion.

Additional Key Words and Phrases: bidirectional typechecking, higher-rank polymorphism, indexed types, GADTs, equality types, existential types

1 INTRODUCTION

Consider a list type Vec with a numeric index representing its length, in Agda-like notation:

\[
data \text{Vec} : \text{Nat} \rightarrow \text{Type} \rightarrow \text{Type} \where
\]
- \(\text{[]} : \forall A. \text{Vec} 0 A\)
- \((::) : \forall A. \text{Vec} n A \rightarrow \text{Vec} (\text{succ} n) A\)

We can use this definition to write a head function that always gives us an element of type \(A\) when the length is at least one:

\[
\text{head} : \forall n, A. \text{Vec} (\text{succ} n) A \rightarrow A
\]
\[
\text{head} (x :: xs) = x
\]

This clausal definition omits the clause for \(\text{[]}\), which has an index of 0. The type annotation tells us that head’s argument has an index of \(\text{succ} (n)\) for some \(n\). Since there is no natural number \(n\) such that \(0 = \text{succ} (n)\), the nil case cannot occur and can be omitted.

This is an entirely reasonable explanation for programmers, but language designers and implementors will have more questions. First, designers of functional languages are accustomed to the
benefits of the Curry–Howard correspondence, and expect a logical reading of type systems to accompany the operational reading. So what is the logical reading of GADTs? Second, how can we implement such a type system? Clearly we needed some equality reasoning to justify leaving off the nil case, which is not trivial in general.

Since we relied on equality information to omit the nil case, it seems reasonable to look to logical accounts of equality. In proof theory, it is possible to formulate equality in (at least) two different ways. The better-known is the identity type of Martin-Löf, but GADTs actually correspond best to the equality of Schroeder-Heister [1994] and Girard [1992]. The Girard–Schroeder-Heister (GSH) approach introduces equality via the reflexivity principle:

\[ \Gamma \vdash t = t \]

The GSH elimination rule was originally formulated in a sequent calculus style, as follows:

\[ \text{for all } \theta, \text{ if } \theta \in \text{csu}(s, t) \text{ then } \theta(\Gamma) \vdash \theta(C) \]

\[ \Gamma, (s = t) \vdash C \]

Here, we write csu(s, t) for a complete set of unifiers of s and t. So the rule says that we can eliminate an equality s = t if, for every substitution \( \theta \) that makes s and t equal, we can prove the goal C.

This rule has three important features, two good and one bad.

1. The GSH elimination rule is an invertible left rule. By “left rule”, we mean that the rule decomposes the assumptions to the left of the turnstile (in this case, the assumption that s = t), and by “invertible”, we mean the conclusion of the rule implies the premises.\(^1\) Invertible left rules are interesting, because they are known to correspond (via Curry–Howard) to the deconstruction steps carried out by pattern-matching rules [Krishnaswami 2009]. This is our first hint that the GSH rule has something to do with GADTs; programming languages like Haskell and OCaml indeed use pattern matching to propagate equality information.

2. Observe that if we have an inconsistent equation, we can immediately prove the goal. If we specialize the rule above to the equality 0 = 1, we get:

\[ \Gamma, (0 = 1) \vdash C \]

Because 0 and 1 have no unifiers, the complete set of unifiers is the empty set. As a result, the GSH rule for this instance has no premises, and the elimination rule for an absurd equation ends up looking exactly like the elimination rule for the empty type:

\[ \Gamma, \bot \vdash C \]

Moreover, recall that when we eliminate an empty type, we can view the eliminator \texttt{abort(e)} as a pattern match with no clauses. Together, these two features line up nicely with our definition of \texttt{head}, where the impossibility of the case for [] was indicated by the absence of a pattern clause. The use of equality in GADTs corresponds perfectly with the GSH equality.

3. Alas, we cannot simply give a proof term assignment for first-order logic and call it a day. The third important feature of the GSH equality rule is its use of unification: it works by treating the free variables of the two terms as unification variables. But type inference algorithms also use unification, introducing unification variables to stand for unknown types. These two uses of unification are entirely different! Type inference introduces unification variables to stand for the specific instantiations of universal quantifiers. In contrast, the Girard–Schroeder-Heister rule uses unification to constrain the universal parameters. As a

---

\(^1\)The invertibility of equality elimination is certainly not obvious; see Schroeder-Heister [1994] and Girard [1992].
result, we need to understand how to integrate these two uses of unification, or at least how to keep them decently separated, in order to take this logical specification and implement type inference for it.

This problem—formulating indexed types in as logical a style as feasible, while retaining the ability to implement type inference algorithms for them—is the subject of this paper.

**Contributions.** It has long been known that GADTs are equivalent to the combination of existential types and equality constraints [Xi et al. 2003]. Our key contribution is to reduce GADTs to standard logical ingredients, while retaining the implementability of the type system. We manage this by formulating a system of indexed types in a bidirectional style (combining type synthesis with checking against a known type), which is both practically implementable and theoretically tidy.

- Our language supports implicit higher-rank polymorphism (in which quantifiers can be nested under arrows) including existential types. While algorithms for higher-rank universal polymorphism are well-known [Peyton Jones et al. 2007; Dunfield and Krishnaswami 2013], our approach to supporting existential types is novel.

  We go beyond the standard practice of tying existentials to datatype declarations [Läuffer and Odersky 1994], in favour of a first-class treatment of implicit existential types. This approach has historically been thought difficult, since treating existentials in a first-class way opens the door to higher-rank polymorphism that mixes universal and existential quantifiers.

  Our approach extends existing bidirectional methods for handling higher-rank polymorphism, by adapting the proof-theoretic technique of focusing to give a novel polarized subtyping judgment, which lets us treat mixed quantifiers in a way that retains decidability while maintaining the essential properties of subtyping, such as stability under substitution and transitivity.

- Our language includes equality types in the style of Girard and Schroeder-Heister, but without an explicit introduction form for equality. Instead, we treat equalities as property types, in the style of intersection or refinement types: we do not write explicit equality proofs in our syntax, permitting us to more closely model how equalities are used in OCaml and Haskell.

- The use of focusing also lets us equip our calculus with nested pattern matching. This fits in neatly with our bidirectional framework, and permits us to give a formal specification of coverage checking with GADTs, which is easy to understand, easy to implement, and theoretically well-motivated.

- In contrast to systems which globally possess or lack principal types, our declarative system tracks whether or not a derivation has a principal type.

  Our system includes an unusual “higher-order principality” rule, which says that if only a single type can be synthesized for a term, then that type is principal. While this style of hypothetical reasoning is natural to explain to programmers, formalizing it requires giving an inference rule with universal quantification over possible typing derivations in the premise. This is an extremely non-algorithmic rule (even its well-foundedness is not immediate).

  As a result, the soundness and completeness proofs for our implementation have to be done in a new style. It is no longer possible to prove soundness and completeness independently, and instead we must prove them mutually recursively.

- We formulate an algorithmic type system (Section 5) for our declarative calculus, and prove that typechecking is decidable, deterministic (5.4), and sound and complete (Sections 6–7) with respect to the declarative system.

  The resulting type system is relatively easy to implement (an undergraduate implemented most of it on his own from a draft of the paper, with minimal contact with the authors), and
is close in style to languages such as Haskell or OCaml. As a result, it seems like a reasonable basis for implementing new languages with expressive type systems.

Our algorithmic system (and, to a lesser extent, our declarative system) uses some techniques developed by Dunfield and Krishnaswami [2013], but we extend these to a far richer type language (existentials, indexed types, sums, products, equations over type variables), and we differ by supporting pattern matching, polarized subtyping, and principality tracking.

Supplementary material. The appendix contains rules omitted for space reasons, and full proofs.

2 OVERVIEW
To orient the reader, we give an overview and rationale of the novelties in our type system, before getting into the details of the typing rules and algorithm. As is well-known [Cheney and Hinze 2003; Xi et al. 2003], GADTs can be desugared into type expressions that use equality and existential types to express the return type constraints. These two features lead to difficulties in type-checking for GADTs.

Universal, existentials, and type inference. Practical typed functional languages must support some degree of type inference, most critically the inference of type arguments. That is, if we have a function $f$ of type $\forall a. a \to a$, and we want to apply it to the argument 3, then we want to write $f\,3$, and not $f\,[\text{Nat}]\,3$ (as we would in pure System F). Even with a single type argument, the latter style is noisy, and programs using even moderate amounts of polymorphism rapidly become unreadable.

However, omitting type arguments has significant metatheoretical implications. In particular, it forces us to include subtyping in our typing rules, so that (for instance) the polymorphic type $\forall a. a \to a$ is a subtype of its instantiations (like $\text{Nat} \to \text{Nat}$).

The subtype relation induced by System F-style polymorphism and function contravariance is already undecidable [Tiuryn and Urzyczyn 1996; Chrząszcz 1998], so even at the first step we must introduce restrictions on type inference to ensure decidability. In our case, matters are further complicated by the fact that we need to support existential types in addition to universal types.

Existentials are required to encode GADTs [Xi and Pfenning 1999], but programming languages have traditionally stringently restricted the use of existential types. Following the approach of Läufer and Odersky [1994], languages such as OCaml and Haskell tie existential introduction and elimination to datatype declarations, so that there is always a syntactic marker for when to introduce or eliminate existential types. This choice permits leaving existentials out of subtyping altogether, at the price of no longer permitting implicit subtyping (such as using $\lambda x. x + 1$ at type $\exists a. a \to a$).

While this is a practical solution, it increases the distance between a surface language and its type-theoretic core. Our goal is to give a direct type-theoretic account of as many features of our surface languages as possible. In addition to the theoretical tidiness, this also has practical language design benefits. By avoiding a complex elaboration step, we are forced to specify the type inference algorithm in terms of a language close to a programmer-visible surface language. This does increase the complexity of the approach, but in a productive way: we are forced to analyze and understand how type inference will look to the end programmer.

The key problem is that when both universal and existential quantifiers are permitted, the order in which to instantiate quantifiers when computing subtype entailments becomes unclear. For example, suppose we need to decide $\Gamma \vdash \forall a_1. \exists a_2. A(a_1, a_2) \leq \exists b_1. \forall b_2. B(b_1, b_2)$. An algorithm to solve this must either first introduce a unification variable for $a_1$ and a parameter for $a_2$ first, and only then introduce a unification variable for $b_1$ and a parameter for $b_2$, or the other way around—and the order in which we make these choices matters! With the first order, the instantiation for

$b_1$ may refer to $a_2$, but the instantiation for $a_1$ cannot have $b_2$ as a free variable. With the second order, the instantiation for $a_1$ may have $b_2$ as a free variable, but $b_1$ may not refer to $a_2$.

In some cases, depending on what $A(a_1, a_2)$ and $B(b_1, b_2)$ are, only one choice of order works. For example, if we are trying to decide $\Gamma \vdash \forall a_1. \exists a_2. a_1 \rightarrow a_2 \leq \exists b_1, \forall b_2. b_2 \rightarrow b_1$, we must choose the first order: we must pick an instantiation for $a_1$, and then make $a_2$ into a parameter before we can instantiate $b_1$ as $a_2$. The second order will not work, because $b_1$ must depend on $a_2$. Conversely, if we are trying to solve $\Gamma \vdash \forall a_1. \exists a_2. a_1 \rightarrow a_2 \leq \exists b_1, \forall b_2, b_1 \times b_2 \rightarrow b_3$, the first order will not work; we must instantiate $b_1$ (say, to int) and quantify over $b_2$ before instantiating $a_1$ as int $\times b_2$.

As a result, the outermost connectives do not reliably determine which side of a subtype judgement $\Gamma \vdash \forall a. A \leq \exists b. B$ to specialize first.

One implementation strategy is simply to give up determinism: an algorithm could backtrack when faced with deciding subtype entailments of this form. Unfortunately, backtracking is dangerous for a practical implementation, since it potentially causes type-checking to take exponential time. This tends to defeat the benefit of a complete declarative specification, since different implementations with different backtracking strategies could have radically different running times when checking the same program. So we may end up with an implementation that is theoretically complete, but incomplete in practice.

Instead, we turn to ideas from proof theory—specifically, polarized type theory. In the language of polarization, universals are a negative type, and existentials are a positive type. So we introduce two mutually recursive subtype relations: $\Gamma \vdash A \leq^+ B$ for positive types and $\Gamma \vdash A \leq^- B$ for negative types. The positive subtype relation only deconstructs existentials, and the negative subtype relation only deconstructs universals. This fixes the order in which quantifiers are instantiated, making the problem decidable (in fact, rather straightforward).

The price we pay is that fewer subtype entailments are derivable. Fortunately, any program typeable under a more liberal subtyping judgement can be made typable in our discipline by $\eta$-expanding it. Moreover, the lost subtype entailments seem to be rare in practice: most of the types we see in practice are of the form $\forall a. \tilde{A} \rightarrow \exists \tilde{b}. B$, and this fits perfectly with the kinds of types our polarized subtyping judgement works best on. As a result, we keep fundamental expressivity, and also efficient decidability.

Equality as a property. The usual convention in Haskell and OCaml is to make equality proofs in GADT definitions implicit. We would like to model this feature directly, so that our calculus stays close to surface languages, without sacrificing the logical reading of the system. In this case, the appropriate logical concepts come from the theory of intersection types. A typing judgment such as $e : A \times B$ can be viewed as giving instructions on how to construct a value (pair an $A$ with a $B$). But types can also be viewed as properties, where $e : X$ is read "$e$ has property $X$".

To model GADTs, we need both of these readings! For example, a term of vector type is constructed from nil and cons constructors, but also has a property governing its index. To support this combination, we introduce a type constructor $A \land P$, read "$A$ with $P$", to model elements of type $A$ satisfying the property (equation) $P$. (We also introduce $P \supset A$, read "$P$ implies $A$", for its adjoint dual, consisting of terms which have the type $A$ conditionally under the assumption that $P$ holds.) Then we make equality $t = t'$ into a property, and make use of standard rules for property types (which omit explicit proof terms) to type equality constraints [Dunfield 2007b, Section 2.4].

This gives us a logical account of how OCaml and Haskell avoid requiring explicit equality proofs in their syntax. The benefit of handling equality constraints through intersection types is that certain restrictions on typing that are useful for decidability, such as restricting property introduction to values, arise naturally from the semantic point of view—via the value restriction needed for soundly modeling intersection and union types [Davies and Pfenning 2000;
In addition, the appropriate approach to take when combining GADTs and effects is clear.²

*Bidirectionality, pattern matching, and principality.* Something that is not by itself novel in our approach is our decision to formulate both the declarative and algorithmic systems in a bidirectional style. Bidirectional checking [Pierce and Turner 2000] is a popular implementation choice for systems ranging from dependent types [Coquand 1996; Abel et al. 2008] and contextual types [Pientka 2008] to object-oriented languages [Odersky et al. 2001], but also has good proof-theoretic foundations [Watkins et al. 2004], making it useful both for specifying and implementing type systems. Bidirectional approaches make it clear to programmers where annotations are needed (which is good for specification), and can also remove unneeded nondeterminism from typing (which is good for both implementation and proving its correctness).

However, it is worth highlighting that because both bidirectionality and pattern matching arise from focalization, these two features fit together extremely well. In fact, by following the blueprint of focalization-based pattern matching, we can give a coverage-checking algorithm that explains when it is permissible to omit clauses in GADT pattern matching.

In the propositional case, the type synthesis judgment of a bidirectional type system generates principal types: if a type can be inferred for a term, that type is the most specific possible type for that term. (Indeed, in many cases, including the current system, the inferred type will even be unique.) This property is lost once quantifiers are introduced into the system, which is why it is not much remarked upon. However, prior work on GADTs, starting with Simonet and Pottier [2007], has emphasized the importance of the fact that handling equality constraints is much easier when the type of a scrutinee is principal. Essentially, this ensures that no existential variables can appear in equations, which prevents equation solving from interfering with unification-based type inference. The OutsideIn algorithm takes this consequence as a definition, permitting non-principal types just so long as they do not change the values of equations. However, Vytiniotis et al. [2011] note that while their system is sound, they no longer have a completeness result for their type system.

We use this insight to extend our bidirectional typechecking algorithm to track principality: The judgments we give track whether types are principal, and we use this to give a relatively simple specification for whether or not type annotations are needed. We are able to give a very natural spec to programmers—cases on GADTs must scrutinize terms with principal types, and an inferred type is principal just when it is the only type that can be inferred for that term—which soundly and completely corresponds to the implementation-side constraints: a type is principal when it contains no existential unification variables.

### 3 EXAMPLES

In this section, we give some examples of terms from our language, which illustrate the key features of our system and give a sense of how many type annotations are needed in practice. To help make this point clearly, all of the examples which follow are unsugared: they are the actual terms from our core calculus.

*Mapping over lists.* First, we begin with the traditional `map` function, which takes a function and applies it to every element of a list.

²The traditional eq GADT and its constructor `ref 1` can be encoded into our system as the type `1 ∧ (s = t)`, which which can be constructed as a unit value only under the constraint that `s` equals `t`.  

This function case-analyzes its second argument `xs`. Given an empty `xs`, it returns the empty list; given a cons cell `y :: ys`, it applies the argument function `f` to the head `y` and makes a recursive call on the tail `ys`.

In addition, we annotate the definition with a type. We have two type parameters `α` and `β` for the input and output element types. Since we are working with length-indexed lists, we also have a length index parameter `n`, which lets us show by typing that the input and output of `map` have the same length.

In our system, this type annotation is mandatory. Full type inference for definitions using GADTs requires polymorphic recursion, which is undecidable. As a result, this example also requires annotation in OCaml and GHC Haskell. However, Haskell and OCaml infer polymorphic types when no polymorphic recursion is needed. We adopt the simpler rule that all polymorphic definitions are annotated. This choice is motivated by Vytiniotis et al. [2010], who analyzed a large corpus of Haskell code and showed that implicit let-generalization was used primarily only for top-level definitions, and even then it is typically considered good practice to annotate top-level definitions for documentation purposes. Furthermore, experience with languages such as Agda and Idris (which do not implicitly generalize) show this is a modest burden in practice.

**Nested patterns and GADTs.** Now, we consider the `zip` function, which converts a pair of lists into a list of pairs. In ordinary ML or Haskell, we must consider what to do when the two lists are not the same length. However, with length-indexed lists, we can statically reject passing two lists of differing length:

```ml
rec zip. λp. λxs. case(xs, ([]) ⇒ [])
  | y :: ys ⇒ (f y) :: map f ys
: ∀n : ℕ. ∀α : ⋆. ∀β : ⋆. (α → β) → Vec n α → Vec n β
```

This case expression has only two patterns, one for when both lists are empty and one for when both lists have elements, with the type annotation indicating that both lists must be of length `n`. Typing shows that the cases where one list is empty and the other is non-empty are impossible, so our coverage checking rules accept this as a complete set of patterns. This example also illustrates that we support nested pattern matching.

**Existential Types.** Now, we consider the `filter` function, which takes a predicate and a list, and returns a list containing the elements satisfying that predicate. This example makes a nice showcase for supporting existential types, since the size of the return value is not predictable statically.

```ml
rec filter. λp. λxs. case(xs, ([]) ⇒ []
  | x :: xs ⇒ let tl = filter p xs in
case(p xs,
    inj1_ ⇒ tl
  | inj2_ ⇒ x :: tl))
: ∀n : ℕ. ∀α : ⋆. (α → 1 + 1) → Vec n α → ∃k : ℕ. Vec k α
```

So, this function takes predicate and a vector of arbitrary size, and then returns a list of unknown size (represented by the existential type `∃k : ℕ. Vec k α`). Note that we did not need to package the existential in another datatype, as one would have to in OCaml or GHC Haskell—we are free to use existential types as “just another type constructor”.

Expressions  
\[ e ::= x \mid \lambda x. e \mid e \; s^+ \mid \text{rec} \; x. \; v \mid (e : A) \]
\[ \mid \langle e_1, e_2 \rangle \mid \text{inj}_1 \; e \mid \text{inj}_2 \; e \mid \text{case}(e, \Pi) \]
\[ \mid [] \mid e_1 :: e_2 \]

Values  
\[ v ::= x \mid \lambda x. e \mid \text{rec} \; x. \; v \mid (v : A) \]
\[ \mid \langle v_1, v_2 \rangle \mid \text{inj}_1 \; v \mid \text{inj}_2 \; v \mid [] \mid v_1 :: v_2 \]

Spines  
\[ s ::= \cdot \mid e \; s \]
Nonempty spines  
\[ s^+ ::= e \; s \]

Patterns  
\[ \rho ::= x \mid \langle \rho_1, \rho_2 \rangle \mid \text{inj}_1 \; \rho \mid \text{inj}_2 \; \rho \mid [] \mid \rho_1 :: \rho_2 \]

Branches  
\[ \pi ::= \vec{\rho} \Rightarrow e \]

Branch lists  
\[ \Pi ::= \cdot \mid (\pi \mid \Pi) \]

Fig. 1. Source syntax

Universal variables  
\[ \alpha, \beta, \gamma \]

Sorts  
\[ \kappa ::= \ast \mid \mathbb{N} \]

Types  
\[ A, B, C ::= 1 \mid A \rightarrow B \mid A + B \mid A \times B \]
\[ \mid \alpha \mid \forall \alpha : \kappa. \; A \mid \exists \alpha : \kappa. \; A \]
\[ \mid P \supset A \mid A \land P \mid \text{Vec} \; t \; A \]

Terms/monotypes  
\[ t, \tau, \sigma ::= \text{zero} \mid \text{succ}(t) \mid 1 \mid \alpha \]
\[ \mid \tau \rightarrow \sigma \mid \tau + \sigma \mid \tau \times \sigma \]

Propositions  
\[ P, Q ::= t = t' \]

Contexts  
\[ \Psi ::= \cdot \mid \Psi, \alpha : \kappa \mid \Psi, x : Ap \]

Polarities  
\[ \mathcal{P} ::= + \mid - \mid \circ \]

Binary connectives  
\[ \oplus ::= \rightarrow \mid + \mid \times \]

Principalities  
\[ p, q ::= ! \mid \bigcirc \]

sometimes omitted

Fig. 2. Syntax of declarative types and contexts

checking, eq. elim.  
\[ \Psi \parallel P \vdash e \leftrightarrow C \; p \]

subtyping  
\[ \Psi \vdash A \leq B \]

coverage  
\[ \Psi \vdash \Pi \text{covers} \; A \; q \]

spine typing  
\[ \Psi \vdash s : Ap \Rightarrow Bq \]

type checking  
\[ \Psi \vdash e \leftrightarrow A \; p \]

match, eq. elim.  
\[ \Psi / P \vdash \Pi :: A! \leftrightarrow C \; p \]

principality-recovering  
\[ \Psi \vdash s : A \; p \Rightarrow B \; [q] \]

type synthesis  
\[ \Psi \vdash e \Rightarrow B \; p \]

pattern matching  
\[ \Psi \vdash \Pi :: A! \leftrightarrow C \; p \]

Fig. 3. Dependency structure of the declarative judgments

4 DECLARATIVE TYPING

Expressions. Expressions (Figure 1) are variables \(x\); the unit value \((\cdot)\); functions \(\lambda x. \; e\); applications to a spine \(e \; s^+\); fixed points \(\text{rec} \; x. \; v\); annotations \(e : A\); pairs \(\langle e_1, e_2 \rangle\); injections into a sum

\[\begin{array}{|c|c|}
\hline
\text{pol}(A) & \text{Determine the polarity of a type} \\
\text{nonpos}(A) & \text{Check if } A \text{ is not positive} \\
\text{nonneg}(A) & \text{Check if } A \text{ is not negative} \\
\hline
\end{array}\]

\[\begin{array}{|c|c|}
\hline
\text{join}(\mathcal{P}_1, \mathcal{P}_2) & \text{Join polarities} \\
\hline
\text{join}(+, \mathcal{P}_2) = + & \text{join}(-, \mathcal{P}_2) = - \\
\hline
\text{join}(\circ, +) = + & \text{join}(\circ, -) = - \\
\hline
\text{join}(\circ, \circ) = - & \\
\hline
\end{array}\]

\[\Psi \vdash A \leq^\mathcal{P} B \quad \text{Under context } \Psi, \text{ type } A \text{ is a subtype of } B, \text{ decomposing head connectives of polarity } \mathcal{P}\]

\[\begin{array}{c}
\Psi \vdash A \text{ type} \\
\Psi \vdash A \leq^\mathcal{P} A \\
\Psi \vdash A \leq B & \frac{\text{nonpos}(A)}{\Psi \vdash A \leq^+ B} & \frac{\text{nonneg}(A)}{\Psi \vdash A \leq^- B} \\
\hline
\Psi, \alpha : \kappa \vdash A \leq^+ B & \frac{\exists \alpha : \kappa \cdot A \leq^+ B}{\Psi, \alpha : \kappa \vdash A \leq^+ B} & \frac{\exists \alpha : \kappa \cdot A \leq^+ B}{\Psi, \alpha : \kappa \vdash A \leq^+ B} \\
\hline
\end{array}\]

Fig. 4. Subtyping in the declarative system

Type \(\text{inj}_k e\); case expressions \(\text{case}(e, \Pi)\) where \(\Pi\) is a list of branches \(\pi\), which can eliminate pairs and injections (see below); the empty vector \([]\); and consing a head \(e_1\) to a tail vector \(e_2\).

Values \(\nu\) are standard for a call-by-value semantics; the variables introduced by fixed points are considered values, because we only allow fixed points of values. A spine \(s\) is a list of expressions—arguments to a function. Allowing empty spines (written \(\cdot\)) is convenient in the typing rules, but would be strange in the source syntax, so (in the grammar of expressions \(e\)) we require a nonempty spine \(s^+\). We usually omit the empty spine \(\cdot\), writing \(e_1 e_2\) instead of \(e_1 (\cdot) e_2\). Since we use juxtaposition for both application \(e^s\) and spines, some strings are ambiguous; we resolve this ambiguity in favour of the spine, so \(e_1 e_2 e_3\) is parsed as the application of \(e_1\) to the spine \(e_2 e_3\), which is technically \(e_2 (e_3 \cdot)\). Patterns \(\rho\) consist of pattern variables, pairs, and injections. A branch \(\pi\) is a sequence of patterns \(\overline{\rho}\) with a branch body \(e\). We represent patterns as sequences, which enables us to deconstruct tuple patterns.

Types. We write types as \(A, B\) and \(C\). We have the unit type \(1\), functions \(A \to B\), sums \(A + B\), and products \(A \times B\). We have universal and existential types \(\forall \alpha : \kappa \cdot A\) and \(\exists \alpha : \kappa \cdot A\); these are predicative quantifiers over monotypes (see below). We write \(\alpha, \beta\), etc. for type variables; these are universal, except when bound within an existential type. We also have a guarded type \(P \supset A\), read “\(P\) implies \(A\)”. This implication corresponds to type \(A\), provided \(P\) holds. Its dual is the asserting type \(A \land P\), read “\(A\) with \(P\)”, which witnesses the proposition \(P\). In both, \(P\) has no runtime content.

Sorts, terms, monotypes, and propositions. Terms and monotypes \(t, \tau, \sigma\) share a grammar but are distinguished by their sorts \(\kappa\). Natural numbers zero and \(\text{succ}(t)\) are terms and have sort \(\mathbb{N}\). Unit \(1\) has the sort \(\star\) of monotypes. A variable \(\alpha\) stands for a term or a monotype, depending on the sort.
κ annotating its binder. Functions, sums, and products of monotypes are monotypes and have sort ♦. We tend to prefer t for terms and σ, τ for monotypes.

A proposition P or Q is simply an equation t = t’. Note that terms, which represent runtime-irrelevant information, are distinct from expressions; however, an expression may include type annotations of the form P ⊃ A and A ∧ P, where P contains terms.

Contexts. A declarative context Ψ is an ordered sequence of universal variable declarations α : κ and expression variable typings x : Ap, where p denotes whether the type A is principal (Section 4.2). A variable α can be free in a type A only if α was declared to the left: α : ♦, x : α p is well-formed, but x : α p, α : ♦ is not.

4.1 Subtyping
We give our two subtyping relations, ≤+ and ≤−, in Figure 4. We treat the universal quantifier as a negative type (since it is a function in System F), and the existential as a positive type (since it is a pair in System F). We have two typing rules for each of these connectives, corresponding to the left and right rules for universals and existentials in the sequent calculus. We treat all other types as having no polarity. The positive and negative subtype judgments are mutually recursive, and the ≤− rule permits switching the polarity of subtyping from positive to negative when both of the types are non-positive, and conversely for ≤+. When both types are neither positive nor negative, we require them to be equal (≤Refl).

In logical terms, functions and guarded types are negative; sums, products and assertion types are positive. We could potentially operate on these types in the negative and positive subtype relations, respectively. Leaving out (for example) function subtyping means that we will have to do some η-expansions to get programs to typecheck; we omit these rules to keep the implementation complexity low. (The idea that η-expansion can substitute for subsumption dates to Barendregt el al. [1983].)

This also illustrates a nice feature of bidirectional typing: we are relatively free to adjust the subtype relation to taste. Moreover, the structure of polarization makes it easy to work out just what the rules should be. E.g., to add function subtyping to our system, we would use the rule:

$$\frac{\Psi \vdash A \leq^+ A \quad \Psi \vdash B \leq^- B'}{\Psi \vdash A \rightarrow B \leq^- A' \rightarrow B'}$$

As polarized function types are a negative type of the form X+ → Y−, we see (1) the rule as a whole lives in the negative subtyping judgement, (2) argument types compare in the positive judgement (with the usual contravariant twist), and (3) result types compare in the negative judgement.

4.2 Typing judgments

Principality. Our typing judgments carry principalities: A! means that A is principal, and A! means A is not principal. Note that a principality is part of a judgment, not part of a type. In the checking judgment Ψ ⊨ e ⇐ A p the type A is input; if p = !, we know that A is not the result of guessing. For example, the e in (e : A) is checked against A !. In the synthesis judgment

Sound and Complete Bidirectional Typechecking

Fig. 6. Declarative typing, omitting rules for ×, +, and Vec

\[ \Psi \vdash P \text{ true} \] Under context \( \Psi \), check \( P \)

\[ \Psi \vdash (t = t) \text{ true} \]

\[ \Psi \vdash e \iff A p \] Under context \( \Psi \), expression \( e \) checks against input type \( A \)

\[ \Psi \vdash e \Rightarrow A p \] Under context \( \Psi \), expression \( e \) synthesizes output type \( A \)

\[ x : Ap \in \Psi \quad \Psi \vdash x \Rightarrow Ap \quad \text{DeclVar} \]
\[ \Psi \vdash e \Rightarrow A q \quad \Psi \vdash A \leq \text{join}(\text{pol}(B), \text{pol}(A)) B \quad \text{DeclSub} \]

\[ \Psi \vdash e \iff B p \]

\[ \Psi \vdash (e : A) \Rightarrow A ! \quad \text{DecAnno} \]

\[ \Psi, x : Ap \vdash v \iff Ap \quad \text{DeclRec} \]
\[ \Psi \vdash \tau : \kappa \quad \Psi \vdash e \iff [\tau / \alpha] A \quad \text{DeclEl} \]

\[ \Psi \vdash \tau : \kappa \quad \Psi \vdash (\exists \alpha : \kappa. A) p \quad \text{DeclEI} \]

\[ \psi \vdash \neg \psi \quad \text{Decl1l} \]

\[ \Psi, x : Ap \vdash v \iff Ap \quad \text{DeclVI} \]
\[ \Psi \vdash \tau : \kappa \quad \Psi \vdash e \iff (\exists \alpha : \kappa. A) p \quad \text{DeclEl} \]

\[ \Psi \vdash (P \Rightarrow A) ! \quad \text{DeclCI} \]
\[ \Psi \vdash \tau : \kappa \quad \Psi \vdash e \iff (\exists \alpha : \kappa. A) p \quad \text{DeclEl} \]

\[ \Psi, x : Ap \vdash e \iff B p \quad \text{DeclCI} \]
\[ \Psi \vdash e \Rightarrow Ap \quad \Psi \vdash s : A p \gg C [\tau] \quad \text{DeclRI} \]

\[ \Psi \vdash \kappa \quad \Psi \vdash \xi \Rightarrow Ap \quad \text{DeclEmptySpine} \]
\[ \Psi \vdash e \iff Ap \quad \Psi \vdash s : B p \gg C q \quad \text{DeclSpine} \]

\[ \Psi \vdash e \iff Ap \quad \Psi \vdash s : A p \gg C [\tau] \quad \text{DeclSpinePass} \]
\[ \Psi \vdash e \iff Ap \quad \Psi \vdash s : B p \gg C q \quad \text{DeclSpine} \]

\[ \Psi / P \vdash e \iff C p \quad \text{DeclSpineRecover} \]

\[ \Psi / (\sigma = \tau) \vdash e \iff C p \quad \text{DeclCheck\bot} \]

\[ \Psi / (\sigma = \tau) \vdash e \iff C p \quad \text{DeclCheckUnify} \]

We sometimes omit a principality when it is \( ! \) ("not principal"). We write \( p \subseteq q \), read "\( p \) at least as principal as \( q \)" for the reflexive closure of \( ! \subseteq ! \).
**Spine judgments.** The ordinary form of spine judgment, $\Psi \vdash s : A \triangleright C \triangleright q$, says that if arguments $s$ are passed to a function of type $A$, the function returns type $C$. For a function $e$ applied to one argument $e_1$, we write $e \ e_1$ as syntactic sugar for $e \ (e_1 \cdot)$. Supposing $e$ synthesizes $A_1 \to A_2$, we apply $\text{Decl} \to \text{Spine}$, checking $e_1$ against $A_1$ and using $\text{DeclEmptySpine}$ to derive $\Psi \vdash \cdot : A_2 \triangleright A_2 \ p$.

Rule $\text{DeclSpine}$ does not decompose $e \ s$ but instantiates a $\forall$. Note that, even if the given type $\forall \alpha : \kappa. \ A$ is principal ($p = !$), the type $[\tau/\alpha]A$ in the premise is not principal—we could choose a different $\tau$. In fact, the $q$ in $\text{DeclSpine}$ is also always $\forall$, because no rule deriving the ordinary spine judgment can recover principality.

The recovery spine judgment $\Psi \vdash s : A \triangleright C \ [q]$, however, can restore principality in situations where the choice of $\tau$ in $\text{DeclSpine}$ cannot affect the result type $C$. If $A$ is principal ($p = !$) but the ordinary spine judgment produces a non-principal $C$, we can try to recover principality with $\text{DeclSpineRecover}$. Its first premise is $\Psi \vdash s : A ! \triangleright C ! j$; its second premise (really, an infinite set of premises) quantifies over all derivations of $\Psi \vdash s : A ! \triangleright C' j$. If $C' = C$ in all such derivations, then the ordinary spine rules erred on the side of caution: $C$ is actually principal, so we can set $q = !$ in the conclusion of $\text{DeclSpineRecover}$.

If some $C' \neq C$, then $C$ is certainly not principal, and we must apply $\text{DeclSpinePass}$, which simply transitions from the ordinary judgment to the recovery judgment.

Figure 3 shows the dependencies between the declarative judgments. Given the cycle containing the spine typing judgments, we need to stop and ask: Is $\text{DeclSpineRecover}$ well-founded? For well-foundedness of type systems, we can often make a straightforward argument that, as we move from the conclusion of a rule to its premises, either the expression gets smaller, or the expression stays the same but the type gets smaller. In $\text{DeclSpineRecover}$, neither the expression nor the type get smaller. Fortunately, the rule that gives rise to the arrow from “spine typing” to “type checking” in Figure 3—$\text{DeclSpine}$—does decompose its subject, and any derivations of a recovery judgment lurking within the second premise of $\text{DeclSpineRecover}$ must be for a smaller spine. In the appendix (Lemma 1, p. 38), we prove that the recovery judgment, and all the other declarative judgments, are well-founded.

**Example.** In Section 5.1 we present some example derivations that illustrate how the spine typing rules work to recover principality.

**Subtyping.** Rule $\text{DeclSub}$ invokes the subtyping judgment, at the join of the polarities of $B$ (the type being checked against) and $A$ (the type being synthesized). Using the join ensures that the polarity of $B$ takes precedence over $A$’s, which means the programmer control which subtyping mode to begin with via a type annotation.

Furthermore, the subtyping rule allows $\text{DeclSub}$ to play the role of an existential introduction rule, by applying subtyping rule $\leq \exists \ R$ when $B$ is an existential type.

**Pattern matching.** Rule $\text{DeclCase}$ checks that the scrutinee has a type and principality, and then invokes the two main judgments for pattern matching. The $\Psi \vdash \Pi : \tilde{A} \ q \leftarrow C \ p$ judgement checks that each branch in the list of branches $\Pi$ is well-typed, taking a vector $\tilde{A}$ of pattern types to simplify the specification of coverage checking, as well as a principality annotation covering all of the types (i.e., if any of the types in $\tilde{A}$ is non-principal, the whole vector is not principal).

The $\Psi \vdash \Pi$ covers $\tilde{A} \ q$ judgement does coverage checking for the list of branches. However, the $\text{DeclCase}$ does not simply check that the patterns cover for the inferred type of the scrutinee—it checks that they cover for every possible type that could be inferred for the scrutinee. In the case that the scrutinee is principal, this is the same as checking coverage at the scrutinee’s type, but when the scrutinee is not principal, this rule has the effect of preventing type inference from...
\[
\begin{align*}
\Psi \vdash \Pi :: \bar{A} q \equiv C p & \quad \text{Under context } \Psi, \text{ check branches } \Pi \text{ with patterns of type } \bar{A} \text{ and bodies of type } C \\
\Psi \vdash \cdot : \bar{A} q \equiv C p & \quad \text{DeclMatchEmpty} \\
\Psi \vdash e : C p & \quad \text{DeclMatchBase} \\
\Psi \vdash (\cdot \mapsto e) :: q : C p & \\
\Psi, \alpha : \kappa \vdash \bar{p} \Rightarrow e :: A, \bar{A} q \equiv C p & \quad \text{DeclMatch} \exists \\
\Psi \vdash (\bar{p} \Rightarrow e) :: (\exists \alpha : \kappa \cdot A), \bar{A} q \equiv C p & \\
\Psi \vdash \rho, \bar{p} \Rightarrow e :: A_k, \bar{A} q \equiv C p & \quad \text{DeclMatch} + k \\
\Psi / P \vdash \bar{p} \Rightarrow e :: A, \bar{A} ! \equiv C p & \quad \text{DeclMatch} \land \\
\Psi \vdash \bar{p} \Rightarrow e :: (A \land P), \bar{A} ! \equiv C p & \quad \text{DeclMatch} \land X \\
\Psi, \alpha : \mathbb{N} \vdash \rho_1, \rho_2, \bar{p} \Rightarrow e :: A, (\text{Vec } a A), \bar{A} ! \equiv C p & \quad \text{DeclMatchCons} \land Y \\
\Psi \vdash (\rho_1 :: \rho_2), \bar{p} \Rightarrow e :: (\text{Vec } t A), \bar{A} ! \equiv C p & \\
\Psi \vdash (\rho_1 :: \rho_2), \bar{p} \Rightarrow e :: (\text{Vec } t A), \bar{A} ! \equiv C p & \quad \text{DeclMatchNil} \\
\Psi \vdash \Pi, \bar{p} \Rightarrow e :: (\text{Vec } t A), \bar{A} ! \equiv C p & \\
\Psi \vdash x, \bar{p} \Rightarrow e :: A, \bar{A} q \equiv C p & \quad \text{DeclMatchWild} \\
\Psi / P \vdash \Pi :: \bar{A} ! \equiv C p & \\
\Psi / \sigma = \tau \vdash \bar{p} \Rightarrow e :: \bar{A} ! \equiv C p & \quad \text{DeclMatch} \bot \\
\Psi / \sigma = \tau \vdash \bar{p} \Rightarrow e :: \bar{A} ! \equiv C p & \quad \text{DeclMatchUnify} \\
\end{align*}
\]

Fig. 7. Declarative pattern matching

using the shape of the patterns to infer a type, which is notoriously problematic with GADTs (e.g., whether a missing nil in a list match should be taken as evidence of coverage failure or that the length is non-zero). As with spine recovery, this rule is only well-founded because the universal quantification ranges over synthesized types versus a subterm.

The \( \Psi \vdash \Pi :: \bar{A} q \equiv C p \) judgment (rules in Figure 7) systematically checks the typing of each branch in \( \Pi \); rule \text{DeclMatchEmpty} succeeds on the empty list, and \text{DeclMatchSeq} checks one branch and recurs on the remaining branches. Rules for sums, units, and products break down patterns left to right, one constructor at a time. Products also extend the sequences of patterns and types, with \text{DeclMatch\times} breaking down a pattern vector headed by a pair pattern \( \langle p, p' \rangle \), \( \bar{p} \)}
Patterns $\Pi$ cover the types $\vec{A}$ in context $\Psi$

Patterns $\Psi / P \vdash \Pi$ cover $\vec{A}$!

Pattern list $\Pi$ contains a list pattern constructor at the head position

\[
\begin{align*}
\text{DeclCoversEmpty} & : \Psi \vdash (\Rightarrow e_1) \mid \Pi' \text{ covers } \vec{p} \\
\text{DeclCoversVar} & : \Psi / P \vdash \Pi' \text{ covers } \vec{A} \! \vec{p} \\
\text{DeclCovers1} & : \Psi \vdash (\Rightarrow P) \mid \Pi' \text{ covers } \vec{A} \! \vec{p} \\
\text{DeclCoversPlus} & : \Psi \vdash (\Rightarrow \Pi_1 \mid \Pi_2) \mid \Pi' \text{ covers } \vec{A} \! \vec{p} \\
\text{DeclCoversAnd} & : \Psi \vdash (\Rightarrow \Pi_1 \land \Pi_2) \mid \Pi' \text{ covers } \vec{A} \! \vec{p} \\
\text{DeclCoversVec} & : \Psi \vdash (\Rightarrow \text{Vec} \: t \: A, \vec{A}) \mid \Pi' \text{ covers } \vec{A} \! \vec{p} \\
\text{DeclCoversEq} & : \Psi \vdash (\Rightarrow \text{mgu}(t_1, t_2)) \mid \Pi' \text{ covers } \vec{A} \! \vec{p} \\
\text{DeclCoversEqBot} & : \Psi \vdash (\Rightarrow \bot) \mid \Pi' \text{ covers } \vec{A} \! \vec{p}
\end{align*}
\]

Fig. 8. Match coverage

into $p, p', \vec{p}$, and breaking down the type sequence $(A \times B), \vec{C}$ into $A, B, \vec{C}$. Once all the patterns are eliminated, the DeclMatchBase rule says that if the body typechecks, then the branch typechecks. For completeness, the variable and wildcard rules are restricted so that any top-level existentials and equations are eliminated before discarding the type.

The existential elimination rule DeclMatch存在着 an existential type, and DeclMatchAnd breaks apart a conjunction by eliminating the equality using unification. The DeclMatchAnd rule says that if the equation is false then typing succeeds, because this case is impossible. The DeclMatchUnify rule unifies the two terms of an equation and applies the substitution before continuing to check typing. Together, these two rules implement the Schroeder-Heister equality elimination rule. Because our language of terms has only simple first-order terms, either unification will fail, or there is a most general unifier. Note, however, that DeclMatchAnd only applies when the pattern type is principal. Otherwise, we use the DeclMatchAnd rule, which throws away the equation and does not refine any types at all. In this way, we can ensure that we will only try to eliminate equations which are fully known (i.e., principal). Similar considerations apply to vectors, with length information being used to refine types only when the type of the scrutinee is principal.
Expand vector patterns in $\Pi$

$$\begin{align*}
\Pi & \overset{\text{Vec}}{\leadsto} \Pi_1 \parallel \Pi_2 \\
\overset{\text{Vec}}{\leadsto} & \leadsto \\
\Pi & \overset{\text{Vec}}{\leadsto} \Pi_1 \parallel \Pi_2
\end{align*}$$

Expand head pair patterns in $\Pi$

$$\begin{align*}
\Pi & \overset{\times}{\leadsto} \Pi' \\
\overset{\times}{\leadsto} & \leadsto \\
\Pi & \overset{\times}{\leadsto} \Pi'
\end{align*}$$

Expand head sum patterns in $\Pi$ into left $\Pi_L$ and right $\Pi_R$ sets

$$\begin{align*}
\Pi & \overset{\sum}{\leadsto} \Pi_L \parallel \Pi_R \\
\overset{\sum}{\leadsto} & \leadsto \\
\Pi & \overset{\sum}{\leadsto} \Pi_L \parallel \Pi_R
\end{align*}$$

Remove head variable and wildcard patterns from $\Pi$

$$\begin{align*}
\Pi & \overset{\text{var}}{\leadsto} \Pi' \\
\overset{\text{var}}{\leadsto} & \leadsto \\
\Pi & \overset{\text{var}}{\leadsto} \Pi'
\end{align*}$$

Remove head variable, wildcard, and unit patterns from $\Pi$

$$\begin{align*}
\Pi & \overset{1}{\leadsto} \Pi' \\
\overset{1}{\leadsto} & \leadsto \\
\Pi & \overset{1}{\leadsto} \Pi'
\end{align*}$$

Fig. 9. Pattern expansion

The $\Psi \vdash \Pi$ covers $\tilde{A} \rho$ judgment (in Figure 8) checks whether a set of patterns covers all possible cases. As with typing, we systematically deconstruct the sequence of types in the branch, but we also need auxiliary operations to expand the patterns. For example, the $\Pi \overset{\times}{\leadsto} \Pi'$ operation takes every branch $(\rho, \rho')$, $\rho \Rightarrow e$ and expands it to $\rho, \rho', \rho \Rightarrow e$. To keep the sequence of patterns aligned with the sequence of types, we also expand variables and wildcards patterns into two wildcards: $x, \rho \Rightarrow e$ becomes $\_, \rho \Rightarrow e$. After expanding out all the pairs, DeclCovers× checks coverage by breaking down the pair type.

For sum types, we expand a list of branches into two lists, one for each injection. So $\Pi \overset{\sum}{\leadsto} \Pi_L \parallel \Pi_R$ will send all branches headed by inj$_1 \rho$ into $\Pi_L$ and all branches headed by inj$_2 \rho$ into $\Pi_R$, with variables and wildcards being sent to both sides. Then DeclCovers+ checks the left and right branches independently.

As with typing, DeclCovers∃ just unpacks the existential type. Likewise, DeclCoversEqBot and DeclCoversEq handle the two cases arising from equations. If an equation is unsatisfiable, coverage succeeds since there are no possible values of that type. If it is satisfiable, we apply the substitution and continue coverage checking. Just as when typechecking patterns, we only use property types to refine coverage checking when the equations come from a principal type — the DeclCovers∧j rule simply throws away the equation when the type is not principal. (This is a sound approximation which ends up requiring more patterns when the type is not principal.)

So far, the coverage rules for pattern matching are almost purely type-directed. However, once recursive types like $\text{Vec} \ n \ A$ enter the picture, matters become a little more subtle. The issue is
that if we split a wildcard _ of type Vec n A, the type doesn’t tell us when to stop. That is, we could split a wildcard into a nil [] and cons _ :: _ pattern; or we could turn it into a nil [], singleton _ :: [] and two-or-longer _ :: _ :: _ pattern; and so on. The key issue is that the tail of a list has the same set of possible patterns as the list itself, and so blindly following the type structure will not ensure termination of coverage checking.

In this paper, we take the view that the patterns the programmer wrote should guide how much to split types when doing coverage checking for inductive types. In Figure 8, we introduce the \( \Pi \) guarded judgement, which checks to see if a constructor pattern is present in the leading column of patterns. If it is, then our algorithm will unfold the recursive type as part of type checking, and otherwise it will not. This is by no means a canonical choice: our choice is similar to the choice Agda makes, but other language implementations make other choices. In contrast, the OCaml coverage checking algorithm unfolds wildcard patterns at GADT type one step more than what the programmer wrote [Garrigue and Le Normand 2015]. (They also observe that precise exhaustiveness checking is undecidable, meaning that some choice of heuristic is unavoidable.)

4.2.1 Design Considerations for Pattern Matching.

**Evaluation Order.** Our typing and coverage checking rules are given assuming a call-by-value evaluation strategy. These coverage rules are not sound under a call-by-name evaluation order. Consider the following program, writing \( \bot \) for a looping term:

\[
\text{case}(\bot : A \land (s = t), x \Rightarrow e)
\]

When type-checking this program, the DeclMatch and DeclCovers rules are permitted to eliminate the equality \( s = t \) when checking \( e \). However, one can use a looping program to inhabit \( A \land (s = t) \) for any \( P \), and so we have introduced a spurious equality into the context when checking \( e \). In contrast, in a call-by-value language the scrutinee of a case will be reduced before the match proceeds, so this issue cannot arise. (In a total language such as Koka, these rules would be sound irrespective of evaluation order, since all evaluation strategies are indistinguishable.)

**Redundant Patterns.** These rules do not check for redundancy: DeclCoversEmpty applies even when branches are left over. When DeclCoversEmpty is applied, we could mark the \( \cdot \Rightarrow e_1 \) branch, and issue a warning for unmarked branches. This seems better as a warning than an error, since redundancy is not stable under substitution. For example, a case over (Vec n A) with [] and :: branches is not redundant—but if we substitute 0 for \( n \), the :: branch becomes redundant.

**Synthesis.** Bidirectional typing is a form of partial type inference, which Pierce and Turner [2000] said should “eliminate especially those type annotations that are both common and silly”. But our rules are rather parsimonious in what they synthesize; for instance, () does not synthesize 1, and so might need an annotation. Fortunately, it would be straightforward to add such rules, following the style of Dunfield and Krishnaswami [2013].

5 ALGORITHMIC TYPING

Our algorithmic rules closely mimic our declarative rules, except that whenever a declarative rule would make a guess, the algorithmic rule adds to the context an existential variable (written with a hat \( \hat{\alpha} \)). As typechecking proceeds, we add solutions to the existential variables, reflecting increasing knowledge. Hence, each declarative typing judgment has a corresponding algorithmic judgment with an output context as well as an input context. The algorithmic type checking judgment \( \Gamma \vdash e \iff A \vdash \Delta \) takes an input context \( \Gamma \) and yields an output context \( \Delta \) that includes increased knowledge about what the types have to be. The notion of increasing knowledge is formalized by a judgment \( \Gamma \rightarrow \Delta \) (Section 5.3).
5.1 Examples

To show how the spine typing rules recover principality, we present some example derivations.

Suppose we have an identity function $id$, defined in an algorithmic context $\Gamma$ by the hypothesis $id : (\forall \alpha : \star. \alpha \rightarrow \alpha) : \tau$. Since the hypothesis has $\tau$, the type of $id$ is known to be principal. If we apply $id$ to $()$, we expect to get something of unit type $1$. Despite the $\forall$ in the type of $id$, the resulting type should be principal, because no other type is possible. We can indeed derive that type:

\[
\begin{align*}
(id : (\forall \alpha : \star. \alpha \rightarrow \alpha) : \tau) \in \Gamma \\
\Gamma \vdash id \Rightarrow (\forall \alpha : \star. \alpha \rightarrow \alpha) : \tau \Gamma \\
\Gamma : (()) : (\forall \alpha : \star. \alpha \rightarrow \alpha) : \tau \Rightarrow 1 \ [1] \ : \Gamma, \hat{\alpha} : \star = 1 \\
\Gamma \vdash id (()) \Rightarrow 1 ! \ : \Gamma, \hat{\alpha} : \star = 1
\end{align*}
\]

(Here, we write the application $id ()$ as $id (())$, to show the structure of the spine as analyzed by the typing rules.) In the derivation of the second premise of $\rightarrow E$, shown below, we can follow
the evolution of the principality marker.

\[ \text{EmptySpine} \]
\[ \Gamma, \hat{\alpha} \vdash \ast \implies \hat{\alpha} f + \Gamma, \hat{\alpha} : \ast = 1 \]
\[ \text{Spine} \]
\[ \Gamma, \hat{\alpha} : \ast = 1 \vdash \hat{\alpha} f \implies 1 f + \Gamma, \hat{\alpha} : \ast = 1 \]

\[ \text{SpineRecover} \]
\[ \Gamma \vdash (\hat{\alpha} f : (\forall \alpha : \ast. \alpha \rightarrow \alpha)) ! \implies 1 f + \Gamma, \hat{\alpha} : \ast = 1 \]

- The input principality (marked “input”) is \( ! \), because the input type \((\forall \alpha : \ast. \alpha \rightarrow \alpha)\) was marked as principal in the hypothesis typing \( \text{id} \).
- Rule \( \text{SpineRecover} \) begins by invoking the ordinary (non-recovering) spine judgment, passing all inputs unchanged, including the principality \( ! \).
- Rule \( \forall \text{Spine} \) adds an existential variable \( \hat{\alpha} \) to represent the instantiation of the quantified type variable \( \alpha \), and substitutes \( \hat{\alpha} \) for \( \alpha \). Since this instantiation is, in general, \textit{not} principal, it replaces \( ! \) with \( f \) (highlighted) in its premise. This marks the type \( \hat{\alpha} \rightarrow \hat{\alpha} \) as non-principal.
- Rule \( \rightarrow \text{Spine} \) decomposes \( \hat{\alpha} \rightarrow \hat{\alpha} \) and checks \( \hat{\alpha} \) against \( \hat{\alpha} \), maintaining the principality \( f \). Once principality is lost, it can only be recovered within the \( \text{SpineRecover} \) rule itself.
- Rule \( 1!\hat{\alpha} \) notices that we are checking \( \hat{\alpha} \) against an unknown type \( \hat{\alpha} \); since the expression is \( \hat{\alpha} \), the type \( \hat{\alpha} \) must be \( 1 \), so it adds that solution to its output context.
- Moving to the second premise of \( \rightarrow \text{Spine} \), we analyze the remaining part of the spine. That is just the empty spine \( \cdot \), and rule \( \text{EmptySpine} \) passes its inputs along as outputs. In particular, the principality \( f \) is unchanged.
- The principalities are passed down to the conclusion of \( \forall \text{Spine} \), where \( f \) is highlighted.
- In \( \text{SpineRecover} \), we notice that the output type \( 1 \) has no existential variables \((\text{FEV}(1) = \emptyset)\), which allows us to recover principality of the output type: \([!]\).

In the corresponding derivation in our declarative system, we have, instead, a check that no other types are derivable:

\[ \Psi \vdash (\hat{\alpha} f : (\forall \alpha : \ast. \alpha \rightarrow \alpha)) ! \implies 1 f \]

- **Syntax.** Expressions are the same as in the declarative system.

  - **Existential variables.** The algorithmic system adds existential variables \( \hat{\alpha}, \hat{\beta}, \hat{\gamma} \) to types and terms/monotypes (Figure 11). We use the same meta-variables \( A, \ldots \). We write \( u \) for either a universal variable \( \alpha \) or an existential variable \( \hat{\alpha} \).

  - **Contexts.** An algorithmic context \( \Gamma \) is a sequence that, like a declarative context, may contain universal variable declarations \( \alpha : \kappa \) and expression variable typings \( x : Ap \). However, it may also have (1) \textit{unsolved} existential variable declarations \( \hat{\alpha} : \kappa \) (included in the \( \Gamma, u : \kappa \) production); (2)
Universal variables $\alpha, \beta, \gamma$
Existential variables $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$
Variables $u ::= \alpha \mid \hat{\alpha}$
Types $A, B, C ::= \varepsilon \mid A \rightarrow B \mid A + B \mid A \times B \mid \alpha \mid \hat{\alpha} \mid \forall \alpha : \kappa. A \mid \exists \alpha : \kappa. A \mid P \supset A \mid A \wedge P \mid \text{Vec } t A$

Fig. 11. Syntax of types, contexts, and other objects in the algorithmic system

\[
\begin{align*}
[\Gamma]r & = \begin{cases} 
[\Gamma]r \text{ when } (\alpha = \tau) \in \Gamma & \\
\alpha \text{ otherwise} & \\
\end{cases} \\
[\Gamma](P \supset A) & = ([\Gamma]P) \supset ([\Gamma]A) \\
[\Gamma](A \land P) & = ([\Gamma]A) \land ([\Gamma]P) \\
[\Gamma](A \lor B) & = ([\Gamma]A) \lor ([\Gamma]B) \\
[\Gamma](\text{Vec } t A) & = \text{Vec } ([\Gamma]t) ([\Gamma]A) \\
\end{align*}
\]

Fig. 12. Applying a context, as a substitution, to a type

solved existential variable declarations $\hat{\alpha} : \kappa = \tau$; (3) equations over universal variables $\alpha = \tau$; and (4) markers $\triangleright u$. An equation $\alpha = \tau$ must appear to the right of the universal variable’s declaration $\alpha : \kappa$. We use markers as delimiters within contexts. For example, rule $\supset 1$ adds $\triangleright p$, which tells it how much of its last premise’s output context $(\Delta, \triangleright p, \Delta')$ should be dropped. (We abuse notation by writing $\triangleright p$ rather than cluttering the context with a dummy $\alpha$ and writing $\triangleright \alpha$.)

A complete algorithmic context, denoted by $\Omega$, is an algorithmic context with no unsolved existential variable declarations.

Assuming an equality can yield inconsistency: for example, $\text{zero} = \text{succ} (\text{zero})$. We write $\Delta \perp$ for either a valid algorithmic context $\Delta$ or inconsistency $\perp$.

5.2 Context substitution $[\Gamma]A$ and hole notation $\Gamma[\Theta]$

An algorithmic context can be viewed as a substitution for its solved existential variables. For example, $\hat{\alpha} = 1, \hat{\beta} = \hat{\alpha} = 1$ can be applied as if it were the substitution $1/\hat{\alpha}, (\hat{\alpha} \rightarrow 1)/\hat{\beta}$ (applied right to left), or the simultaneous substitution $1/\hat{\alpha}, (1 \rightarrow 1)/\hat{\beta}$. We write $[\Gamma]A$ for $\Gamma$ applied as a substitution (Figure 12).

Applying a complete context to a type $A$ (provided it is well-formed: $\Omega \vdash A$ type) yields a type $[\Omega]A$ with no existentials. Such a type is well-formed under the declarative context obtained by dropping all the existential declarations and applying $\Omega$ to declarations $x : A$ (to yield $x : [\Omega]A$). We can think of this context as the result of applying $\Omega$ to itself: $[\Omega] \Omega$. More generally, we can apply $\Omega$ to any context $\Gamma$ that it extends: context application $[\Omega] \Gamma$ is given in Figure 13. The application $[\Omega] \Gamma$ is defined if and only if $\Gamma \rightarrow \Omega$ (context extension; see Section 5.3), and applying $\Omega$ to any such $\Gamma$ yields the same declarative context $[\Omega] \Omega$. 
In addition to appending declarations (as in the declarative system), we sometimes insert and replace declarations, so a notation for contexts with a hole is useful: $\Gamma = \Gamma_0[\Theta]$ means $\Gamma$ has the form $(\Gamma_0, \Theta, \Gamma_R)$. For example, if $\Gamma = \Gamma_0[\hat{\beta}] = (\hat{\alpha}, \hat{\beta}, x : \hat{\beta})$, then $\Gamma_0[\hat{\beta}] = \hat{\alpha} = (\hat{\alpha}, \hat{\beta} = \hat{\alpha}, x : \hat{\beta})$.

We also use contexts with two ordered holes: if $\Gamma = \Gamma_0[\Theta_1][\Theta_2]$ then $\Gamma = (\Gamma_L, \Theta_1, \Gamma_M, \Theta_2, \Gamma_R)$.

### 5.3 The context extension relation $\Gamma \rightarrow \Delta$

A context $\Gamma$ is extended by a context $\Delta$, written $\Gamma \rightarrow \Delta$, if $\Delta$ has at least as much information as $\Gamma$, while conforming to the same declarative context—that is, $[\Omega]\Gamma = [\Omega]\Delta$ for some $\Omega$. In a sense, $\Gamma \rightarrow \Delta$ says that $\Gamma$ is entailed by $\Delta$: all positive information derivable from $\Gamma$ can also be derived from $\Delta$ (which may have more information, say, that $\hat{\alpha}$ is equal to a particular type). We give the rules for extension in Figure 15.

The rules deriving the context extension judgment (Figure 15) say that the empty context extends the empty context ($\rightarrow\text{Id}$); a term variable typing with $A'$ extends one with $A$ if applying the extending context $\Delta$ to $A$ and $A'$ yields the same type ($\rightarrow\text{Var}$); universal variable declarations and equations must match ($\rightarrow\text{Uvar}$, $\rightarrow\text{Eqn}$); scope markers must match ($\rightarrow\text{Marker}$); and, existential variables may either match ($\rightarrow\text{Unsolved}$, $\rightarrow\text{Solved}$), get solved by the extending context ($\rightarrow\text{Solve}$), or be added by the extending context ($\rightarrow\text{Add}$, $\rightarrow\text{AddSolved}$).

Extension may change solutions, if information is preserved or increased: $(\hat{\alpha} : \star, \hat{\beta} : \star = \hat{\alpha}) \rightarrow (\hat{\alpha} : \star = 1, \hat{\beta} : \star = \hat{\alpha})$ directly increases information about $\hat{\alpha}$, and indirectly increases information about $\hat{\beta}$. More interestingly, if $\Delta = (\hat{\alpha} : \star = 1, \hat{\beta} : \star = \hat{\alpha})$ and $\Omega = (\hat{\alpha} : \star = 1, \hat{\beta} : \star = 1)$, then $\Delta \rightarrow \Omega$: while the solution of $\hat{\beta}$ in $\Omega$ is different, in the sense that $\Omega$ contains $\hat{\beta} : \star = 1$ while $\Delta$ contains $\hat{\beta} : \star = \hat{\alpha}$, applying $\Omega$ to the solutions gives the same result: $[\Omega]\hat{\alpha} = [\Omega]1 = 1$, the same as $[\Omega]1 = 1$.

Extension is quite rigid, however, in two senses. First, if a declaration appears in $\Gamma$, it appears in all extensions of $\Gamma$. Second, extension preserves order. For example, if $\hat{\beta}$ is declared after $\hat{\alpha}$ in $\Gamma$, then $\hat{\beta}$ will also be declared after $\hat{\alpha}$ in every extension of $\Gamma$. This holds for every variety of declaration, including equations of universal variables. This rigidity aids in enforcing type variable scoping and dependencies, which are nontrivial in a setting with higher-rank polymorphism.

### 5.4 Determinacy

Given appropriate inputs ($\Gamma, e, A, p$) to the algorithmic judgments, only one set of outputs ($C, q, \Delta$) is derivable (Theorem 5 in the supplementary material, p. 30). We use this property (for spine judgments) in the proof of soundness.
Sound and Complete Bidirectional Typechecking

Under input context $\Gamma$, expression $e$ checks against input type $A$, with output context $\Delta$

$$
\begin{align*}
\Gamma &\vdash e \equiv A \pplus \Delta \\
\Gamma &\vdash e \Rightarrow A \pplus \Delta
\end{align*}
$$

Under input context $\Gamma$, expression $e$ synthesizes output type $A$, with output context $\Delta$

$$
\begin{align*}
\Gamma &\vdash e \Rightarrow A q \ni \Theta \\
\Theta &\vdash A <_{\text{join}(\text{pol}(B),\text{pol}(A))} B + \Delta
\end{align*}
$$

Fig. 14. Algorithmic typing, omitting rules for $\times$, $+$, and $\text{Vec}$
6 SOUNDNESS

We show that the algorithmic system is sound with respect to the declarative system. Soundness for the mutually recursive judgments depends on lemmas for the auxiliary judgments (instantiation, equality elimination, checkprop, algorithmic subtyping and match coverage), which are in Appendix J for space reasons. The main soundness result has mutually recursive parts for checking, synthesis, spines and matching—including the principality-recovering spine judgment.

**Theorem 6.8 (Soundness of Algorithmic Typing).** Given $\Delta \rightarrow \Omega$:

(i) If $\Gamma \vdash e \equiv A \ p + \Delta$ and $\Gamma \vdash A \ p$ type then $[\Omega]A \vdash [\Omega]e \equiv [\Omega]A \ p$.
(ii) If $\Gamma \vdash e \Rightarrow A \ p + \Delta$ then $[\Omega]A \vdash [\Omega]e \Rightarrow [\Omega]A \ p$.
(iii) If $\Gamma \vdash s \ : \ A \ p \Rightarrow B \ q + \Delta$ and $\Gamma \vdash A \ p$ type then $[\Omega]A \vdash [\Omega]s : [\Omega]A \ p \Rightarrow [\Omega]B \ q$.
(iv) If $\Gamma \vdash s \ : \ A \ p \Rightarrow B \ q$ and $\Gamma \vdash A \ p$ type then $[\Omega]A \vdash [\Omega]s : [\Omega]A \ p \Rightarrow [\Omega]B \ q$.
(v) If $\Gamma \vdash \Pi \vdash \tilde{A} \ q \equiv C \ p + \Delta$ and $\Gamma \vdash \tilde{A} \ q$ types and $[\Gamma]\tilde{A} = \tilde{A}$ and $\Gamma \vdash C \ p$ type then $[\Omega]\Delta \vdash [\Omega]\Pi : [\Omega]\tilde{A} \ q \equiv [\Omega]C \ p$.
(vi) If $\Gamma \vdash P + \Pi \vdash A \ ! \equiv C \ p + \Delta$ and $\Gamma \vdash P$ prop and $\text{FEV}(P) = \emptyset$ and $[\Gamma]P = P$
and $\Gamma \vdash \tilde{A} \ !$ types and $\Gamma \vdash C \ p$ type then $[\Omega]\Delta / [\Omega]P + [\Omega]\Pi : [\Omega]\tilde{A} \ ! \equiv [\Omega]C \ p$.

Much of this proof “turns the crank”: apply the induction hypothesis to each premise, yielding derivations of corresponding declarative judgments (with $\Omega$ applied everywhere), then apply the corresponding declarative rule; for example, in the Sub case we finish by applying DeclSub. However, in the SpineRecover case we finish by applying DeclSpineRecover, but since DeclSpineRecover contains a premise that quantifies over all declarative derivations of a certain form, we must appeal to completeness! Consequently, soundness and completeness are really one theorem.

These parts are mutually recursive—later, we’ll see that the DeclSpineRecover case of completeness must appeal to soundness (to show that the algorithmic type has no free existential variables). We cannot induce on the given derivation alone, because the derivations in the “for all” part of DeclSpineRecover are not subderivations. So we need a more involved induction measure that can make the leaps between soundness and completeness: lexicographic order with (1) the size of the subject term, (2) the judgment form, with ordinary spine judgments considered smaller than
recovering spine judgments, and (3) the height of the derivation:

\[
\begin{align*}
&\text{ordinary spine judgment} \\
&e/s/\Pi, \quad < \quad \text{recovering spine judgment}
\end{align*}
\]

**Proof sketch—SpineRecover** case. By i.h., \([\Omega]\Gamma \vdash [\Omega]s : [\Omega]A ! \gg [\Omega]C q\). Our goal is to apply \text{DeclSpineRecover}, which requires that we show that for all \(C'\) such that \([\Omega]\Theta \vdash s : [\Omega]A ! \gg C' \Gamma\), we have \(C' = [\Omega]C\). Suppose we have such a \(C'\). By completeness (Theorem 12), \(\Gamma \vdash s : [\Gamma]A ! \gg C'' q + \Delta''\) where \(\Delta'' \rightarrow \Omega''\). We already have (as a subderivation) \(\Gamma \vdash A! \gg C \Gamma \Delta + \Delta\), so by determinacy, \(C'' = C\) and \(q = \Gamma\) and \(\Delta'' = \Delta\). With the help of lemmas about context application, we can show \(C' = [\Omega'']C'' = [\Omega'']C = [\Omega]C\). (Using completeness is permitted since our measure says a non-principality-restoring judgment is smaller.)

### 6.1 Auxiliary Soundness

For several auxiliary judgment forms, soundness is a matter of showing that, given two algorithmic terms, their declarative versions are equal. For example, for the instantiation judgment we have:

**Lemma (Soundness of Instantiation).**

If \(\Gamma \vdash \hat{a} : \tau \uplus \Delta\) and \(\hat{a} \notin \text{FV}([\Gamma]\tau)\) and \([\Gamma]\tau = \tau\) and \(\Delta \rightarrow \Omega\) then \([\Omega]\hat{a} = [\Omega]\tau\).

We have similar lemmas for term equality (\(\Gamma \vdash \sigma \equiv t : \kappa \uplus \Delta\), propositional equivalence (\(\Gamma \vdash P \equiv Q \uplus \Delta\)) and type equivalence (\(\Gamma \vdash A \equiv B \uplus \Delta\)).

Our eliminating judgments incorporate assumptions into the context \(\Gamma\). We show that the algorithmic rules for these judgments just append equations over universal variables:

**Lemma (Soundness of Equality Elimination).** If \([\Gamma][\sigma = \sigma]t = t\) and \(\Gamma \vdash \sigma : \kappa\) and \(\Gamma \vdash \tau : \kappa\) and \(\text{FV}(\sigma) \cup \text{FV}(t) = \emptyset\), then:

1. If \(\Gamma / \sigma \equiv t : \kappa \uplus \Delta\) then \(\Delta = (\Gamma, \Theta)\) where \(\Theta = (\alpha_1 = t_1, \ldots, \alpha_n = t_n)\) and for all \(\Omega\) such that \(\Gamma \rightarrow \Omega\) and all \(\tau'\) s.t. \(\Theta \vdash \tau' : \kappa'\) we have \([\Omega, \Theta]t' = [\Theta][\Omega]t'\) whenever \(\Theta = \text{mgp}(\sigma, t)\).

2. If \(\Gamma / \sigma \equiv t : \kappa \uplus \perp\) then no most general unifier exists.

The last lemmas for soundness move directly from an algorithmic judgment to the corresponding declarative judgment.

**Lemma (Soundness of Checkprop).** If \(\Gamma \vdash P \text{ true} \uplus \Delta\) and \(\Delta \rightarrow \Omega\) then \(\Psi \vdash [\Omega]P \text{ true}\).

**Lemma (Soundness of Match Coverage).**

1. If \(\Gamma / \Pi \text{ covers } \bar{A} q\) and \(\Gamma \rightarrow \Omega\) and \(\Gamma \vdash \bar{A}!\) types and \([\Gamma]\bar{A} = \bar{A}\) then \([\Omega]\Gamma \vdash \Pi \text{ covers } \bar{A} q\).

2. If \(\Gamma / P \vdash \Pi \text{ covers } \bar{A}!\) and \(\Gamma \rightarrow \Omega\) and \(\Gamma \vdash \bar{A}!\) types and \([\Gamma]\bar{A} = \bar{A}\) and \([\Gamma]P = P\) then \([\Omega]\Gamma / P \vdash \Pi \text{ covers } \bar{A}!\).

**Theorem 6.9 (Soundness of Algorithmic Subtyping).** If \([\Gamma]\bar{A} = A\) and \([\Gamma]B = B\) and \(\Gamma \vdash A\) type and \(\Gamma \vdash B\) type and \(\Delta \rightarrow \Omega\) and \(\Gamma \vdash A < [P]B \uplus \Delta\) then \([\Omega]\Delta \vdash [\Omega]A \leq^P [\Omega]B\).

### 7 COMPLETENESS

We show that the algorithmic system is complete with respect to the declarative system. As with soundness, we need to show completeness of the auxiliary algorithmic judgments. We omit the full statements of these lemmas; as an example, if \([\Omega]\hat{a} = [\Omega]\tau\) and \(\hat{a} \notin \text{FV}(\tau)\) then \(\Gamma \vdash \hat{a} : \tau : \kappa \uplus \Delta\).

#### 7.1 Separation

To show completeness, we will need to show that wherever the declarative rule \text{DeclSpineRecover} is applied, we can apply the algorithmic rule \text{SpineRecover}. Thus, we need to show that semantic principality—that no other type can be given—entails that a type has no free existential variables.
The principality-recovering rules are potentially applicable when we start with a principal type $A \vdash$ but produce $C \vdash$, with DeclSpineRecover changing $!$ to $\hat{!}$. Completeness (Thm. 12) will use the “for all” part of DeclSpineRecover, which quantifies over all types produced by the spine rules under a given declarative context $[\Omega]\Gamma$. By i.h. we get an algorithmic spine judgment $\Gamma \vdash s : A' \vdash C' ! \vdash \Delta$. Since $A'$ is principal, unsolved existentials in $C'$ must have been introduced within this derivation—they can’t be in $\Gamma$ already. Thus, we might have $\hat{a} : \kappa \vdash s : A' ! \vdash \hat{!} ! \vdash \hat{a} : \kappa ! \vdash \hat{!} C \vdash$ a DeclSpineRecover subderivation introduced $\hat{!}$, but $\hat{a}$ can’t appear in $C'$. We also can’t equate $\hat{a}$ and $\hat{!} \beta$ in $\Delta$, which would be tantamount to $C' = \hat{a}$. Knowing that unsolved existentials in $C'$ are “new” and independent from those in $\Gamma$ means we can argue that, if there were an unsolved existential in $C'$, it would correspond to an unforced choice in a DeclSpineRecover subderivation, invalidating the “for all” part of DeclSpineRecover. Formalizing “must have been introduced” requires several definitions.

**Definition 7.1 (Separation).** An algorithmic context $\Gamma$ is separable into $\Gamma_L * \Gamma_R$ if (1) $\Gamma = (\Gamma_L, \Gamma_R)$ and (2) for all $(\hat{a} : \kappa = \tau) \in \Gamma_R$ it is the case that $\text{FEV}(\tau) \subseteq \text{dom}(\Gamma_R)$.

If $\Gamma$ is separable into $\Gamma_L * \Gamma_R$, then $\Gamma_R$ is self-contained in the sense that all existential variables declared in $\Gamma_R$ have solutions whose existential variables are themselves declared in $\Gamma_R$. Every context $\Gamma$ is separable into $\cdot * \Gamma$ and into $\Gamma * \cdot$.

**Definition 7.2 (Separation-Preserving Ext.).** Separated context $\Gamma_L * \Gamma_R$ extends to $\Delta_L * \Gamma_R$, written $(\Gamma_L * \Gamma_R) \rightarrow^* (\Delta_L * \Delta_R)$, if $(\Gamma_L, \Gamma_R) \rightarrow (\Delta_L, \Delta_R)$ and $\text{dom}(\Gamma_L) \subseteq \text{dom}(\Delta_L)$ and $\text{dom}(\Gamma_R) \subseteq \text{dom}(\Delta_R)$.

Separation-preserving extension says that variables from one side of $*$ haven’t “jumped” to the other side. Thus, $\Delta_L$ may add existential variables to $\Gamma_L$, and $\Delta_R$ may add existential variables to $\Gamma_R$, but no variable from $\Gamma_L$ ends up in $\Delta_R$ and no variable from $\Gamma_R$ ends up in $\Delta_L$. It is necessary to write $(\Gamma_L * \Gamma_R) \rightarrow^* (\Delta_L * \Delta_R)$ rather than $(\Gamma_L * \Gamma_R) \rightarrow (\Delta_L * \Delta_R)$, because only $\rightarrow^*$ includes the domain conditions. For example, $(\hat{a} * \hat{!} \beta) \rightarrow (\hat{a}, \hat{!} \beta = \hat{a}) * \cdot$, but $\hat{!} \beta$ has jumped to the left of $*$ in the context $(\hat{a} * \hat{!} \beta = \hat{a}) * \cdot$.

We prove many lemmas about separation, but use only one of them in the subsequent development (in the DeclSpineRecover case of typing completeness), and then only the part for spines. It says that if we have a spine whose type $A$ mentions only variables in $\Gamma_R$, then the output context $\Delta$ extends $\Gamma$ and preserves separation, and the output type $C$ mentions only variables in $\Delta_R$.

**Lemma (Separation—Main).** If $\Gamma_L * \Gamma_R \vdash s : A \vdash C \vdash \Delta$ or $\Gamma_L * \Gamma_R \vdash s : A \vdash C \vdash [\Omega] \vdash \Delta$ and $\Gamma_L * \Gamma_R \vdash A \vdash C \vdash \text{FEV}(A) \subseteq \text{dom}(\Gamma_R)$ then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \rightarrow^* (\Delta_L * \Delta_R)$ and $\text{FEV}(C) \subseteq \text{dom}(\Delta_R)$.

### 7.2 Completeness of typing

Like soundness, completeness has several mutually recursive parts (see the appendix, p. 36).

**Theorem 7.11 (Completeness of Algorithmic Typing).** Given $\Gamma \rightarrow \Omega$ s.t. $\text{dom}(\Gamma) = \text{dom}(\Omega)$:

1. If $\Gamma \vdash A \vdash p \text{ type and } [\Omega] \Gamma \vdash [\Omega] e \iff [\Omega] A \vdash p \text{ and } p' \subseteq p \text{ then there exist } \Delta \text{ and } \Omega' \text{ such that } \Delta \rightarrow \Omega' \text{ and } \text{dom}(\Delta) \text{ and } \Omega \rightarrow \Omega' \text{ and } \Gamma \vdash e \iff [\Gamma] A \vdash p' \vdash \Delta$.
2. If $\Gamma \vdash A \vdash p \text{ type and } [\Omega] \Gamma \vdash [\Omega] e \Rightarrow A \vdash p \text{ then there exist } \Delta, \Omega', A', \text{ and } p' \subseteq p \text{ such that } \Delta \rightarrow \Omega' \text{ and } \text{dom}(\Delta) \text{ and } \Omega \rightarrow \Omega' \text{ and } \Gamma \vdash e \Rightarrow A' \vdash p + \Delta \text{ and } A' = [\Delta] A'$.
3. If $\Gamma \vdash A \vdash p \text{ type and } [\Omega] \Gamma \vdash [\Omega] s : [\Omega] A \vdash B \vdash q \text{ and } p' \subseteq p \text{ then there exist } \Delta, \Omega', B', \text{ and } q' \subseteq q \text{ such that } \Delta \rightarrow \Omega' \text{ and } \text{dom}(\Delta) \text{ and } \Omega \rightarrow \Omega' \text{ and } \Gamma \vdash s : [\Gamma] A \vdash B' \vdash q' + \Delta \text{ and } B' = [\Delta] B' \text{ and } B = [\Omega'] B'$.
4. As part (iii), but with $\Rightarrow B \vdash [q] \cdot \cdot \cdot$ and $\Rightarrow B' \vdash [q'] \cdot \cdot \cdot$.

Proof sketch—DeclSpineRecover case. By i.h., $\Gamma \vdash s : [\Gamma] A \vdash C' ! \vdash \Delta$ where $\Delta \rightarrow \Omega'$ and $\Omega \rightarrow \Omega'$ and $\text{dom}(\Delta) = \text{dom}(\Omega')$ and $C = [\Omega'] C'$.
To apply SpineRecover, we need to show FEV([\Delta]|C') = \emptyset. Suppose, for a contradiction, that FEV([\Delta]|C') \neq \emptyset. Construct a variant of \Omega' called \Omega_2 that has a different solution for some \alpha \in FEV([\Delta]|C'). By soundness (Thm. 12), [\Omega_2] \Gamma \vdash [\Omega_2]s : \Delta \rightarrow [\Omega_2] \alpha \Rightarrow [\Omega_2]C' \Gamma. Using a separation lemma with the trivial \Gamma = (\Gamma \ast \cdot) we get \Delta = (\Delta_L \ast \Delta_R) and (\Gamma \ast \cdot) \rightarrow (\Delta_L \ast \Delta_R) and FEV(C') \subseteq dom(\Delta_R). That is, all existentials in C' were introduced within the derivation of the (algorithmic) spine judgment. Thus, applying \Omega_2 to things gives the same result as \Omega, except for C', giving [\Omega] \Gamma \vdash [\Omega]s : \Delta \rightarrow [\Omega] \alpha \Rightarrow [\Omega]C' \Gamma. Now instantiate the “for all C_2” premise with C_2 = [\Omega_2]C', giving C = [\Omega_2]C'. But we chose \Omega_2 to have a different solution for \alpha \in FEV(C'), so we have C \neq [\Omega_2]C': Contradiction. Therefore FEV([\Delta]|C') = \emptyset, so we can apply SpineRecover.

8 DISCUSSION AND RELATED WORK

A staggering amount of work has been done on GADTs and indexed types, and for space reasons we cannot offer a comprehensive survey of the literature. So we compare more deeply to fewer papers, to communicate our understanding of the design space.

Proof theory and type theory. As described in Section 1, there are two logical accounts of equality—the identity type of Martin-Löf and the equality type of Schroeder-Heister [1994] and Girard [1992]. The Girard/Schroeder-Heister equality has a more direct connection to pattern matching, which is why we make use of it. Coquand [1996] pioneered the study of pattern matching in dependent type theory. One perhaps surprising feature of Coquand’s pattern-matching syntax is that it is strictly stronger than Martin-Löf’s eliminators. His rules can derive the uniqueness of identity proofs as well as the disjointness of constructors. Constructor disjointness is also derivable from the Girard/Schroeder-Heister equality, because there is no unifier for two distinct constructors.

In future work, we hope to study the relation between these two notions of equality in more depth; richer equational theories (such as the theory of commutative rings or the \beta\eta-theory of the lambda calculus) do not have decidable unification, but it seems plausible that there are hybrid approaches which might let us retain some of the convenience of the G/SH equality rule while retaining the decidability of Martin-Löf’s J eliminator.

Indexed and refinement types. Dependent ML [Xi and Pfenning 1999] indexed programs with propositional constraints, extending the ML type discipline to maintain additional invariants. DML collected constraints from the program and passed them to a constraint solver, a technique used by systems like Stardust [Dunfield 2007a] and liquid types [Rondon et al. 2008].

From phantom types to GADTs. Leijen and Meijer [1999] introduced the term phantom type to describe a technique for programming in ML/Haskell where additional type parameters are used to constrain when values are well-typed. This idea proved to have many applications, ranging from foreign function interfaces [Blume 2001] to encoding Java-style subtyping [Fluet and Pucella 2006]. Phantom types allow constructing values with constrained types, but do not easily permit learning about type equalities by analyzing them, putting applications such as intensional type analysis [Harper and Morrisett 1995] out of reach. Both Cheney and Hinze [2003] and Xi et al. [2003] proposed treating equalities as a first-class concept, giving explicitly-typed calculi for equalities, but without studying algorithms for type inference.

Simonet and Pottier [2007] gave a constraint-based algorithm for type inference for GADTs. It is this work which first identified the potential intractibility of type inference arising from the interaction of hypothetical constraints and unification variables. To resolve this issue they introduce the notion of tractable constraints (i.e., constraints where hypothetical equations never contain existentials), and require placing enough annotations that all constraints are tractable. In general,
this could require annotations on case expressions, so subsequent work focused on relaxing this requirement. Though quite different in technical detail, stratified inference [Pottier and Régis-Gianas 2006] and wobbly types [Peyton Jones et al. 2006] both work by pushing type information from annotations to case expressions, with stratified type inference literally moving annotations around, and wobbly types tracking which parts of a type have no unification variables. Modern GHC uses the OutsIdeln algorithm [Vytiniotis et al. 2011], which further relaxes the constraint: case analysis is permitted as long as it cannot modify what is known about an equation.

In our type system, the checking judgment of the bidirectional algorithm serves to propagate annotations; our requirement that the scrutinee of a case expression be principal ensures that no equations contain unification variables. The result is close in effect to stratified types, and is less expressive than OutsIdeln. This is a deliberate design choice to keep the meaning of principality—that only a single type can be inferred for a term—clear and easy to understand.

To specify the OutsIdeln approach, the case rule in our declarative system should permit scrutinizing an expression if all types that can be synthesized for it have exactly the same equations, even if they differ in their monotype parts. To achieve this, we would need to introduce a relation \( C' \sim C \) which checks whether the equational constraints in \( C \) and \( C' \) are the same, and then modify the higher-order premise of the DeclSpineRecover rule to check that \( C' \sim C \) (rather than \( C' = C \), as it is currently). However, we thought such a spec is harder for programmers to develop an intuition for than simply saying that a scrutinee must synthesize a unique type.

Garrigue and Rémy [2013] proposed ambivalent types, which are a way of deciding when it is safe to generalize the type of a function using GADTs. This idea is orthogonal to our calculus, simply because we do no generalization at all: every polymorphic function takes an annotation. However, Garrigue and Rémy [2013] also emphasize the importance of monotonicity, which says that substitution should be stable under subtyping, that is, giving a more general type should not cause subtyping to fail. This condition is satisfied by our bidirectional system.

Karachalias et al. [2015] developed a coverage algorithm for GADTs that depends on external constraint solving; we offer a more self-contained but still logically-motivated approach.

**Polarized subtyping.** Barendregt et al. [1983] observed that a program which typechecks under a subtyping discipline can be checked without subtyping, provided that the program is sufficiently \( \eta \)-expanded. This idea of subtyping as \( \eta \)-expansion was investigated in a focused (albeit infinitary) setting by Zeilberger [2009]. Another notion of polarity arises from considering the (co-, contra-, in-)variance of type constructors. It is used by Abel [2006] to give a version of \( F^\omega \) with subtyping, and Dolan and Mycroft [2017] apply this version of polarity to give a complete type inference algorithm for an ML-style language with subtyping. Our polarized subtyping judgment is closest in spirit to the work of Zeilberger [2009]. The restriction on our subtyping relation can be understood in terms of requiring the \( \eta \) expansions our subtyping relation infers to be in a focused normal form.

**Extensions.** To keep our formalization manageable, we left out some features that would be desirable in practice. In particular, we need (1) type constructors which take arguments and (2) recursive types [Pierce 2002, chapter 20]. The issue with both of these features is that they need to permit instantiating quantifiers with existentials and other binders, and our system relies upon monotypes (which do not contain such connectives). This limitation should create no difficulties in typical practice if we treat user-defined type constructors like List as monotypes, expanding the definition only as needed: when checking an expression against a user type constructor, and for pattern matching. Another extension, which we intend as future work, is to replace ordinary unification with pattern or nominal unification, to allow type instantiations containing binders.

Another extension is to increase the amount of type inference done. For instance, a natural question is whether we can extend the bidirectional approach to subsume the inference done by the
algorithm of Damas and Milner [1982]. On the implementation side, this seems easy—to support ML-style type inference, we can add rules to infer types for values:

$$\Gamma, \triangleright \hat{a}, \hat{\beta}, x : \hat{a} + e \Leftarrow \hat{\beta} \triangleright \Delta, \triangleright \hat{a}, \hat{\Delta}' \triangleright \gamma = \text{unsolved}(\hat{\Delta}')$$

$$\Gamma + \lambda x. e \Rightarrow \forall \hat{a}. [\hat{a}/\gamma][\hat{\Delta}'](\hat{a} \rightarrow \hat{\beta}) ! + \Delta$$

This rule adds a marker $\triangleright \hat{a}$ to the context, then checks the body $e$ against the type $\hat{\beta}$. Our output type substitutes away all the solved existential variables to the right of $\triangleright \hat{a}$, and generalizes over all unsolved variables to the right of the marker. Using an ordered context gives precise control over the scope of the existential variables, easily expressing polymorphic generalization.

However, in the presence of generalization, the declarative specification of type inference no longer strictly specifies the order of polymorphic quantifiers (i.e., $\forall \alpha, \beta. \alpha \rightarrow \beta \rightarrow (\alpha \times \beta)$ and $\forall \beta, \alpha. \alpha \rightarrow \beta \rightarrow (\alpha \times \beta)$ should be equivalent) and so our principal synthesis would no longer return types stable up to alpha-equivalence. Fixing this would be straightforward (by relaxing the definition of type equivalence), but we have not pursued this because we do not value let-generalization enough to pay the price of increased complexity in our proofs.

ACKNOWLEDGMENTS

We thank the anonymous reviewers of this version, and of several previous versions, for their comments. We also thank Soham Chowdhury for his work on implementing the system presented in this paper.

REFERENCES


Sound and Complete Bidirectional Typechecking for Higher-Rank Polymorphism with Existentials and Indexed Types: Full definitions, lemmas and proofs

Joshua Dunfield Neelakantan R. Krishnaswami

November 13, 2018

The first part (Sections 1–2) of this supplementary material contains rules, figures and definitions omitted in the main paper for space reasons, and a list of judgment forms (Section 2).

The remainder (Sections A–K) includes statements of all lemmas and theorems, along with full proofs, as well as statements of theorems and a few selected lemmas.

Contents

1 Figures 8
2 List of Judgments 18
A Properties of the Declarative System 19
1 Lemma (Declarative Well-foundedness) 19
2 Lemma (Declarative Weakening) 19
3 Lemma (Declarative Term Substitution) 19
4 Lemma (Reflexivity of Declarative Subtyping) 19
5 Lemma (Subtyping Inversion) 19
6 Lemma (Subtyping Polarity Flip) 19
7 Lemma (Transitivity of Declarative Subtyping) 20
B Substitution and Well-formedness Properties 20
8 Lemma (Substitution—Well-formedness) 20
9 Lemma (Uvar Preservation) 20
10 Lemma (Sorting Implies Typing) 20
11 Lemma (Right-Hand Substitution for Sorting) 20
12 Lemma (Right-Hand Substitution for Propositions) 20
13 Lemma (Right-Hand Substitution for Typing) 20
14 Lemma (Substitution for Sorting) 20
15 Lemma (Substitution for Prop Well-Formedness) 20
16 Lemma (Substitution for Type Well-Formedness) 20
17 Lemma (Substitution Stability) 20
18 Lemma (Equal Domains) 20
C Properties of Extension 20
19 Lemma (Declaration Preservation) 20
20 Lemma (Declaration Order Preservation) 20
21 Lemma (Reverse Declaration Order Preservation) 20
22 Lemma (Extension Inversion) 21
## CONTENTS

<table>
<thead>
<tr>
<th>Page</th>
<th>Lemma (Deep Evar Introduction)</th>
<th>Lemma (Soft Extension)</th>
<th>Lemma (Parallel Admissibility)</th>
<th>Lemma (Parallel Extension Solution)</th>
<th>Lemma (Parallel Variable Update)</th>
<th>Lemma (Substitution Monotonicity)</th>
<th>Lemma (Substitution Invariance)</th>
<th>Lemma (Split Extension)</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>26</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>33</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>34</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>37</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>38</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>39</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>41</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>42</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>43</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>44</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>46</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>47</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>48</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>49</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>51</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>52</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>53</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>54</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>55</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>56</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>57</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>58</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>59</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### C.1 Reflexivity and Transitivity

<table>
<thead>
<tr>
<th>Page</th>
<th>Lemma (Extension Reflexivity)</th>
<th>Lemma (Extension Transitivity)</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### C.2 Weakening

<table>
<thead>
<tr>
<th>Page</th>
<th>Lemma (Suffix Weakening)</th>
<th>Lemma (Suffix Weakening)</th>
<th>Lemma (Extension Weakening (Sorts))</th>
<th>Lemma (Extension Weakening (Props))</th>
<th>Lemma (Extension Weakening (Types))</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### C.3 Principal Typing Properties

<table>
<thead>
<tr>
<th>Page</th>
<th>Lemma (Principal Agreement)</th>
<th>Lemma (Right-Hand Subst. for Principal Typing)</th>
<th>Lemma (Extension Weakening for Principal Typing)</th>
<th>Lemma (Inversion of Principal Typing)</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### C.4 Instantiation Extends

<table>
<thead>
<tr>
<th>Page</th>
<th>Lemma (Instantiation Extension)</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td></td>
</tr>
</tbody>
</table>

### C.5 Equivalence Extends

<table>
<thead>
<tr>
<th>Page</th>
<th>Lemma (Elimeq Extension)</th>
<th>Lemma (Elimprop Extension)</th>
<th>Lemma (Checkeq Extension)</th>
<th>Lemma (Checkprop Extension)</th>
<th>Lemma (Prop Equivalence Extension)</th>
<th>Lemma (Equivalence Extension)</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### C.6 Subtyping Extends

<table>
<thead>
<tr>
<th>Page</th>
<th>Lemma (Subtyping Extension)</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td></td>
</tr>
</tbody>
</table>

### C.7 Typing Extends

<table>
<thead>
<tr>
<th>Page</th>
<th>Lemma (Typing Extension)</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td></td>
</tr>
</tbody>
</table>

### C.8 Unfiled

<table>
<thead>
<tr>
<th>Page</th>
<th>Lemma (Context Partitioning)</th>
<th>Lemma (Completing Stability)</th>
<th>Lemma (Completing Completeness)</th>
<th>Lemma (Confluence of Completeness)</th>
<th>Lemma (Multiple Confluence)</th>
<th>Lemma (Canonical Completion)</th>
<th>Lemma (Split Solutions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### D Internal Properties of the Declarative System

<table>
<thead>
<tr>
<th>Page</th>
<th>Lemma (Interpolating With and Exists)</th>
<th>Lemma (Case Invertibility)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### E Miscellaneous Properties of the Algorithmic System

<table>
<thead>
<tr>
<th>Page</th>
<th>Lemma (Well-Formed Outputs of Typing)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td></td>
</tr>
</tbody>
</table>
### CONTENTS

#### F Decidability of Instantiation  
- Lemma (Left Unsolvedness Preservation)  
- Lemma (Left Free Variable Preservation)  
- Lemma (Instantiation Size Preservation)  
- Lemma (Decidability of Instantiation)  

#### G Separation  
- Lemma (Transitivity of Separation)  
- Lemma (Separation Truncation)  
- Lemma (Separation for Auxiliary Judgments)  
- Lemma (Separation for Subtyping)  
- Lemma (Separation—Main)  

#### H Decidability of Algorithmic Subtyping  
**H.1 Lemmas for Decidability of Subtyping**  
- Lemma (Substitution Isn’t Large)  
- Lemma (Instantiation Solves)  
- Lemma (Checkeq Solving)  
- Lemma (Prop Equiv Solving)  
- Lemma (Equiv Solving)  
- Lemma (Decidability of Propositional Judgments)  
- Lemma (Decidability of Equivalence)  

**H.2 Decidability of Subtyping**  
1. Theorem (Decidability of Subtyping)  

**H.3 Decidability of Matching and Coverage**  
- Lemma (Decidability of Guardedness Judgment)  
- Lemma (Decidability of Expansion Judgments)  
- Lemma (Expansion Shrinks Size)  
2. Theorem (Decidability of Coverage)  

**H.4 Decidability of Typing**  
3. Theorem (Decidability of Typing)  

#### I Determinacy  
- Lemma (Determinacy of Auxiliary Judgments)  
- Lemma (Determinacy of Equivalence)  
4. Theorem (Determinacy of Subtyping)  
5. Theorem (Determinacy of Typing)  

#### J Soundness  
**J.1 Soundness of Instantiation**  
- Lemma (Soundness of Instantiation)  

**J.2 Soundness of Checkeq**  
- Lemma (Soundness of Checkeq)  

**J.3 Soundness of Equivalence (Propositions and Types)**  
- Lemma (Soundness of Propositional Equivalence)  
- Lemma (Soundness of Algorithmic Equivalence)  

**J.4 Soundness of Checkprop**  
- Lemma (Soundness of Checkprop)  

**J.5 Soundness of Eliminations (Equality and Proposition)**  
- Lemma (Soundness of Equality Elimination)  

**J.6 Soundness of Subtyping**  
- Theorem (Soundness of Algorithmic Subtyping)  

**J.7 Soundness of Typing**  
- Theorem (Soundness of Match Coverage)  
- Lemma (Well-formedness of Algorithmic Typing)  

November 13, 2018
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>K</strong></td>
<td>Completeness</td>
<td>34</td>
</tr>
<tr>
<td>K.1</td>
<td>Completeness of Auxiliary Judgments</td>
<td>34</td>
</tr>
<tr>
<td>92</td>
<td>Lemma (Completeness of Instantiation)</td>
<td>34</td>
</tr>
<tr>
<td>93</td>
<td>Lemma (Completeness of Checkeq)</td>
<td>34</td>
</tr>
<tr>
<td>94</td>
<td>Lemma (Completeness of Elimeq)</td>
<td>34</td>
</tr>
<tr>
<td>95</td>
<td>Lemma (Substitution Upgrade)</td>
<td>34</td>
</tr>
<tr>
<td>96</td>
<td>Lemma (Completeness of Propequiv)</td>
<td>35</td>
</tr>
<tr>
<td>97</td>
<td>Lemma (Completeness of Checkprop)</td>
<td>35</td>
</tr>
<tr>
<td>K.2</td>
<td>Completeness of Equivalence and Subtyping</td>
<td>35</td>
</tr>
<tr>
<td>98</td>
<td>Lemma (Completeness of Equiv)</td>
<td>35</td>
</tr>
<tr>
<td>99</td>
<td>Theorem (Completeness of Subtyping)</td>
<td>35</td>
</tr>
<tr>
<td>K.3</td>
<td>Completeness of Typing</td>
<td>35</td>
</tr>
<tr>
<td>99</td>
<td>Lemma (Variable Decomposition)</td>
<td>35</td>
</tr>
<tr>
<td>100</td>
<td>Lemma (Pattern Decomposition and Substitution)</td>
<td>35</td>
</tr>
<tr>
<td>101</td>
<td>Lemma (Pattern Decomposition Functionality)</td>
<td>36</td>
</tr>
<tr>
<td>102</td>
<td>Lemma (Decidability of Variable Removal)</td>
<td>36</td>
</tr>
<tr>
<td>103</td>
<td>Lemma (Variable Inversion)</td>
<td>36</td>
</tr>
<tr>
<td>104</td>
<td>Theorem (Completeness of Match Coverage)</td>
<td>36</td>
</tr>
<tr>
<td><strong>Proofs</strong></td>
<td></td>
<td>38</td>
</tr>
<tr>
<td>A'</td>
<td>Properties of the Declarative System</td>
<td>38</td>
</tr>
<tr>
<td>1</td>
<td>Proof of Lemma (Declarative Well-foundedness)</td>
<td>38</td>
</tr>
<tr>
<td>2</td>
<td>Proof of Lemma (Declarative Weakening)</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>Proof of Lemma (Declarative Term Substitution)</td>
<td>41</td>
</tr>
<tr>
<td>4</td>
<td>Proof of Lemma (Reflexivity of Declarative Subtyping)</td>
<td>41</td>
</tr>
<tr>
<td>5</td>
<td>Proof of Lemma (Subtyping Inversion)</td>
<td>41</td>
</tr>
<tr>
<td>6</td>
<td>Proof of Lemma (Subtyping Polarity Flip)</td>
<td>42</td>
</tr>
<tr>
<td>7</td>
<td>Proof of Lemma (Transitivity of Declarative Subtyping)</td>
<td>42</td>
</tr>
<tr>
<td>B'</td>
<td>Substitution and Well-formedness Properties</td>
<td>45</td>
</tr>
<tr>
<td>8</td>
<td>Proof of Lemma (Substitution—Well-formedness)</td>
<td>45</td>
</tr>
<tr>
<td>9</td>
<td>Proof of Lemma (Uvar Preservation)</td>
<td>45</td>
</tr>
<tr>
<td>10</td>
<td>Proof of Lemma (Sorting Implies Typing)</td>
<td>45</td>
</tr>
<tr>
<td>11</td>
<td>Proof of Lemma (Right-Hand Substitution for Sorting)</td>
<td>45</td>
</tr>
<tr>
<td>12</td>
<td>Proof of Lemma (Right-Hand Substitution for Propositions)</td>
<td>46</td>
</tr>
<tr>
<td>13</td>
<td>Proof of Lemma (Right-Hand Substitution for Typing)</td>
<td>46</td>
</tr>
<tr>
<td>14</td>
<td>Proof of Lemma (Substitution for Sorting)</td>
<td>46</td>
</tr>
<tr>
<td>15</td>
<td>Proof of Lemma (Substitution for Prop Well-Formedness)</td>
<td>47</td>
</tr>
<tr>
<td>16</td>
<td>Proof of Lemma (Substitution for Type Well-Formedness)</td>
<td>48</td>
</tr>
<tr>
<td>17</td>
<td>Proof of Lemma (Substitution Stability)</td>
<td>49</td>
</tr>
<tr>
<td>18</td>
<td>Proof of Lemma (Equal Domains)</td>
<td>49</td>
</tr>
<tr>
<td>C'</td>
<td>Properties of Extension</td>
<td>49</td>
</tr>
<tr>
<td>19</td>
<td>Proof of Lemma (Declaration Preservation)</td>
<td>49</td>
</tr>
<tr>
<td>20</td>
<td>Proof of Lemma (Declaration Order Preservation)</td>
<td>50</td>
</tr>
<tr>
<td>21</td>
<td>Proof of Lemma (Reverse Declaration Order Preservation)</td>
<td>51</td>
</tr>
<tr>
<td>22</td>
<td>Proof of Lemma (Extension Inversion)</td>
<td>51</td>
</tr>
<tr>
<td>23</td>
<td>Proof of Lemma (Deep Evar Introduction)</td>
<td>62</td>
</tr>
<tr>
<td>24</td>
<td>Proof of Lemma (Parallel Admissibility)</td>
<td>66</td>
</tr>
<tr>
<td>25</td>
<td>Proof of Lemma (Parallel Extension Solution)</td>
<td>67</td>
</tr>
<tr>
<td>Page</td>
<td>Lemma Title</td>
<td></td>
</tr>
<tr>
<td>------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>Proof of Lemma (Parallel Variable Update)</td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>Proof of Lemma (Substitution Monotonicity)</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>Proof of Lemma (Substitution Invariance)</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>Proof of Lemma (Soft Extension)</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>Proof of Lemma (Split Extension)</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>Proof of Lemma (Extension Reflexivity)</td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>Proof of Lemma (Extension Transitivity)</td>
<td></td>
</tr>
<tr>
<td>34</td>
<td>Proof of Lemma (Suffix Weakening)</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>Proof of Lemma (Suffix Weakening)</td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>Proof of Lemma (Extension Weakening (Sorts))</td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>Proof of Lemma (Extension Weakening (Props))</td>
<td></td>
</tr>
<tr>
<td>38</td>
<td>Proof of Lemma (Extension Weakening (Types))</td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>Proof of Lemma (Principal Agreement)</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>Proof of Lemma (Right-Hand Subst. for Principal Typing)</td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>Proof of Lemma (Extension Weakening for Principal Typing)</td>
<td></td>
</tr>
<tr>
<td>42</td>
<td>Proof of Lemma (Inversion of Principal Typing)</td>
<td></td>
</tr>
<tr>
<td>43</td>
<td>Proof of Lemma (Instantiation Extension)</td>
<td></td>
</tr>
<tr>
<td>44</td>
<td>Proof of Lemma (Elimeq Extension)</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>Proof of Lemma (Elimprop Extension)</td>
<td></td>
</tr>
<tr>
<td>46</td>
<td>Proof of Lemma (Checkeq Extension)</td>
<td></td>
</tr>
<tr>
<td>47</td>
<td>Proof of Lemma (Checkprop Extension)</td>
<td></td>
</tr>
<tr>
<td>48</td>
<td>Proof of Lemma (Prop Equivalence Extension)</td>
<td></td>
</tr>
<tr>
<td>49</td>
<td>Proof of Lemma (Equivalence Extension)</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>Proof of Lemma (Subtyping Extension)</td>
<td></td>
</tr>
<tr>
<td>51</td>
<td>Proof of Lemma (Typing Extension)</td>
<td></td>
</tr>
<tr>
<td>52</td>
<td>Proof of Lemma (Context Partitioning)</td>
<td></td>
</tr>
<tr>
<td>53</td>
<td>Proof of Lemma (Completing Stability)</td>
<td></td>
</tr>
<tr>
<td>54</td>
<td>Proof of Lemma (Completing Completeness)</td>
<td></td>
</tr>
<tr>
<td>55</td>
<td>Proof of Lemma (Confluence of Completeness)</td>
<td></td>
</tr>
<tr>
<td>56</td>
<td>Proof of Lemma (Multiple Confluence)</td>
<td></td>
</tr>
<tr>
<td>57</td>
<td>Proof of Lemma (Canonical Completion)</td>
<td></td>
</tr>
<tr>
<td>58</td>
<td>Proof of Lemma (Split Solutions)</td>
<td></td>
</tr>
<tr>
<td>59</td>
<td>Proof of Lemma (Interpolating With and Exists)</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>Proof of Lemma (Case Invertibility)</td>
<td></td>
</tr>
<tr>
<td>61</td>
<td>Proof of Lemma (Well-Formed Outputs of Typing)</td>
<td></td>
</tr>
<tr>
<td>62</td>
<td>Proof of Lemma (Left Unsolvedness Preservation)</td>
<td></td>
</tr>
<tr>
<td>63</td>
<td>Proof of Lemma (Left Free Variable Preservation)</td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>Proof of Lemma (Instantiation Size Preservation)</td>
<td></td>
</tr>
<tr>
<td>65</td>
<td>Proof of Lemma (Decidability of Instantiation)</td>
<td></td>
</tr>
</tbody>
</table>
## CONTENTS

### G' Separation

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>68</td>
<td>Proof of Lemma (Transitivity of Separation)</td>
<td>93</td>
</tr>
<tr>
<td>69</td>
<td>Proof of Lemma (Separation Truncation)</td>
<td>94</td>
</tr>
<tr>
<td>70</td>
<td>Proof of Lemma (Separation for Auxiliary Judgments)</td>
<td>94</td>
</tr>
<tr>
<td>71</td>
<td>Proof of Lemma (Separation for Subtyping)</td>
<td>95</td>
</tr>
<tr>
<td>72</td>
<td>Proof of Lemma (Separation—Main)</td>
<td>95</td>
</tr>
</tbody>
</table>

### H' Decidability of Algorithmic Subtyping

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>73</td>
<td>Proof of Lemma (Substitution isn’t Large)</td>
<td>103</td>
</tr>
<tr>
<td>74</td>
<td>Proof of Lemma (Instantiation Solves)</td>
<td>103</td>
</tr>
<tr>
<td>75</td>
<td>Proof of Lemma (Checkeq Solving)</td>
<td>104</td>
</tr>
<tr>
<td>76</td>
<td>Proof of Lemma (Prop Equiv Solving)</td>
<td>105</td>
</tr>
<tr>
<td>77</td>
<td>Proof of Lemma (Equiv Solving)</td>
<td>105</td>
</tr>
<tr>
<td>78</td>
<td>Proof of Lemma (Decidability of Propositional Judgments)</td>
<td>106</td>
</tr>
<tr>
<td>79</td>
<td>Proof of Lemma (Decidability of Equivalence)</td>
<td>107</td>
</tr>
</tbody>
</table>

### I' Determinacy

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>83</td>
<td>Proof of Lemma (Determinacy of Auxiliary Judgments)</td>
<td>115</td>
</tr>
<tr>
<td>84</td>
<td>Proof of Lemma (Determinacy of Equivalence)</td>
<td>116</td>
</tr>
<tr>
<td>85</td>
<td>Proof of Theorem (Determinacy of Subtyping)</td>
<td>116</td>
</tr>
<tr>
<td>86</td>
<td>Proof of Theorem (Determinacy of Typing)</td>
<td>117</td>
</tr>
</tbody>
</table>

### J' Soundness

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>85</td>
<td>Proof of Lemma (Soundness of Instantiation)</td>
<td>119</td>
</tr>
<tr>
<td>86</td>
<td>Proof of Lemma (Soundness of Checkeq)</td>
<td>119</td>
</tr>
<tr>
<td>87</td>
<td>Proof of Lemma (Soundness of Propositional Equivalence)</td>
<td>120</td>
</tr>
<tr>
<td>88</td>
<td>Proof of Lemma (Soundness of Algorithmic Equivalence)</td>
<td>121</td>
</tr>
<tr>
<td>89</td>
<td>Proof of Lemma (Soundness of Checkprop)</td>
<td>123</td>
</tr>
<tr>
<td>90</td>
<td>Proof of Lemma (Soundness of Equality Elimination)</td>
<td>123</td>
</tr>
<tr>
<td>91</td>
<td>Proof of Theorem (Soundness of Algorithmic Subtyping)</td>
<td>127</td>
</tr>
<tr>
<td>92</td>
<td>Proof of Theorem (Soundness of Match Coverage)</td>
<td>129</td>
</tr>
<tr>
<td>93</td>
<td>Proof of Lemma (Well-formedness of Algorithmic Typing)</td>
<td>130</td>
</tr>
<tr>
<td>94</td>
<td>Proof of Theorem (Eagerness of Types)</td>
<td>131</td>
</tr>
<tr>
<td>95</td>
<td>Proof of Theorem (Soundness of Algorithmic Typing)</td>
<td>134</td>
</tr>
</tbody>
</table>

### K' Completeness

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>92</td>
<td>Proof of Lemma (Completeness of Instantiation)</td>
<td>149</td>
</tr>
<tr>
<td>93</td>
<td>Proof of Lemma (Completeness of Checkeq)</td>
<td>149</td>
</tr>
<tr>
<td>94</td>
<td>Proof of Lemma (Completeness of Elimeq)</td>
<td>153</td>
</tr>
<tr>
<td>95</td>
<td>Proof of Lemma (Substitution Upgrade)</td>
<td>155</td>
</tr>
<tr>
<td>Page</td>
<td>Section</td>
<td>Title</td>
</tr>
<tr>
<td>------</td>
<td>---------</td>
<td>-------</td>
</tr>
<tr>
<td>96</td>
<td>2</td>
<td>Proof of Lemma (Completeness of Propequiv)</td>
</tr>
<tr>
<td>97</td>
<td>2</td>
<td>Proof of Lemma (Completeness of Checkprop)</td>
</tr>
<tr>
<td>98</td>
<td>2</td>
<td>Proof of Lemma (Completeness of Equiv)</td>
</tr>
<tr>
<td>99</td>
<td>2</td>
<td>Proof of Theorem (Completeness of Subtyping)</td>
</tr>
<tr>
<td>100</td>
<td>2</td>
<td>Proof of Lemma (Pattern Decomposition and Substitution)</td>
</tr>
<tr>
<td>101</td>
<td>2</td>
<td>Proof of Lemma (Pattern Decomposition Functionality)</td>
</tr>
<tr>
<td>102</td>
<td>2</td>
<td>Proof of Lemma (Decidability of Variable Removal)</td>
</tr>
<tr>
<td>103</td>
<td>2</td>
<td>Proof of Lemma (Variable Inversion)</td>
</tr>
<tr>
<td>111</td>
<td>3</td>
<td>Proof of Theorem (Completeness of Match Coverage)</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
<td>Proof of Theorem (Completeness of Algorithmic Typing)</td>
</tr>
</tbody>
</table>

November 13, 2018
1 Figures

We repeat some figures from the main paper. In Figures 6 and 14, we include rules omitted from the main paper for space reasons.

Figure 6: Declarative typing, including rules omitted from main paper
Under input context $\Gamma$, expression $e$ checks against input type $A$, with output context $\Delta$
Under input context $\Gamma$, expression $e$ synthesizes output type $A$, with output context $\Delta$

\[
\begin{align*}
(\cdot) & \vdash e \iff A \vdash \Delta \\
\Gamma \vdash e \implies A \vdash \Delta & \quad \text{Anno} \\
\Gamma \vdash e \implies B \vdash \Delta & \quad \text{Rec}
\end{align*}
\]

Figure 14: Algorithmic typing, including rules omitted from main paper
\[ \Psi \vdash t : \kappa \] Under context $\Psi$, term $t$ has sort $\kappa$

\[ \frac{\alpha : \kappa \in \Psi}{\Psi \vdash \alpha : \kappa} \quad \text{UvarSort} \quad \frac{\Psi \vdash 1 : \star}{\Psi \vdash \alpha : \star} \quad \text{UnitSort} \quad \frac{\Psi \vdash t_1 : \star \quad \Psi \vdash t_2 : \star}{\Psi \vdash t_1 \oplus t_2 : \star} \quad \text{BinSort} \]

\[ \Psi \vdash \text{zero} : \mathbb{N} \quad \Psi \vdash t : \mathbb{N} \quad \Psi \vdash \text{succ}(t) : \mathbb{N} \quad \text{SuccSort} \]

\[ \Psi \vdash P \quad \text{prop} \] Under context $\Psi$, proposition $P$ is well-formed

\[ \frac{\Psi \vdash t : \mathbb{N}}{\Psi \vdash t = t' : \mathbb{N}} \quad \text{EqDeclProp} \]

\[ \Psi \vdash A \quad \text{type} \] Under context $\Psi$, type $A$ is well-formed

\[ \frac{(\alpha : \star) \in \Psi}{\Psi \vdash \alpha : \text{type}} \quad \text{DeclUvarWF} \quad \frac{\Psi \vdash \alpha : \star}{\Psi \vdash \alpha : \text{type}} \quad \text{DeclUnitWF} \]

\[ \frac{\Psi \vdash P \quad \Psi \vdash A \quad \Psi \vdash \lnot A \quad \Psi \vdash \top \quad \Psi \vdash \bot}{\Psi \vdash P \lor A \quad \Psi \vdash P \land A \quad \Psi \vdash P \land \lnot A \quad \Psi \vdash P \lor \lnot A} \quad \text{DeclImplesWF} \]

\[ \frac{\Psi, \alpha : \kappa \vdash A \quad \Psi \vdash (\forall \alpha : \kappa. A) \quad \Psi \vdash (\exists \alpha : \kappa. A) \quad \Psi \vdash (\forall \alpha : \kappa. A) \text{ type}}{\Psi \vdash (\forall \alpha : \kappa. A) \text{ type}} \quad \text{DeclAllWF} \]

\[ \Psi \vdash A \quad \Psi \vdash \lnot A \quad \Psi \vdash A \quad \Psi \vdash \top \quad \Psi \vdash \bot \quad \text{DeclExistsWF} \]

\[ \frac{\Psi \vdash (\alpha : \kappa) \quad \Psi \vdash A \quad \Psi \vdash (\forall \alpha : \kappa. A) \quad \Psi \vdash (\exists \alpha : \kappa. A) \quad \Psi \vdash (\forall \alpha : \kappa. A) \text{ type}}{\Psi \vdash (\forall \alpha : \kappa. A) \text{ type}} \quad \text{DeclWithWF} \]

\[ \Psi \vdash \vec{A} \quad \text{types} \] Under context $\Psi$, types in $\vec{A}$ are well-formed

\[ \text{for all } A \in \vec{A}, \quad \frac{\Psi \vdash A \text{ type}}{\Psi \vdash \vec{A} \text{ types}} \quad \text{DeclTypevecWF} \]

\[ \Psi \text{ ctx} \] Declarative context $\Psi$ is well-formed

\[ \frac{\Psi \text{ ctx} \quad x \notin \text{dom}(\Psi)}{\Psi, x : A \text{ ctx}} \quad \text{HypDeclCtx} \quad \frac{\Psi \text{ ctx} \quad \alpha \notin \text{dom}(\Psi)}{\Psi, \alpha : \kappa \text{ ctx}} \quad \text{VarDeclCtx} \]

Figure 16: Sorting; well-formedness of propositions, types, and contexts in the declarative system
\[ \Gamma \vdash \tau : \kappa \] Under context \( \Gamma \), term \( \tau \) has sort \( \kappa \)

\[ \frac{(u : \kappa) \in \Gamma}{\Gamma \vdash u : \kappa} \] \hspace{1cm} \text{VarSort}

\[ \frac{(\alpha : \kappa = \tau) \in \Gamma}{\Gamma \vdash \alpha : \kappa} \] \hspace{1cm} \text{SolvedVarSort}

\[ \Gamma \vdash 1 : \ast \] \hspace{1cm} \text{UnitSort}

\[ \frac{\Gamma \vdash \tau_1 : \ast}{\Gamma \vdash \tau_1 \oplus \tau_2 : \ast} \] \hspace{1cm} \text{BinSort}

\[ \frac{\Gamma \vdash \text{zero} : \mathbb{N}}{\Gamma \vdash \text{succ}(t) : \mathbb{N}} \] \hspace{1cm} \text{SuccSort}

\[ \Gamma \vdash P \text{ prop} \] Under context \( \Gamma \), proposition \( P \) is well-formed

\[ \frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash t' : \mathbb{N}} \] \hspace{1cm} \text{EqProp}

\[ \frac{\Gamma \vdash \tau_1 : \mathbb{N} \Gamma \vdash \tau_2 : \mathbb{N}}{\Gamma \vdash \tau_1 \oplus \tau_2 : \mathbb{N}} \] \hspace{1cm} \text{BinWF}

\[ \frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \text{vec}(t) : A} \] \hspace{1cm} \text{VecWF}

\[ \frac{\Gamma, \alpha : \kappa \vdash A \text{ type}}{\Gamma \vdash \forall \alpha : \kappa. A \text{ type}} \] \hspace{1cm} \text{ForallWF}

\[ \frac{\Gamma, \alpha : \kappa \vdash A \text{ type}}{\Gamma \vdash \exists \alpha : \kappa. A \text{ type}} \] \hspace{1cm} \text{ExistsWF}

\[ \frac{\Gamma \vdash P \text{ prop} \Gamma \vdash A \text{ type}}{\Gamma \vdash A \rightarrow P \text{ type}} \] \hspace{1cm} \text{ImpliesWF}

\[ \frac{\Gamma \vdash A \text{ type} \Gamma \vdash P \text{ prop}}{\Gamma \vdash A \land P \text{ type}} \] \hspace{1cm} \text{WithWF}

\[ \Gamma \vdash A \text{ type} \] Under context \( \Gamma \), type \( A \) is well-formed and respects principality \( p \)

\[ \frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \text{FEV}(\Gamma) = \emptyset} \] \hspace{1cm} \text{PrincipalWF}

\[ \frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash A \not\not \text{ type}} \] \hspace{1cm} \text{NonPrincipalWF}

\[ \Gamma \vdash \vec{A} \text{ [p] types} \] Under context \( \Gamma \), types in \( \vec{A} \) are well-formed [with principality \( p \)]

\[ \frac{\text{for all } A \in \vec{A}. \ \Gamma \vdash A \text{ type}}{\Gamma \vdash \vec{A} \text{ types}} \] \hspace{1cm} \text{TypevecWF}

\[ \frac{\text{for all } A \in \vec{A}. \ \Gamma \vdash A \text{ p type}}{\Gamma \vdash \vec{A} \text{ p types}} \] \hspace{1cm} \text{PrincipalTypevecWF}

\[ \Gamma \text{ ctx} \] Algorithmic context \( \Gamma \) is well-formed

\[ \frac{\chi \notin \text{dom}(\Gamma)}{\text{EmptyCtx}} \hspace{1cm} \frac{\chi \notin \text{dom}(\Gamma)}{\text{HypCtx}} \hspace{1cm} \frac{\chi \notin \text{dom}(\Gamma)}{\text{HypCtx}} \frac{\text{FEV}(\Gamma) = \emptyset}{\text{HypCtx}} \]

\[ \frac{\mu \notin \text{dom}(\Gamma)}{\text{VarCtx}} \hspace{1cm} \frac{\chi \notin \text{dom}(\Gamma)}{\text{SolvedCtx}} \hspace{1cm} \frac{\chi \notin \text{dom}(\Gamma)}{\text{SolvedCtx}} \frac{\mu \notin \Gamma}{\text{MarkerCtx}} \]

Figure 17: Well-formedness of types and contexts in the algorithmic system.
\[ \Gamma \vdash \mathbf{P} \mathbf{true} \vdash \Delta \] Under context \( \Gamma \), check \( \mathbf{P} \), with output context \( \Delta \)

\[
\begin{align*}
\Gamma &\vdash t_1 = t_2 : \mathbb{N} \vdash \Delta & \text{CheckpropEq} \\
\Gamma &\vdash t_1 = t_2 \mathbf{true} \vdash \Delta
\end{align*}
\]

\[ \Gamma \vdash \Delta \perp \] Incorporate hypothesis \( \mathbf{P} \) into \( \Gamma \), producing \( \Delta \) or inconsistency \( \perp \)

\[
\begin{align*}
\Gamma &\vdash t_1 \bowtie t_2 : \mathbb{N} \vdash \Delta \perp & \text{ElimpropEq} \\
\Gamma &\vdash t_1 = t_2 \mathbf{true} \vdash \Delta \perp
\end{align*}
\]

Figure 18: Checking and assuming propositions

\[ \Gamma \vdash t_1 \equiv t_2 : \kappa \vdash \Delta \] Check that \( t_1 \) equals \( t_2 \), taking \( \Gamma \) to \( \Delta \)

\[
\begin{align*}
\Gamma &\vdash u \equiv u : \kappa \vdash \Gamma & \text{CheckeqVar} \\
\Gamma &\vdash 1 \equiv 1 : \ast \vdash \Gamma & \text{CheckeqUnit} \\
\Gamma &\vdash \tau_1 \equiv \tau_1' : \ast \vdash \Theta & \Theta \vdash |\Theta| \tau_2 \equiv |\Theta| \tau_2' : \ast \vdash \Delta & \text{CheckeqBin} \\
\Gamma &\vdash (\tau_1 \oplus \tau_2) \equiv (\tau_1' \oplus \tau_2') : \ast \vdash \Delta
\end{align*}
\]

\[
\begin{align*}
\Gamma &\vdash \text{zero} \equiv \text{zero} : \mathbb{N} \vdash \Gamma & \text{CheckeqZero} \\
\Gamma &\vdash \text{succ}(t_1) \equiv \text{succ}(t_2) : \mathbb{N} \vdash \Gamma & \text{CheckeqSucc} \\
\Gamma &\vdash \hat{\alpha} : \kappa \vdash \hat{\alpha} \equiv t : \kappa \vdash \Delta & \text{CheckeqInstL} \\
\Gamma &\vdash \hat{\alpha} : \kappa \vdash \hat{\alpha} \equiv t : \kappa \vdash \Delta & \text{CheckeqInstR}
\end{align*}
\]

Figure 19: Checking equations

\[ t_1 \neq t_2 \] \( t_1 \) and \( t_2 \) have incompatible head constructors

\[
\begin{align*}
\text{zero} &\neq \text{succ}(t) \\
\text{succ}(t) &\neq \text{zero} \\
1 &\neq (\tau_1 \oplus \tau_2) \\
(\tau_1 \oplus \tau_2) &\neq 1 \\
\oplus_1 &\neq \oplus_2 \\
(\sigma_1 \oplus_1 \tau_1) &\neq (\sigma_2 \oplus_2 \tau_2)
\end{align*}
\]

Figure 20: Head constructor clash
Figure 21: Eliminating equations
\begin{proof}
\begin{align*}
\Gamma \vdash A <:^P B \vdash \Delta & \quad \text{Under input context } \Gamma, \text{ type } A \text{ is a subtype of } B, \text{ with output context } \Delta \\
A \text{ not headed by } \forall/\exists & \quad \frac{\Gamma \vdash A \equiv B \vdash \Delta}{\Gamma \vdash A <:^E \vdash \Delta} \quad \text{\textless :Equiv} \\
B \text{ not headed by } \forall & \quad \frac{\Gamma, \alpha : \kappa \vdash \alpha \colon B \vdash \Delta, \alpha : \kappa, \Theta}{\Gamma \vdash \forall \alpha : \kappa \vdash B \vdash \Delta} \quad \text{\textless :\forallL} \\
& \quad \frac{\Gamma \vdash \forall \alpha : \kappa \vdash A \vdash B \vdash \Delta, \alpha : \kappa, \Theta}{\Gamma \vdash A \vdash B \vdash \Delta} \quad \text{\textless :\forallR} \\
A \text{ not headed by } \exists & \quad \frac{\Gamma, \beta : \kappa \vdash \beta \vdash B \vdash \Delta, \beta : \kappa, \Theta}{\Gamma \vdash \exists \beta : \kappa \vdash B \vdash \Delta} \quad \text{\textless :\existsL} \\
& \quad \frac{\Gamma \vdash \exists \beta : \kappa \vdash A \vdash B \vdash \Delta, \beta : \kappa, \Theta}{\Gamma \vdash A \vdash B \vdash \Delta} \quad \text{\textless :\existsR} \\
\Gamma \vdash \neg (A) & \quad \frac{\Gamma \vdash A \vdash \Delta}{\Gamma \vdash \neg (A) \vdash \Delta} \quad \text{\textless :\neg} \\
\Gamma \vdash \varphi (B) & \quad \frac{\Gamma \vdash A \vdash \Delta}{\Gamma \vdash \varphi (B) \vdash \Delta} \quad \text{\textless :\varphi} \\
\Gamma \vdash \varphi (A) & \quad \frac{\Gamma \vdash A \vdash \Delta}{\Gamma \vdash \varphi (A) \vdash \Delta} \quad \text{\textless :\varphi} \\
\Gamma \vdash 1 = 1 \vdash \Gamma & \quad \text{\textless :\equiv} \\
\Gamma \vdash \alpha = \alpha \vdash \Gamma & \quad \text{\textless :\equivVar} \\
\Gamma \vdash \alpha \equiv \alpha \vdash \Gamma & \quad \text{\textless :\equivExvar} \\
\Gamma \vdash (A_1 \oplus A_2) = (B_1 \oplus B_2) \vdash \Delta & \quad \text{\textless :\equivPropEq} \\
\Gamma \vdash (\exists \alpha : \kappa \vdash A) = (\exists \alpha : \kappa \vdash B) \vdash \Delta & \quad \text{\textless :\equivVec} \\
\Gamma \vdash (A \land P) = (B \land Q) \vdash \Delta & \quad \text{\textless :\equivL} \\
\Gamma \vdash (A \lor P) \equiv (B \lor Q) \vdash \Delta & \quad \text{\textless :\equivR} \\
\hat{\alpha} \notin \text{FV}(\tau) & \quad \frac{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} \equiv \tau \vdash \Delta}{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} \equiv \tau \vdash \Delta} \quad \text{\textless :\equiv\InstantiationL} \\
\hat{\alpha} \notin \text{FV}(\tau) & \quad \frac{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} \equiv \tau \vdash \Delta}{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} \equiv \tau \vdash \Delta} \quad \text{\textless :\equiv\InstantiationR} \\
\end{align*}
\end{proof}

Figure 22: Algorithmic subtyping and equivalence
Γ ⊢ \hat{\alpha} := t : \kappa \vdash_{\Delta}

Under input context Γ, instantiate \hat{\alpha} such that \hat{\alpha} = t with output context Δ

\[\Gamma_0 \vdash \tau : \kappa\]
\[\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \vdash \hat{\alpha} := \tau : \kappa \vdash \Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1\]

\[\hat{\beta} \in \text{unsolved}(\Gamma[\hat{\alpha} : \kappa][\hat{\beta} : \kappa])\]
\[\Gamma[\hat{\alpha} : \kappa][\hat{\beta} : \kappa] \vdash \hat{\alpha} := \hat{\beta} : \kappa \vdash \Gamma[\hat{\alpha} : \kappa][\hat{\beta} : \kappa = \hat{\alpha}]\]

\[\Gamma[\hat{\alpha}_2 : \ast, \hat{\alpha}_1 : \ast = \hat{\alpha}_1 \oplus \hat{\alpha}_2] \vdash \hat{\alpha}_1 := \tau_1 : \ast \vdash \Theta \vdash \hat{\alpha}_2 := [\Theta] \tau_2 : \ast \vdash \Delta\]

\[\Gamma[\hat{\alpha}_2] \vdash \hat{\alpha} := \tau_1 \oplus \tau_2 : \ast \vdash \Delta\]

\[\Gamma[\hat{\alpha}] : \mathbb{N} \vdash \hat{\alpha} := \text{zero} : \mathbb{N} \vdash \Gamma[\hat{\alpha} : \mathbb{N} = \text{zero}]\]

\[\Gamma[\hat{\alpha}] : \mathbb{N} \vdash \hat{\alpha} := \text{succ}(t_1) : \mathbb{N} \vdash \Delta\]

Figure 23: Instantiation
Under context $\Gamma$, incorporate proposition $P$ while checking branches $\Pi$ with patterns of type $\vec{A}$ and bodies of type $C$ 

$$\Gamma \vdash \Pi :: \vec{A} q \iff C p \vdash \Delta$$

Figure 24: Algorithmic pattern matching
\[ \Gamma \vdash \Pi \text{ covers } \vec{A}, q \]

Under context \( \Gamma \), patterns \( \Pi \) cover the types \( \vec{A} \)

\[ \Gamma / \mathcal{P} \vdash \Pi \text{ covers } \vec{A}, ! \]

Under context \( \Gamma \), patterns \( \Pi \) cover the types \( \vec{A} \) assuming \( \mathcal{P} \)

Pattern list \( \Pi \) contains a list pattern constructor at the head position

\[ \Pi \text{ guarded} \]

\[ \Gamma \vdash (\rightarrow e_1) \cdot \Pi \text{ covers } q \quad \text{CoversEmpty} \]

\[ \Pi \vdash \Pi' \text{ covers } \vec{A}, q \quad \frac{\Pi \vdash \Pi' \text{ covers } \vec{A}, q}{\Gamma \vdash \Pi \text{ covers } \vec{A}, q} \quad \text{CoversVar} \]

\[ \frac{\Pi \vdash \Pi' \text{ covers } \vec{A}, q}{\Gamma \vdash \Pi \text{ covers } 1, \vec{A}, q} \quad \text{Covers1} \]

\[ \frac{\Pi \vdash \Pi' \text{ covers } \vec{A}, q}{\Gamma \vdash \Pi \text{ covers } (A_1 \times A_2), \vec{A}, q} \quad \text{Covers\times} \]

\[ \frac{\Pi \vdash \Pi_1 \text{ covers } A_1, \vec{A}, q \quad \Pi \vdash \Pi_2 \text{ covers } A_2, \vec{A}, q}{\Gamma \vdash \Pi \text{ covers } (A_1 + A_2), \vec{A}, q} \quad \text{Covers+} \]

\[ \frac{\Gamma, \alpha : \kappa \vdash \Pi \text{ covers } \vec{A}, q}{\Gamma \vdash \Pi \text{ covers } (\exists \alpha : \kappa. A), \vec{A}, q} \quad \text{Covers\exists} \]

\[ \frac{\Pi \vdash \Pi_1 \text{ covers } A_1, \vec{A}, q}{\Gamma \vdash \Pi \text{ covers } (A_0 \land (t_1 = t_2)), \vec{A}, !} \quad \text{Covers\land} \]

\[ \frac{\Gamma \vdash \Pi \text{ covers } (A_0 \land (t_1 = t_2)), \vec{A}, !}{\Gamma / t = \text{zero} \vdash \Pi \text{ covers } \vec{A}, !} \quad \text{CoversVec} \]

\[ \frac{\Gamma, n : \mathbb{N} \vdash \Pi \text{ covers } (A, \text{Vec } n \cdot A, \vec{A}), !}{\Gamma \vdash \Pi \text{ covers } \text{Vec } t \cdot A, \vec{A}, !} \quad \text{CoversVec\_n} \]

\[ \frac{\Pi \vdash \Pi \text{ covers } \vec{A}, !}{\Gamma \vdash \Pi \text{ covers } \text{Vec } t \cdot A, \vec{A}, !} \quad \text{CoversVec\_t} \]

\[ \frac{\Gamma / [\Delta] t_1 \equiv [\Delta] t_2 : \kappa \vdash \Delta \vdash [\Delta] \Pi \text{ covers } [\Delta] \vec{A}, q}{\Gamma / t_1 = t_2 \vdash \Pi \text{ covers } \vec{A}, !} \quad \text{CoversEq} \]

\[ \frac{\Gamma / [\Delta] t_1 \equiv [\Delta] t_2 : \kappa \vdash \Delta \vdash [\Delta] \Pi \text{ covers } [\Delta] \vec{A}, q}{\Gamma / t_1 = t_2 \vdash \Pi \text{ covers } \vec{A}, !} \quad \text{CoversEq\_Bot} \]

\[ \varnothing, \vec{p} \vdash e \quad \Pi \text{ guarded} \]

\[ \frac{\vec{p} : \vec{p'}, \vec{p} \vdash e \quad \Pi \text{ guarded}}{\vec{p} : \vec{p'}, \vec{p} \vdash e \quad \Pi \text{ guarded}} \]

\[ x, \vec{p} \vdash e \quad \Pi \text{ guarded} \]

Figure 25: Algorithmic match coverage
## 2 List of Judgments

For convenience, we list all the judgment forms:

<table>
<thead>
<tr>
<th>Judgment</th>
<th>Description</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Psi \vdash t : \kappa)</td>
<td>Index term/monotype is well-formed</td>
<td>Figure 16</td>
</tr>
<tr>
<td>(\Psi \vdash P \text{ prop})</td>
<td>Proposition is well-formed</td>
<td>Figure 16</td>
</tr>
<tr>
<td>(\Psi \vdash A \text{ type})</td>
<td>Type is well-formed</td>
<td>Figure 16</td>
</tr>
<tr>
<td>(\Psi \vdash \vec{A} \text{ types})</td>
<td>Type vector is well-formed</td>
<td>Figure 16</td>
</tr>
<tr>
<td>(\Psi \text{ ctx})</td>
<td>Declarative context is well-formed</td>
<td>Figure 16</td>
</tr>
<tr>
<td>(\Psi \vdash A \leq^P B)</td>
<td>Declarative subtyping</td>
<td>Figure 4</td>
</tr>
<tr>
<td>(\Psi \vdash P \text{ true})</td>
<td>Declarative truth</td>
<td>Figure 6</td>
</tr>
<tr>
<td>(\Psi \vdash e \iff A\ p)</td>
<td>Declarative checking</td>
<td>Figure 6</td>
</tr>
<tr>
<td>(\Psi \vdash e \implies A\ p)</td>
<td>Declarative synthesis</td>
<td>Figure 6</td>
</tr>
<tr>
<td>(\Psi \vdash s : A\ p \gg C\ q)</td>
<td>Declarative spine typing</td>
<td>Figure 6</td>
</tr>
<tr>
<td>(\Psi \vdash s : A\ p \gg C\ q[\ ])</td>
<td>Declarative spine typing, recovering principality</td>
<td>Figure 6</td>
</tr>
<tr>
<td>(\Psi \vdash \Pi \Downarrow \vec{A}!)</td>
<td>Declarative pattern matching</td>
<td>Figure 7</td>
</tr>
<tr>
<td>(\Psi / P \vdash \Pi \Downarrow \vec{A}!)</td>
<td>Declarative proposition assumption</td>
<td>Figure 7</td>
</tr>
<tr>
<td>(\Psi \vdash \Pi \text{ covers } \vec{A}!)</td>
<td>Declarative match coverage</td>
<td>Figure 8</td>
</tr>
<tr>
<td>(\Gamma \vdash \tau : \kappa)</td>
<td>Index term/monotype is well-formed</td>
<td>Figure 17</td>
</tr>
<tr>
<td>(\Gamma \vdash P \text{ prop})</td>
<td>Proposition is well-formed</td>
<td>Figure 17</td>
</tr>
<tr>
<td>(\Gamma \vdash A \text{ type})</td>
<td>Polytype is well-formed</td>
<td>Figure 17</td>
</tr>
<tr>
<td>(\Gamma \text{ ctx})</td>
<td>Algorithmic context is well-formed</td>
<td>Figure 17</td>
</tr>
<tr>
<td>([\Gamma]\ A\ ))</td>
<td>Applying a context, as a substitution, to a type</td>
<td>Figure 12</td>
</tr>
<tr>
<td>(\Gamma \vdash P \text{ true } \dashv \Delta)</td>
<td>Check proposition</td>
<td>Figure 18</td>
</tr>
<tr>
<td>(\Gamma / P \vdash \Delta\downarrow)</td>
<td>Assume proposition</td>
<td>Figure 18</td>
</tr>
<tr>
<td>(\Gamma \vdash s \equiv t : \kappa \dashv \Delta)</td>
<td>Check equation</td>
<td>Figure 19</td>
</tr>
<tr>
<td>(s # t)</td>
<td>Head constructors clash</td>
<td>Figure 20</td>
</tr>
<tr>
<td>(\Gamma / s \vdash t : \kappa \dashv \Delta\downarrow)</td>
<td>Assume/eliminate equation</td>
<td>Figure 21</td>
</tr>
<tr>
<td>(\Gamma \vdash A \lessdot^P B \dashv \Delta)</td>
<td>Algorithmic subtyping</td>
<td>Figure 22</td>
</tr>
<tr>
<td>(\Gamma / P \vdash A \lessdot^P B \dashv \Delta)</td>
<td>Assume/eliminate proposition</td>
<td>Figure 22</td>
</tr>
<tr>
<td>(\Gamma \vdash P \equiv Q \dashv \Delta)</td>
<td>Equivalence of propositions</td>
<td>Figure 22</td>
</tr>
<tr>
<td>(\Gamma \vdash A \equiv B \dashv \Delta)</td>
<td>Equivalence of types</td>
<td>Figure 22</td>
</tr>
<tr>
<td>(\Gamma \vdash \alpha := t : \kappa \dashv \Delta)</td>
<td>Instantiate</td>
<td>Figure 23</td>
</tr>
<tr>
<td>(e \text{ chk-I})</td>
<td>Checking intro form</td>
<td>Figure 5</td>
</tr>
<tr>
<td>(\Gamma \vdash e \equiv A\ p \dashv \Delta)</td>
<td>Algorithmic checking</td>
<td>Figure 14</td>
</tr>
<tr>
<td>(\Gamma \vdash e \implies A\ p \dashv \Delta)</td>
<td>Algorithmic synthesis</td>
<td>Figure 14</td>
</tr>
<tr>
<td>(\Gamma \vdash s : A\ p \gg C\ q \dashv \Delta)</td>
<td>Algorithmic spine typing</td>
<td>Figure 14</td>
</tr>
<tr>
<td>(\Gamma \vdash s : A\ p \gg C\ q[\ ] \dashv \Delta)</td>
<td>Algorithmic spine typing, recovering principality</td>
<td>Figure 14</td>
</tr>
<tr>
<td>(\Gamma \vdash \Pi \Downarrow \vec{A} q \iff C\ p \dashv \Delta)</td>
<td>Algorithmic pattern matching</td>
<td>Figure 24</td>
</tr>
<tr>
<td>(\Gamma / P \vdash \Pi \Downarrow \vec{A}! \iff C\ p \dashv \Delta)</td>
<td>Algorithmic pattern matching (assumption)</td>
<td>Figure 24</td>
</tr>
<tr>
<td>(\Gamma \dashv \Pi \text{ covers } \vec{A} q)</td>
<td>Algorithmic match coverage</td>
<td>Figure 25</td>
</tr>
<tr>
<td>(\Gamma \dashv \Delta)</td>
<td>Context extension</td>
<td>Figure 15</td>
</tr>
<tr>
<td>([\Omega] \Gamma)</td>
<td>Apply complete context</td>
<td>Figure 13</td>
</tr>
</tbody>
</table>
A Properties of the Declarative System

Lemma 1 (Declarative Well-foundedness). Go to proof
The inductive definition of the following judgments is well-founded:

(i) synthesis \( \Psi \vdash e \Rightarrow B \ p \)
(ii) checking \( \Psi \vdash e \Leftarrow A \ p \)
(iii) checking, equality elimination \( \Psi / P \vdash e \Leftarrow C \ p \)
(iv) ordinary spine \( \Psi \vdash s : A \ p \gg B \ q \)
(v) recovery spine \( \Psi \vdash s : A \ p \gg B \ [q] \)
(vi) pattern matching \( \Psi \vdash \Pi :: \vec{A} ! \Leftarrow C \ p \)
(vii) pattern matching, equality elimination \( \Psi / P \vdash \Pi :: \vec{A} ! \Leftarrow C \ p \)

Lemma 2 (Declarative Weakening). Go to proof

(i) If \( \Psi_0, \Psi_1 \vdash t : \kappa \) then \( \Psi_0, \alpha : \kappa, \Psi_1 \vdash t : \kappa \).
(ii) If \( \Psi_0, \Psi_1 \vdash P \) prop then \( \Psi_0, \Psi, \Psi_1 \vdash P \) prop.
(iii) If \( \Psi_0, \Psi_1 \vdash P \) true then \( \Psi_0, \Psi, \Psi_1 \vdash P \) true.
(iv) If \( \Psi_0, \Psi_1 \vdash A \) type then \( \Psi_0, \Psi, \Psi_1 \vdash A \) type.

Lemma 3 (Declarative Term Substitution). Go to proof
Suppose \( \Psi \vdash t : \kappa \). Then:

1. If \( \Psi_0, \alpha : \kappa, \Psi_1 \vdash t' : \kappa \) then \( \Psi_0, [t/\alpha] \Psi_1 \vdash [t/\alpha] t' : \kappa \).
2. If \( \Psi_0, \alpha : \kappa, \Psi_1 \vdash P \) prop then \( \Psi_0, [t/\alpha] \Psi_1 \vdash [t/\alpha] P \) prop.
3. If \( \Psi_0, \alpha : \kappa, \Psi_1 \vdash A \) type then \( \Psi_0, [t/\alpha] \Psi_1 \vdash [t/\alpha] A \) type.
4. If \( \Psi_0, \alpha : \kappa, \Psi_1 \vdash A \leq^P B \) then \( \Psi_0, [t/\alpha] \Psi_1 \vdash [t/\alpha] A \leq^P [t/\alpha] B \).
5. If \( \Psi_0, \alpha : \kappa, \Psi_1 \vdash P \) true then \( \Psi_0, [t/\alpha] \Psi_1 \vdash [t/\alpha] P \) true.

Lemma 4 (Reflexivity of Declarative Subtyping). Go to proof
Given \( \Psi \vdash A \) type, we have that \( \Psi \vdash A \leq^P A \).

Lemma 5 (Subtyping Inversion). Go to proof

- If \( \Psi \vdash \exists \alpha : \kappa. A \leq^+ B \) then \( \Psi, \alpha : \kappa \vdash A \leq^+ B \).
- If \( \Psi \vdash A \leq^\neg \forall \beta : \kappa. B \) then \( \Psi, \beta : \kappa \vdash A \leq^\neg B \).

Lemma 6 (Subtyping Polarity Flip). Go to proof

- If nonpos\( (A) \) and nonpos\( (B) \) and \( \Psi \vdash A \leq^+ B \) then \( \Psi \vdash A \leq^\neg B \) by a derivation of the same or smaller size.
- If nonneg\( (A) \) and nonneg\( (B) \) and \( \Psi \vdash A \leq^\neg B \) then \( \Psi \vdash A \leq^+ B \) by a derivation of the same or smaller size.
- If nonpos\( (A) \) and nonneg\( (A) \) and nonpos\( (B) \) and nonneg\( (B) \) and \( \Psi \vdash A \leq^P B \) then \( A = B \).
Lemma 7 (Transitivity of Declarative Subtyping). 

Given $\Psi \vdash A$ type and $\Psi \vdash B$ type and $\Psi \vdash C$ type:

(i) If $D_1 : \Psi \vdash A \leq^P B$ and $D_2 : \Psi \vdash B \leq^P C$
then $\Psi \vdash A \leq^P C$.

Property 1. We assume that all types mentioned in annotations in expressions have no free existential variables. By the grammar, it follows that all expressions have no free existential variables, that is, FEV(e) = $\emptyset$.

B Substitution and Well-formedness Properties

Definition 1 (Softness). A context $\Theta$ is soft iff it consists only of $\alpha : \kappa$ and $\check{\alpha} : \kappa = \tau$ declarations.

Lemma 8 (Substitution—Well-formedness). 

(i) If $\Gamma \vdash A p$ type and $\Gamma \vdash \tau p$ type then $\Gamma \vdash [\tau/\alpha]A p$ type.

(ii) If $\Gamma \vdash P$ prop and $\Gamma \vdash \tau p$ type then $\Gamma \vdash [\tau/\alpha]P$ prop.
Moreover, if $p = !$ and FEV(|$\Gamma$|P) = $\emptyset$ then FEV(|$\Gamma$|[$\tau/\alpha$]|P) = $\emptyset$.

Lemma 9 (Uvar Preservation). 

If $\Delta \longrightarrow \Omega$ then:

(i) If $(\alpha : \kappa) \in \Omega$ then $(\alpha : \kappa) \in |\Omega|\Delta$.

(ii) If $(\alpha : A p) \in \Omega$ then $(\alpha : |\Omega|A p) \in |\Omega|\Delta$.

Lemma 10 (Sorting Implies Typing). 

If $\Gamma \vdash t : *$ then $\Gamma \vdash t$ type.

Lemma 11 (Right-Hand Substitution for Sorting). 

If $\Gamma \vdash t : \kappa$ then $\Gamma \vdash [\Gamma]t : \kappa$.

Lemma 12 (Right-Hand Substitution for Propositions). 

If $\Gamma \vdash P$ prop then $\Gamma \vdash [\Gamma]P$ prop.

Lemma 13 (Right-Hand Substitution for Typing). 

If $\Gamma \vdash A$ type then $\Gamma \vdash [\Gamma]A$ type.

Lemma 14 (Substitution for Sorting). 

If $\Omega \vdash t : \kappa$ then $|\Omega|\Omega \vdash |\Omega|t : \kappa$.

Lemma 15 (Substitution for Prop Well-Formedness). 

If $\Omega \vdash P$ prop then $|\Omega|\Omega \vdash |\Omega|P$ prop.

Lemma 16 (Substitution for Type Well-Formedness). 

If $\Omega \vdash A$ type then $|\Omega|\Omega \vdash |\Omega|A$ type.

Lemma 17 (Substitution Stability). 

If $(\Omega, \Omega_Z)$ is well-formed and $\Omega_Z$ is soft and $\Omega \vdash A$ type then $|\Omega|A = |\Omega, \Omega_Z|A$.

Lemma 18 (Equal Domains). 

If $\Omega_1 \vdash A$ type and $\text{dom}(\Omega_1) = \text{dom}(\Omega_2)$ then $\Omega_2 \vdash A$ type.

C Properties of Extension

Lemma 19 (Declaration Preservation). 

If $\Gamma \longrightarrow \Delta$ and $u$ is declared in $\Gamma$, then $u$ is declared in $\Delta$.

Lemma 20 (Declaration Order Preservation). 

If $\Gamma \longrightarrow \Delta$ and $u$ is declared to the left of $v$ in $\Gamma$, then $u$ is declared to the left of $v$ in $\Delta$.

Lemma 21 (Reverse Declaration Order Preservation). 

If $\Gamma \longrightarrow \Delta$ and $u$ and $v$ are both declared in $\Gamma$ and $u$ is declared to the left of $v$ in $\Delta$, then $u$ is declared to the left of $v$ in $\Gamma$.

An older paper had a lemma
“Substitution Extension Invariance”
If \( \Theta \vdash A \) type and \( \Theta \rightarrow \Gamma \) then \( \Gamma \vdash A \Longleftrightarrow (\Theta \vdash A) \) and \( \Gamma \vdash A = (\Theta \vdash A) \).

For the second part, \( \Gamma \vdash A = (\Theta \vdash A) \), use Lemma 29 (Substitution Monotonicity) (i) or (iii) instead. The first part \( \Gamma \vdash A = (\Theta \vdash A) \) hasn’t been proved in this system.

Lemma 22 (Extension Inversion). \( \text{Go to proof} \)

(i) If \( D : \Gamma_0, \alpha : \kappa, \Gamma_1 \rightarrow \Delta \)

then there exist unique \( \Delta_0 \) and \( \Delta_1 \)
such that \( \Delta = (\Delta_0, \alpha : \kappa, \Delta_1) \) and \( D' : \Gamma_0 \rightarrow \Delta_0 \) where \( D' \prec D \).

Moreover, if \( \Gamma_1 \) is soft, then \( \Delta_1 \) is soft.

(ii) If \( D : \Gamma_0, \llcorner, \Gamma_1 \rightarrow \Delta \)

then there exist unique \( \Delta_0 \) and \( \Delta_1 \)
such that \( \Delta = (\Delta_0, \llcorner, \Delta_1) \) and \( D' : \Gamma_0 \rightarrow \Delta_0 \) where \( D' \prec D \).

Moreover, if \( \Gamma_1 \) is soft, then \( \Delta_1 \) is soft.

Moreover, if \( \text{dom}(\Gamma_0, \llcorner, \Gamma_1) = \text{dom}(\Delta) \) then \( \text{dom}(\Gamma_0) = \text{dom}(\Delta_0) \).

(iii) If \( D : \Gamma_0, \alpha = \tau, \Gamma_1 \rightarrow \Delta \)

then there exist unique \( \Delta_0, \tau', \) and \( \Delta_1 \)
such that \( \Delta = (\Delta_0, \alpha = \tau', \Delta_1) \) and \( D' : \Gamma_0 \rightarrow \Delta_0 \) and \( |\Delta_0|\tau = |\Delta_0|\tau' \) where \( D' \prec D \).

(iv) If \( D : \Gamma_0, \delta : \kappa = \tau, \Gamma_1 \rightarrow \Delta \)

then there exist unique \( \Delta_0, \tau', \) and \( \Delta_1 \)
such that \( \Delta = (\Delta_0, \delta : \kappa = \tau', \Delta_1) \) and \( D' : \Gamma_0 \rightarrow \Delta_0 \) and \( |\Delta_0|\tau = |\Delta_0|\tau' \) where \( D' \prec D \).

(v) If \( D : \Gamma_0, \chi : A, \Gamma_1 \rightarrow \Delta \)

then there exist unique \( \Delta_0, A', \) and \( \Delta_1 \)
such that \( \Delta = (\Delta_0, \chi : A', \Delta_1) \) and \( D' : \Gamma_0 \rightarrow \Delta_0 \) and \( |\Delta_0|A = |\Delta_0|A' \) where \( D' \prec D \).

Moreover, if \( \Gamma_1 \) is soft, then \( \Delta_1 \) is soft.

Moreover, if \( \text{dom}(\Gamma_0, \chi : A, \Gamma_1) = \text{dom}(\Delta) \) then \( \text{dom}(\Gamma_0) = \text{dom}(\Delta_0) \).

(vi) If \( D : \Gamma_0, \delta : \kappa, \Gamma_1 \rightarrow \Delta \) then either

- there exist unique \( \Delta_0, \tau', \) and \( \Delta_1 \)
such that \( \Delta = (\Delta_0, \delta : \kappa = \tau', \Delta_1) \) and \( D' : \Gamma_0 \rightarrow \Delta_0 \) where \( D' \prec D \),
or
- there exist unique \( \Delta_0 \) and \( \Delta_1 \)
such that \( \Delta = (\Delta_0, \delta : \kappa, \Delta_1) \) and \( D' : \Gamma_0 \rightarrow \Delta_0 \) where \( D' \prec D \).

Lemma 23 (Deep Evar Introduction). \( \text{Go to proof} \)

(i) If \( \Gamma_0, \Gamma_1 \) is well-formed and \( \delta \) is not declared in \( \Gamma_0, \Gamma_1 \) then \( \Gamma_0, \Gamma_1 \rightarrow \Gamma_0, \delta : \kappa, \Gamma_1 \).

(ii) If \( \Gamma_0, \delta : \kappa, \Gamma_1 \) is well-formed and \( \Gamma \vdash t : \kappa \) then \( \Gamma_0, \delta : \kappa, \Gamma_1 \rightarrow \Gamma_0, \delta : \kappa = t, \Gamma_1 \).

(iii) If \( \Gamma_0, \Gamma_1 \) is well-formed and \( \Gamma \vdash t : \kappa \) then \( \Gamma_0, \Gamma_1 \rightarrow \Gamma_0, \delta : \kappa = t, \Gamma_1 \).

Lemma 24 (Soft Extension). \( \text{Go to proof} \)
If \( \Gamma \rightarrow \Delta \) and \( \Gamma, \Theta \text{ ctx and } \Theta \) is soft, then there exists \( \Omega \) such that \( \text{dom}(\Theta) = \text{dom}(\Omega) \) and \( \Gamma, \Theta \rightarrow \Delta, \Omega \).

Definition 2 (Filling). The filling of a context \( \Gamma \) solves all unsolved variables:
Lemma 25 (Filling Completes). If $\Gamma \rightarrow \Omega$ and $(\Gamma, \Theta)$ is well-formed, then $\Gamma, \Theta \rightarrow \Omega, |\Theta|$.  

Proof. By induction on $|\cdot|$ and applying the rules for $\rightarrow$.  

Lemma 26 (Parallel Admissibility). If $\Gamma_{L} \rightarrow \Delta_{L}$ and $\Gamma_{R} = \Delta_{R}$ then:  

(i) $\Gamma_{L}, \alpha : \kappa, \Gamma_{R} \rightarrow \Delta_{L}, \alpha : \kappa_{L}$  

(ii) If $\Delta_{L} \vdash \tau' : \kappa$ then $\Gamma_{L}, \alpha : \kappa, \Gamma_{R} \rightarrow \Delta_{L}, \alpha : \kappa = \tau', \Delta_{R}$.  

(iii) If $\Gamma_{L} \vdash \tau : \kappa$ and $\Delta_{L} \vdash \tau'$ type and $|\Delta_{L}|\tau = |\Delta_{L}|\tau'$, then $\Gamma_{L}, \alpha : \kappa = \tau, \Gamma_{R} \rightarrow \Delta_{L}, \alpha : \kappa = \tau', \Delta_{R}$.  

Lemma 27 (Parallel Extension Solution). If $\Gamma_{L}, \alpha : \kappa, \Gamma_{R} \rightarrow \Delta_{L}, \alpha : \kappa = \tau_{0}, \Delta_{R}$ and $\Gamma_{L} \vdash \tau : \kappa$ and $|\Delta_{L}|\tau_{0} = |\Delta_{L}|\tau_{1} = |\Delta_{L}|\tau_{2}$ then $\Gamma_{L}, \alpha : \kappa = \tau_{1}, \Gamma_{R} \rightarrow \Delta_{L}, \alpha : \kappa = \tau_{2}, \Delta_{R}$.  

Lemma 28 (Parallel Variable Update). If $\Gamma_{L}, \alpha : \kappa, \Gamma_{R} \rightarrow \Delta_{L}, \alpha : \kappa = \tau_{0}, \Delta_{R}$ and $\Gamma_{L} \vdash \tau : \kappa$ and $\Delta_{L} \vdash \tau_{2} : \kappa$ and $|\Delta_{L}|\tau_{0} = |\Delta_{L}|\tau_{1} = |\Delta_{L}|\tau_{2}$ then $\Gamma_{L}, \alpha : \kappa = \tau_{1}, \Gamma_{R} \rightarrow \Delta_{L}, \alpha : \kappa = \tau_{2}, \Delta_{R}$.  

Lemma 29 (Substitution Monotonicity).  

(i) If $\Gamma \rightarrow \Delta$ and $\Gamma \vdash t : \kappa$ then $|\Delta|\Gamma|t = |\Delta|t$.

(ii) If $\Gamma \rightarrow \Delta$ and $\Gamma \vdash P$ prop then $|\Delta|\Gamma|P = |\Delta|P$.

(iii) If $\Gamma \rightarrow \Delta$ and $\Gamma \vdash A$ type then $|\Delta|\Gamma|A = |\Delta|A$.  

Lemma 30 (Substitution Invariance).  

(i) If $\Gamma \rightarrow \Delta$ and $\Gamma \vdash t : \kappa$ and $\text{FEV}(|\Gamma|t) = \emptyset$ then $|\Delta|\Gamma|t = |\Gamma|t$.

(ii) If $\Gamma \rightarrow \Delta$ and $\Gamma \vdash P$ prop and $\text{FEV}(|\Gamma|P) = \emptyset$ then $|\Delta|\Gamma|P = |\Gamma|P$.

(iii) If $\Gamma \rightarrow \Delta$ and $\Gamma \vdash A$ type and $\text{FEV}(|\Gamma|A) = \emptyset$ then $|\Delta|\Gamma|A = |\Gamma|A$.  

Definition 3 (Canonical Contexts). A (complete) context $\Omega$ is canonical iff, for all $(\alpha : \kappa = t)$ and $(\alpha = t) \in \Omega$, the solution $t$ is ground $(\text{FEV}(t) = \emptyset)$.  

Lemma 31 (Split Extension). If $\Delta \rightarrow \Omega$ and $\alpha \in \text{unsolved}(\Delta)$ and $\Omega = \Omega_{1}[\alpha : \kappa = t_{1}]$ and $\Omega$ is canonical (Definition 3) and $\Omega \vdash t_{2} : \kappa$ then $\Delta \rightarrow \Omega_{1}[\alpha : \kappa = t_{2}]$.  

November 13, 2018
C.1 Reflexivity and Transitivity

Lemma 32 (Extension Reflexivity). Go to proof
If $\Gamma \vdash\text{ctx}$ then $\Gamma \rightarrow \Gamma$.

Lemma 33 (Extension Transitivity). Go to proof
If $D :: \Gamma \rightarrow \Theta$ and $D' :: \Theta \rightarrow \Delta$ then $\Gamma \rightarrow \Delta$.

C.2 Weakening

The “suffix weakening” lemmas take a judgment under $\Gamma$ and produce a judgment under $(\Gamma, \Theta)$. They do not require $\Gamma \rightarrow \rightarrow \Gamma, \Theta$.

Lemma 34 (Suffix Weakening). Go to proof
If $\Gamma \vdash t : \kappa$ then $\Gamma, \Theta \vdash t : \kappa$.

Lemma 35 (Suffix Weakening). Go to proof
If $\Gamma \vdash A \text{ type}$ then $\Gamma, \Theta \vdash A \text{ type}$.

The following proposed lemma is false.

“Extension Weakening (Truth)”
If $\Gamma \vdash P \text{ true} \rightarrow \Delta$ and $\Gamma \rightarrow \rightarrow \Gamma'$ then there exists $\Delta'$ such that $\Delta \rightarrow \Delta'$ and $\Gamma' \vdash P \text{ true} \rightarrow \Delta'$.

Counterexample: Suppose $\hat{\alpha} \vdash \hat{\alpha} = 1 \text{ true} \rightarrow \hat{\alpha} = 1$ and $\hat{\alpha} \rightarrow (\hat{\alpha} = (1 \rightarrow 1))$. Then there does not exist such a $\Delta'$.

Lemma 36 (Extension Weakening (Sorts)). Go to proof
If $\Gamma \vdash t : \kappa$ and $\Gamma \rightarrow \rightarrow \Delta$ then $\Delta \vdash t : \kappa$.

Lemma 37 (Extension Weakening (Props)). Go to proof
If $\Gamma \vdash P \text{ prop}$ and $\Gamma \rightarrow \rightarrow \Delta$ then $\Delta \vdash P \text{ prop}$.

Lemma 38 (Extension Weakening (Types)). Go to proof
If $\Gamma \vdash A \text{ type}$ and $\Gamma \rightarrow \rightarrow \Delta$ then $\Delta \vdash A \text{ type}$.

C.3 Principal Typing Properties

Lemma 39 (Principal Agreement). Go to proof
(i) If $\Gamma \vdash A \text{ ! type}$ and $\Gamma \rightarrow \rightarrow \Delta$ then $\Delta A = \Gamma A$.
(ii) If $\Gamma \vdash P \text{ prop}$ and $\text{FEV}(P) = \emptyset$ and $\Gamma \rightarrow \rightarrow \Delta$ then $\Delta P = \Gamma P$.

Lemma 40 (Right-Hand Subst. for Principal Typing). Go to proof
If $\Gamma \vdash A \text{ p type}$ then $\Gamma \vdash [\Gamma] A \text{ p type}$.

Lemma 41 (Extension Weakening for Principal Typing). Go to proof
If $\Gamma \vdash A \text{ p type}$ and $\Gamma \rightarrow \rightarrow \Delta$ then $\Delta \vdash A \text{ p type}$.

Lemma 42 (Inversion of Principal Typing). Go to proof
(1) If $\Gamma \vdash (A \rightarrow B) \text{ p type}$ then $\Gamma \vdash A \text{ p type}$ and $\Gamma \vdash B \text{ p type}$.
(2) If $\Gamma \vdash (P \supset A) \text{ p type}$ then $\Gamma \vdash P \text{ prop}$ and $\Gamma \vdash A \text{ p type}$.
(3) If $\Gamma \vdash (A \land P) \text{ p type}$ then $\Gamma \vdash P \text{ prop}$ and $\Gamma \vdash A \text{ p type}$.

C.4 Instantiation Extends

Lemma 43 (Instantiation Extension). Go to proof
If $\Gamma \vdash \Delta := \tau : \kappa \rightarrow \Delta$ then $\Gamma \rightarrow \Delta$. 

November 13, 2018
C.5 Equivalence Extends

**Lemma 44** (Elimeq Extension). If \( \Gamma \vdash s \Rightarrow t : \kappa \vdash \Delta \) then there exists \( \Theta \) such that \( \Gamma, \Theta \rightarrow \Delta \).

**Lemma 45** (Elimprop Extension). If \( \Gamma \vdash P \Rightarrow \Delta \) then there exists \( \Theta \) such that \( \Gamma, \Theta \rightarrow \Delta \).

**Lemma 46** (Checkeq Extension). If \( \Gamma \vdash A \equiv B \vdash \Delta \) then \( \Gamma \rightarrow \Delta \).

**Lemma 47** (Checkprop Extension). If \( \Gamma \vdash P \equiv Q \vdash \Delta \) then \( \Gamma \rightarrow \Delta \).

**Lemma 48** (Prop Equivalence Extension). If \( \Gamma \vdash A \equiv B \vdash \Delta \) then \( \Gamma \rightarrow \Delta \).

**Lemma 49** (Equivalence Extension). If \( \Gamma \vdash A \equiv B \vdash \Delta \) then \( \Gamma \rightarrow \Delta \).

C.6 Subtyping Extends

**Lemma 50** (Subtyping Extension). If \( \Gamma \vdash A \prec B \vdash \Delta \) then \( \Gamma \rightarrow \Delta \).

C.7 Typing Extends

**Lemma 51** (Typing Extension). If \( \Gamma \vdash e \leftarrow A \vdash \Delta \) or \( \Gamma \vdash e \rightarrow A \vdash \Delta \) or \( \Gamma \vdash s : A \vdash B \vdash q \vdash \Delta \) or \( \Gamma \vdash \Pi :: \vec{A} \vdash C \vdash p \vdash \Delta \) then \( \Gamma \rightarrow \Delta \).

C.8 Unfiled

**Lemma 52** (Context Partitioning). If \( \Delta, \triangleright \vec{a}, \Theta \rightarrow \Omega, \triangleright \vec{a}, \Omega_Z \) then there is a \( \Psi \) such that \( \left[\Omega, \triangleright \vec{a}, \Omega_Z \mid \Delta, \triangleright \vec{a}, \Theta\right] = \left[\Omega|\Delta, \Psi\right] \).

**Lemma 53** (Softness Goes Away). If \( \Delta, \Theta \rightarrow \Omega, \Omega_Z \) where \( \Delta \rightarrow \Omega \) and \( \Theta \) is soft, then \( \left[\Omega, \Omega_Z \mid \Delta, \Theta\right] = \left[\Omega|\Delta\right] \).

**Proof.** By induction on \( \Theta \), following the definition of \( \left[\Omega|\Gamma\right] \).

**Lemma 54** (Completing Stability). If \( \Gamma \rightarrow \Omega \) then \( \left[\Omega|\Gamma\right] = \left[\Omega|\Omega\right] \).

**Lemma 55** (Completing Completeness). (i) If \( \Omega \rightarrow \Omega' \) and \( \Omega \vdash t : \kappa \) then \( \left[\Omega|t\right] = \left[\Omega'|t\right] \).

(ii) If \( \Omega \rightarrow \Omega' \) and \( \Omega \vdash A \) type then \( \left[\Omega|A\right] = \left[\Omega'|A\right] \).

(iii) If \( \Omega \rightarrow \Omega' \) then \( \left[\Omega|\Omega\right] = \left[\Omega'|\Omega\right] \).

**Lemma 56** (Confluence of Completeness). If \( \Delta_1 \rightarrow \Omega \) and \( \Delta_2 \rightarrow \Omega \) then \( \left[\Omega|\Delta_1\right] = \left[\Omega|\Delta_2\right] \).

**Lemma 57** (Multiple Confluence). If \( \Delta \rightarrow \Omega \) and \( \Omega \rightarrow \Omega' \) and \( \Delta' \rightarrow \Omega' \) then \( \left[\Omega|\Delta\right] = \left[\Omega'|\Delta'\right] \).
Lemma 58 (Bundled Substitution for Sorting). If $\Gamma \vdash t : \kappa$ and $\Gamma \rightarrow \Omega$ then $[\Omega] \Gamma \vdash [\Omega] t : \kappa$.

Proof.
\[
\begin{align*}
\Gamma \vdash t : \kappa & \quad \text{Given} \\
\Omega \vdash t : \kappa & \quad \text{By Lemma 36 (Extension Weakening (Sorts))} \\
[\Omega] \Omega \vdash [\Omega] t : \kappa & \quad \text{By Lemma 14 (Substitution for Sorting)} \\
\Omega \rightarrow \Omega & \quad \text{By Lemma 32 (Extension Reflexivity)} \\
[\Omega] \Omega = [\Omega] \Gamma & \quad \text{By Lemma 56 (Confluence of Completeness)} \\
\Leftrightarrow [\Omega] \Gamma \vdash [\Omega] t : \kappa & \quad \text{By above equality}
\end{align*}
\]

Lemma 59 (Canonical Completion). Go to proof

If $\Gamma \rightarrow \Omega$ then there exists $\Omega_{\text{canon}}$ such that $\Gamma \rightarrow \Omega_{\text{canon}}$ and $\Omega_{\text{canon}} \rightarrow \Omega$ and $\text{dom}(\Omega_{\text{canon}}) = \text{dom}(\Gamma)$ and, for all $\hat{\alpha} : \kappa = \tau$ and $\alpha = \tau$ in $\Omega_{\text{canon}}$, we have $\text{FEV}(\tau) = \emptyset$.

The completion $\Omega_{\text{canon}}$ is “canonical” because (1) its domain exactly matches $\Gamma$ and (2) its solutions $\tau$ have no evars. Note that it follows from Lemma 57 (Multiple Confluence) that $[\Omega_{\text{canon}}] \Gamma = [\Omega] \Gamma$.

Lemma 60 (Split Solutions). Go to proof

If $\Delta \rightarrow \Omega$ and $\alpha \in \text{unsolved}(\Delta)$ then there exists $\Omega_1 = \Omega_1[\hat{\alpha} : \kappa = t_1]$ such that $\Omega_1 \rightarrow \Omega$ and $\Omega_2 = \Omega_2[\hat{\alpha} : \kappa = t_2]$ where $\Delta \rightarrow \Omega_2$ and $t_2 \neq t_1$ and $\Omega_2$ is canonical.

D Internal Properties of the Declarative System

Lemma 61 (Interpolating With and Exists). Go to proof

(1) If $D : \Psi \vdash \Pi : A ! \Leftarrow C p$ and $\Psi \vdash P_0 \text{ true}$ then $D' : \Psi \vdash \Pi : A ! \Leftarrow C \land P_0 \ p$.

(2) If $D : \Psi \vdash \Pi : A ! \Leftarrow [\tau/\alpha] C_0 \ p$ and $\Psi \vdash \tau : \kappa$ then $D' : \Psi \vdash \Pi : A ! \Leftarrow (\exists \alpha : \kappa. C_0) \ p$.

In both cases, the height of $D'$ is one greater than the height of $D$. Moreover, similar properties hold for the eliminating judgment $\Psi / P \vdash \Pi : A ! \Leftarrow C \ p$.

Lemma 62 (Case Invertibility). Go to proof

If $\Psi \vdash \text{case}(e_0, \Pi) \Leftarrow C \ p$ then $\Psi \vdash e_0 \Rightarrow \Lambda !$ and $\Psi \vdash \Pi : A ! \Leftarrow C \ p$ and $\Psi \vdash \Pi \ covers \ A !$ where the height of each resulting derivation is strictly less than the height of the given derivation.

E Miscellaneous Properties of the Algorithmic System

Lemma 63 (Well-Formed Outputs of Typing). Go to proof

(Spines) If $\Gamma \vdash s : A q \Rightarrow C p \rightarrow \Delta$ or $\Gamma \vdash s : A q \Rightarrow C [p] \rightarrow \Delta$ and $\Gamma \vdash A q \ type$ then $\Delta \vdash C \ p \ type$.

(Synthesis) If $\Gamma \vdash e \Rightarrow A \ p \rightarrow \Delta$ then $A \vdash p \ type$. November 13, 2018
F Decidability of Instantiation

Lemma 64 (Left Unsolvedness Preservation). \[\text{If } \Gamma_0, \alpha, \Gamma_1 \vdash \hat{\alpha} := \Lambda : \kappa \vdash \Delta \text{ and } \hat{\beta} \in \text{unsolved}(\Gamma_0) \text{ then } \hat{\beta} \in \text{unsolved}(\Delta).\]

Lemma 65 (Left Free Variable Preservation). \[\text{If } \Gamma_0, \alpha : \kappa, \Gamma_1 \vdash \hat{\alpha} := t : \kappa \vdash \Delta \text{ and } \hat{\beta} \notin \text{FV}(|\Gamma|) \text{ and } \hat{\beta} \in \text{unsolved}(\Gamma_0) \text{ and } \hat{\beta} \notin \text{FV}(|\Delta|), \text{ then } \hat{\beta} \notin \text{FV}(|\Delta|).\]

Lemma 66 (Instantiation Size Preservation). \[\text{If } \Gamma_0, \alpha, \Gamma_1 \vdash \hat{\alpha} := \tau : \kappa \vdash \Delta \text{ and } \hat{\beta} \notin \text{FV}(|\Gamma|), \text{ then } |\Gamma| = |\Delta|, \text{ where } |C| \text{ is the plain size of the term } C.\]

Lemma 67 (Decidability of Instantiation). \[\text{If } \Gamma = \Gamma_0[\hat{\alpha} : \kappa'] \text{ and } \Gamma \vdash t : \kappa \text{ such that } |\Gamma|t = t \text{ and } \hat{\alpha} \notin \text{FV}(t), \text{ then:}\]

(1) Either there exists \(\Delta\) such that \(\Gamma_0[\hat{\alpha} : \kappa'] \vdash \hat{\alpha} := t : \kappa \vdash \Delta\), or not.

G Separation

Definition 4 (Separation). An algorithmic context \(\Gamma\) is separable and written \(\Gamma_L \ast \Gamma_R\) if (1) \(\Gamma = (\Gamma_L, \Gamma_R)\) and (2) for all \((\hat{\alpha} : \kappa = \tau) \in \Gamma_R\) it is the case that \(\text{FEV}(\tau) \subseteq \text{dom}(\Gamma_R)\).

Any context \(\Gamma\) is separable into, at least, \(\ast \Gamma\) and \(\ast \cdot \cdot \).

Definition 5 (Separation-Preserving Extension). The separated context \(\Gamma_L \ast \Gamma_R\) extends to \(\Delta_L \ast \Gamma_R\), written

\[(\Gamma_L \ast \Gamma_R) \longrightarrow (\Delta_L \ast \Delta_R) \text{ if } (\Gamma_L, \Gamma_R) \longrightarrow (\Delta_L, \Delta_R) \text{ and } \text{dom}(\Gamma_L) \subseteq \text{dom}(\Delta_L) \text{ and } \text{dom}(\Gamma_R) \subseteq \text{dom}(\Delta_R).\]

Separation-preserving extension says that variables from one half don’t “cross” into the other half. Thus, \(\Delta_L\) may add existential variables to \(\Gamma_L\), and \(\Delta_R\) may add existential variables to \(\Gamma_R\), but no variable from \(\Gamma_L\) ends up in \(\Delta_R\) and no variable from \(\Gamma_R\) ends up in \(\Delta_L\).

It is necessary to write \((\Gamma_L \ast \Gamma_R) \longrightarrow (\Delta_L \ast \Delta_R)\) rather than \((\Gamma_L \ast \Gamma_R) \longrightarrow (\Delta_L \ast \Delta_R)\), because only \(\longrightarrow\) includes the domain conditions. For example, \((\hat{\alpha} \ast \hat{\beta}) \longrightarrow (\hat{\alpha}, \hat{\beta} = \hat{\alpha}) \ast \cdot\cdot\), but the variable \(\hat{\beta}\) has “crossed over” to the left of \(\ast\) in the context \((\hat{\alpha}, \hat{\beta} = \hat{\alpha}) \ast \cdot \cdot\).

Lemma 68 (Transitivity of Separation). \[\text{If } (\Gamma_L \ast \Gamma_R) \longrightarrow (\Theta_L \ast \Theta_R) \text{ and } (\Theta_L \ast \Theta_R) \longrightarrow (\Delta_L \ast \Delta_R) \text{ then } (\Gamma_L \ast \Gamma_R) \longrightarrow (\Delta_L \ast \Delta_R).\]

Lemma 69 (Separation Truncation). \[\text{If } H \text{ has the form } \alpha : \kappa \text{ or } \varphi \alpha \text{ or } \varphi \beta \text{ or } x : A \varphi \
and (\Gamma_L \ast (\Gamma_R, H)) \longrightarrow (\Delta_L \ast \Delta_R) \text{ then } (\Gamma_L \ast \Gamma_R) \longrightarrow (\Delta_L \ast \Delta_R) \text{ where } \Delta_R = (\Delta_0, H, \Theta).\]

Lemma 70 (Separation for Auxiliary Judgments). \[\text{(i) If } \Gamma_L \ast \Gamma_R \vdash \sigma \equiv \tau : \kappa \vdash \Delta \text{ and } \text{FEV}(\sigma) \cup \text{FEV}(\tau) \subseteq \text{dom}(\Gamma_R) \text{ then } \Delta = (\Delta_L \ast \Delta_R) \text{ and } (\Gamma_L \ast \Gamma_R) \longrightarrow (\Delta_L \ast \Delta_R).\]

\[\text{and } (\Gamma_L \ast \Gamma_R) \longrightarrow (\Delta_L \ast \Delta_R) \text{ then } \Delta_L \ast \Delta_R \text{ where } \Delta_R = (\Delta_0, H, \Theta).\]

(i) \(\text{If } \Gamma_L \ast \Gamma_R \vdash P \text{ true } \vdash \Delta \text{ and } \text{FEV}(P) \subseteq \text{dom}(\Gamma_R) \text{ then } \Delta = (\Delta_L \ast \Delta_R) \text{ and } (\Gamma_L \ast \Gamma_R) \longrightarrow (\Delta_L \ast \Delta_R).\)
(iii) If $\Gamma_L * \Gamma_R / \sigma \vdash \kappa \vdash \Delta$
and $FEV(\sigma) \cup FEV(\tau) = \emptyset$
then $\Delta = (\Delta_L * (\Delta_R, \Theta))$ and $(\Gamma_L * (\Gamma_R, \Theta)) \not\vdash (\Delta_L * \Delta_R)$.

(iv) If $\Gamma_L * \Gamma_R / P \vdash \Delta$
and $FEV(P) = \emptyset$
then $\Delta = (\Delta_L * (\Delta_R, \Theta))$ and $(\Gamma_L * (\Gamma_R, \Theta)) \not\vdash (\Delta_L * \Delta_R)$.

(v) If $\Gamma_L * \Gamma_R \vdash \lambda := \tau : \kappa \vdash \Delta$
and $(FEV(\tau) \cup \{\lambda\}) \subseteq dom(\Gamma_R)$
then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \not\vdash (\Delta_L * \Delta_R)$.

(vi) If $\Gamma_L * \Gamma_R \vdash P \equiv Q \vdash \Delta$
and $FEV(P) \cup FEV(Q) \subseteq dom(\Gamma_R)$
then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \not\vdash (\Delta_L * \Delta_R)$.

(vii) If $\Gamma_L * \Gamma_R \vdash A \equiv B \vdash \Delta$
and $FEV(A) \cup FEV(B) \subseteq dom(\Gamma_R)$
then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \not\vdash (\Delta_L * \Delta_R)$.

Lemma 71 (Separation for Subtyping). Go to proof
If $\Gamma_L * \Gamma_R \vdash A \narrow P \vdash B \vdash \Delta$
and $FEV(A) \subseteq dom(\Gamma_R)$
and $FEV(B) \subseteq dom(\Gamma_R)$
then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \not\vdash (\Delta_L * \Delta_R)$.

Lemma 72 (Separation—Main). Go to proof
(Spines) If $\Gamma_L * \Gamma_R \vdash s : A \narrow p \narrow C \narrow q \vdash \Delta$
or $\Gamma_L * \Gamma_R \vdash s : A \narrow C \narrow [q] \vdash \Delta$
and $\Gamma_L * \Gamma_R \vdash A \narrow p \narrow type$
and $FEV(A) \subseteq dom(\Gamma_R)$
then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \not\vdash (\Delta_L * \Delta_R)$ and $FEV(C) \subseteq dom(\Delta_R)$.

(Checking) If $\Gamma_L * \Gamma_R \vdash e \narrow C \narrow p \narrow \Delta$
and $\Gamma_L * \Gamma_R \vdash C \narrow p \narrow type$
and $FEV(C) \subseteq dom(\Gamma_R)$
then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \not\vdash (\Delta_L * \Delta_R)$.

(Synthesis) If $\Gamma_L * \Gamma_R \vdash e \narrow A \narrow p \narrow \Delta$
then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \not\vdash (\Delta_L * \Delta_R)$.

(Match) If $\Gamma_L * \Gamma_R \vdash \Pi :: A \narrow q \narrow C \narrow p \narrow \Delta$
and $FEV(A) = \emptyset$
and $FEV(C) \subseteq dom(\Gamma_R)$
then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \not\vdash (\Delta_L * \Delta_R)$.

(Match Elim.) If $\Gamma_L * \Gamma_R / P \vdash \Pi :: A \narrow ! \narrow C \narrow p \narrow \Delta$
and $FEV(P) = \emptyset$
and $FEV(A) = \emptyset$
and $FEV(C) \subseteq dom(\Gamma_R)$
then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \not\vdash (\Delta_L * \Delta_R)$.

H Decidability of Algorithmic Subtyping

Definition 6. The following connectives are large:

$\forall \therefore \land$
A type is large iff its head connective is large. (Note that a non-large type may contain large connectives, provided they are not in head position.)

The number of these connectives in a type $A$ is denoted by $\#\text{large}(A)$.

### H.1 Lemmas for Decidability of Subtyping

**Lemma 73** (Substitution Isn’t Large). Go to proof
For all contexts $\Theta$, we have $\#\text{large}(\Theta \uplus A) = \#\text{large}(A)$.

**Lemma 74** (Instantiation Solves). Go to proof
If $\Gamma \vdash A : \Delta$ and $[\Gamma] \tau = \tau$ and $\Delta \not\in FV([\Gamma] \tau)$ then $|\text{unsolved}(\Gamma)| = |\text{unsolved}(\Delta)| + 1$.

**Lemma 75** (Checkeq Solving). Go to proof
If $\Gamma \vdash s = t : \kappa \vdash \Delta$ then either $\Delta = \Gamma$ or $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Gamma)|$.

**Lemma 76** (Prop Equiv Solving). Go to proof
If $\Gamma \vdash P \equiv Q : \kappa \vdash \Delta$ then either $\Delta = \Gamma$ or $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Gamma)|$.

**Lemma 77** (Equiv Solving). Go to proof
If $\Gamma \vdash A \equiv B : \kappa \vdash \Delta$ then either $\Delta = \Gamma$ or $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Gamma)|$.

**Lemma 78** (Decidability of Propositional Judgments). Go to proof
The following judgments are decidable, with $\Delta$ as output in (1)–(3), and $\Delta^\perp$ as output in (4) and (5).

We assume $\sigma = [\Gamma] \sigma$ and $t = [\Gamma] t$ in (1) and (4). Similarly, in the other parts we assume $P = [\Gamma] P$ and (in part (3)) $Q = [\Gamma] Q$.

(1) $\Gamma \vdash \sigma = t : \kappa \vdash \Delta$

(2) $\Gamma \vdash P \text{ true } \vdash \Delta$

(3) $\Gamma \vdash P = Q : \kappa \vdash \Delta$

(4) $\Gamma / \sigma = t : \kappa \vdash \Delta^\perp$

(5) $\Gamma / P = \Delta^\perp$

**Lemma 79** (Decidability of Equivalence). Go to proof
Given a context $\Gamma$ and types $A$, $B$ such that $\Gamma \vdash A : \kappa \vdash \Delta$ and $\Gamma \vdash B : \kappa \vdash \Delta$, it is decidable whether there exists $\Delta$ such that $\Gamma \vdash A \equiv B : \kappa \vdash \Delta$.

### H.2 Decidability of Subtyping

**Theorem 1** (Decidability of Subtyping). Go to proof
Given a context $\Gamma$ and types $A$, $B$ such that $\Gamma \vdash A : \kappa \vdash \Delta$ and $\Gamma \vdash B : \kappa \vdash \Delta$, it is decidable whether there exists $\Delta$ such that $\Gamma \vdash A \ll B : \kappa \vdash \Delta$.

### H.3 Decidability of Matching and Coverage

**Lemma 80** (Decidability of Guardedness Judgment). Go to proof
For any set of branches $\Pi$, the relation $\Pi \text{ guarded}$ is decidable.

**Lemma 81** (Decidability of Expansion Judgments). Go to proof
Given branches $\Pi$, it is decidable whether:

(1) there exists a unique $\Pi'$ such that $\Pi \sim \Pi'$;

(2) there exist unique $\Pi_L$ and $\Pi_R$ such that $\Pi \not\sim \Pi_L \parallel \Pi_R$;

(3) there exists a unique $\Pi'$ such that $\Pi \vartriangleright \Pi'$;
(4) there exists a unique $\Pi'$ such that $\Pi \overset{1}{\sim} \Pi'$.

(5) there exist unique $\Pi_\ell$ and $\Pi_\ell'$ such that $\Pi \overset{\downarrow}{\exists} \Pi_\ell \parallel \Pi_\ell'$.

**Lemma 82** (Expansion Shrinks Size). Go to proof

We define the size of a pattern $|p|$ as follows:

\[
\begin{align*}
|x| &= 0 \\
|\lambda| &= 0 \\
|\langle p, p' \rangle| &= 1 + |p| + |p'| \\
|\ell| &= 1 \\
|\text{inj}_1 p| &= 1 + |p| \\
|\text{inj}_2 p| &= 1 + |p| \\
|\ell| &= 1 \\
|p : p'| &= 1 + |p| + |p'|
\end{align*}
\]

We lift size to branches $\pi = \overrightarrow{p} \Rightarrow e$ as follows:

\[|p_1, \ldots, p_n \Rightarrow e| = |p_1| + \ldots + |p_n|\]

We lift size to branch lists $\Pi = \pi_1 \mid \ldots \mid \pi_n$ as follows:

\[|\pi_1 \mid \ldots \mid \pi_n| = |\pi_1| + \ldots + |\pi_n|\]

Now, the following properties hold:

1. If $\Pi \overset{\exists}{\rightarrow} \Pi'$ then $|\Pi| = |\Pi'|$.
2. If $\Pi \overset{\sim}{\rightarrow} \Pi'$ then $|\Pi| = |\Pi'|$.
3. If $\Pi \overset{\exists}{\rightarrow} \Pi'$ then $|\Pi| \leq |\Pi'|$.
4. If $\Pi \overset{\sim}{\rightarrow} \Pi_\ell \parallel \Pi_R$ then $|\Pi| \leq |\Pi_1|$ and $|\Pi_2| \leq |\Pi|$.
5. If $\Pi \overset{\exists}{\rightarrow} \Pi_\ell \parallel \Pi_\ell'$ then $|\Pi| \leq |\Pi_1|$ and $|\Pi_2| \leq |\Pi|$.
6. If $\Pi$ guarded and $\Pi \overset{\exists}{\rightarrow} \Pi_\ell \parallel \Pi_\ell'$ then $|\Pi_1| < |\Pi|$ and $|\Pi_2| < |\Pi|$.

**Theorem 2** (Decidability of Coverage). Go to proof

Given a context $\Gamma$, branches $\Pi$ and types $\Lambda$, it is decidable whether $\Gamma \vdash \Pi$ covers $\overrightarrow{A} \, q$ is derivable.

### H.4 Decidability of Typing

**Theorem 3** (Decidability of Typing). Go to proof

(i) Synthesis: Given a context $\Gamma$, a principality $p$, and a term $e$,

\[\text{it is decidable whether there exist a type } A \text{ and a context } \Delta \text{ such that }\]

\[\Gamma \vdash e \Rightarrow A \, p \vdash \Delta.\]

(ii) Spines: Given a context $\Gamma$, a spine $s$, a principality $p$, and a type $A$ such that $\Gamma \vdash A$ type,

\[\text{it is decidable whether there exist a type } B, \text{ a principality } q \text{ and a context } \Delta \text{ such that }\]

\[\Gamma \vdash s : A \, p \Rightarrow B \, q \vdash \Delta.\]

(iii) Checking: Given a context $\Gamma$, a principality $p$, a term $e$, and a type $B$ such that $\Gamma \vdash B$ type,

\[\text{it is decidable whether there is a context } \Delta \text{ such that }\]

\[\Gamma \vdash e \Leftarrow B \, p \vdash \Delta.\]

(iv) Matching: Given a context $\Gamma$, branches $\Pi$, a list of types $\overrightarrow{A}$, a type $C$, and a principality $p$, it is decidable whether there exists $\Delta$ such that $\Gamma \vdash \Pi : \overrightarrow{A} \, q \Leftarrow C \, p \vdash \Delta$.

Also, if given a proposition $P$ as well, it is decidable whether there exists $\Delta$ such that $\Gamma \vdash P \vdash \Pi : \overrightarrow{A} ! \Leftarrow C \, p \vdash \Delta$. 

November 13, 2018
I Determinacy

Lemma 83 (Determinacy of Auxiliary Judgments). Go to proof

1. Elimeq: Given $\Gamma, \sigma, t, \kappa$ such that $\text{FEV}(\sigma) \cup \text{FEV}(t) = \emptyset$ and $D_1 ::= \Gamma / \sigma \triangleright t : \kappa \vdash \Delta_1^\perp$ and $D_2 ::= \Gamma / \sigma \triangleright t : \kappa \vdash \Delta_2^\perp$

   \[ \kappa \vdash \Delta_1^\perp \]

   it is the case that $\Delta_1^\perp = \Delta_2^\perp$.

2. Instantiation: Given $\Gamma, \hat{\alpha}, t, \kappa$ such that $\hat{\alpha} \in \text{unsolved}(\Gamma)$ and $\Gamma \vdash \kappa$ and $\hat{\alpha} \notin \text{FV}(t)$

   and $D_1 ::= \Gamma / \hat{\alpha} := t : \kappa \vdash \Delta_1$ and $D_2 ::= \Gamma / \hat{\alpha} := t : \kappa \vdash \Delta_2$

   it is the case that $\Delta_1 = \Delta_2$.

3. Symmetric instantiation:

   Given $\hat{\alpha}, \hat{\beta}, \kappa$ such that $\hat{\alpha}, \hat{\beta} \in \text{unsolved}(\Gamma)$ and $\hat{\alpha} \neq \hat{\beta}$

   and $D_1 ::= \Gamma / \hat{\alpha} := \hat{\beta} : \kappa \vdash \Delta_1$ and $D_2 ::= \Gamma / \hat{\beta} := \hat{\alpha} : \kappa \vdash \Delta_2$

   it is the case that $\Delta_1 = \Delta_2$.

4. Checkeq: Given $\Gamma, \sigma, t, \kappa$ such that $D_1 ::= \Gamma / \sigma \triangleright t : \kappa \vdash \Delta_1$ and $D_2 ::= \Gamma / \sigma \triangleright t : \kappa \vdash \Delta_2$

   it is the case that $\Delta_1 = \Delta_2$.

5. Elimprop: Given $\Gamma, P$ such that $D_1 ::= \Gamma / P \vdash \Delta_1^\perp$ and $D_2 ::= \Gamma / P \vdash \Delta_2^\perp$

   it is the case that $\Delta_1 = \Delta_2$.

6. Checkprop: Given $\Gamma, P$ such that $D_1 ::= \Gamma / P \vdash \text{true} \vdash \Delta_1$ and $D_2 ::= \Gamma / P \vdash \text{true} \vdash \Delta_2$,

   it is the case that $\Delta_1 = \Delta_2$.

Lemma 84 (Determinacy of Equivalence). Go to proof

1. Propositional equivalence: Given $\Gamma, P, Q$ such that $D_1 ::= \Gamma / P \equiv Q \vdash \Delta_1$ and $D_2 ::= \Gamma / P \equiv Q \vdash \Delta_2$,

   it is the case that $\Delta_1 = \Delta_2$.

2. Type equivalence: Given $\Gamma, A, B$ such that $D_1 ::= \Gamma / A \equiv B \vdash \Delta_1$ and $D_2 ::= \Gamma / A \equiv B \vdash \Delta_2$,

   it is the case that $\Delta_1 = \Delta_2$.

Theorem 4 (Determinacy of Subtyping). Go to proof

1. Subtyping: Given $\Gamma, e, A, B$ such that $D_1 ::= \Gamma / e : A \triangleleft P \vdash B \vdash \Delta_1$ and $D_2 ::= \Gamma / e : A \triangleleft P \vdash B \vdash \Delta_2$,

   it is the case that $\Delta_1 = \Delta_2$.

Theorem 5 (Determinacy of Typing). Go to proof

1. Checking: Given $\Gamma, e, A, p$ such that $D_1 ::= \Gamma / e \iff A \vdash p \vdash \Delta_1$ and $D_2 ::= \Gamma / e \iff A \vdash p \vdash \Delta_2$,

   it is the case that $\Delta_1 = \Delta_2$.

2. Synthesis: Given $\Gamma, e$ such that $D_1 ::= \Gamma / e \Rightarrow B_1 p_1 \vdash \Delta_1$ and $D_2 ::= \Gamma / e \Rightarrow B_2 p_2 \vdash \Delta_2$,

   it is the case that $B_1 = B_2$ and $p_1 = p_2$ and $\Delta_1 = \Delta_2$.

3. Spine judgments:

   Given $\Gamma, e, A, p$ such that $D_1 ::= \Gamma / e : A \vdash C_1 q_1 \vdash \Delta_1$ and $D_2 ::= \Gamma / e : A \vdash C_2 q_2 \vdash \Delta_2$,\n
   it is the case that $C_1 = C_2$ and $q_1 = q_2$ and $\Delta_1 = \Delta_2$.

   The same applies for derivations of the principality-recovering judgments $\Gamma / e : A \vdash C_k [q_k] \vdash \Delta_k$.

4. Match judgments:

   Given $\Gamma, \Pi, \tilde{A}, p, C$ such that $D_1 ::= \Gamma / \Pi \vdash \tilde{A} q \iff C p \vdash \Delta_1$ and $D_2 ::= \Gamma / \Pi \vdash \tilde{A} q \iff C p \vdash \Delta_2$,

   it is the case that $\Delta_1 = \Delta_2$.

   Given $\Gamma, \Pi, \tilde{A}, p, C$

   such that $D_1 ::= \Gamma / \Pi \vdash \tilde{A} ! \iff C p \vdash \Delta_1$ and $D_2 ::= \Gamma / \Pi \vdash \tilde{A} ! \iff C p \vdash \Delta_2$,

   it is the case that $\Delta_1 = \Delta_2$.
J Soundness

J.1 Soundness of Instantiation

Lemma 85 (Soundness of Instantiation). \( \text{Go to proof} \)
If \( \Gamma \vdash \alpha := \tau : \kappa \vdash \Delta \) and \( \alpha \notin \text{FV}(\Gamma[\tau]) \) and \( \Gamma[\tau] = \tau \) and \( \Delta \rightarrow \Omega \) then \( \Omega[\alpha] = \Omega[\tau] \).

J.2 Soundness of Checkeq

Lemma 86 (Soundness of Checkeq). \( \text{Go to proof} \)
If \( \Gamma \vdash \sigma \triangleright t : \kappa \vdash \Delta \) where \( \Delta \rightarrow \Omega \) then \( \Omega[\sigma] = \Omega[t] \).

J.3 Soundness of Equivalence (Propositions and Types)

Lemma 87 (Soundness of Propositional Equivalence). \( \text{Go to proof} \)
If \( \Gamma \vdash P \equiv Q \vdash \Delta \) where \( \Delta \rightarrow \Omega \) then \( \Omega[P] = \Omega[Q] \).

Lemma 88 (Soundness of Algorithmic Equivalence). \( \text{Go to proof} \)
If \( \Gamma \vdash A \equiv B \vdash \Delta \) where \( \Delta \rightarrow \Omega \) then \( \Omega[A] = \Omega[B] \).

J.4 Soundness of Checkprop

Lemma 89 (Soundness of Checkprop). \( \text{Go to proof} \)
If \( \Gamma \vdash P \ \text{true} \vdash \Delta \) and \( \Delta \rightarrow \Omega \) then \( \Psi \vdash \Omega[\text{true}] \).

J.5 Soundness of Eliminations (Equality and Proposition)

Lemma 90 (Soundness of Equality Elimination). \( \text{Go to proof} \)
If \( \Gamma[\sigma] = \sigma \) and \( \Gamma[t] = t \) and \( \Gamma \vdash \sigma : \kappa \) and \( \Gamma \vdash t : \kappa \) and \( \text{FEV}(\sigma) \cup \text{FEV}(t) = \emptyset \), then:

1. If \( \Gamma / \sigma \triangleright t : \kappa \vdash \Delta \)
   then \( \Delta = (\Gamma, \Theta) \) where \( \Theta = (\alpha_1 = t_1, \ldots, \alpha_n = t_n) \) and
   for all \( \Omega \) such that \( \Gamma \rightarrow \Omega \)
   and all \( t' \) such that \( \Omega[\alpha] = [\Theta][\Omega][t'] \),
   it is the case that \( \Omega[\Theta][t'] = [\Theta][\Omega][t'] \), where \( \Theta = \text{mgu}(\sigma, t) \).

2. If \( \Gamma / \sigma \triangleright t : \kappa \vdash \perp \) then \( \text{mgu}(\sigma, t) = \perp \) (that is, no most general unifier exists).

J.6 Soundness of Subtyping

Theorem 6 (Soundness of Algorithmic Subtyping). \( \text{Go to proof} \)
If \( \Gamma[A] = A \) and \( \Gamma[B] = B \) and \( \Gamma \vdash A \) type and \( \Gamma \vdash B \) type and \( \Delta \rightarrow \Omega \) and \( \Gamma \vdash A <: P \) \( B \vdash \Delta \) then \( \Omega[\Delta] \vdash \Omega[A] \leq P \) \( \Omega[B] \).

J.7 Soundness of Typing

Theorem 7 (Soundness of Match Coverage). \( \text{Go to proof} \)
1. If \( \Gamma \vdash \Pi \) covers \( \tilde{A} \) \( q \) and \( \Gamma \vdash \tilde{A} \) \( q \) types and \( \Gamma[\tilde{A}] = \tilde{A} \) and \( \Gamma \rightarrow \Omega \) then \( \Omega[\Gamma] \vdash \Pi \) covers \( \tilde{A} \) \( q \).
2. If \( \Gamma / P \vdash \Pi \) covers \( \tilde{A} \) ! and \( \Gamma \rightarrow \Omega \) and \( \Gamma \vdash \tilde{A} \) ! types and \( \Gamma[\tilde{A}] = \tilde{A} \) and \( \Gamma[P] = P \) then \( \Omega[\Gamma / P] \vdash \Pi \) covers \( \tilde{A} \) !.

Lemma 91 (Well-formedness of Algorithmic Typing). \( \text{Go to proof} \)
Given \( \Gamma \) \text{ctx}:
(i) If $\Gamma \vdash e \Rightarrow A : \Delta$ then $\Delta \vdash A$ type.

(ii) If $\Gamma \vdash s : A : \Delta$ and $\Gamma \vdash A$ type then $\Delta \vdash B$ type.

**Definition 7** (Measure). Let measure $M$ on typing judgments be a lexicographic ordering:

1. first, the subject expression $e$, spine $s$, or matches $\Pi$—regarding all types in annotations as equal in size;
2. second, the partial order on judgment forms where an ordinary spine judgment is smaller than a principality-recovering spine judgment—and with all other judgment forms considered equal in size; and,
3. third, the derivation height.

\[ \left( e / s / \Pi, \text{ordinary spine judgment} < \text{recovering spine judgment}, \text{height}(D) \right) \]

Note that this definition doesn’t take notice of whether a spine judgment is declarative or algorithmic.

This measure works to show soundness and completeness. We list each rule below, along with a 3-tuple. For example, for Sub we write $\langle =, =, < \rangle$, meaning that each judgment to which we need to apply the i.h. has a subject of the same size ($=$), a judgment form of the same size ($=$), and a smaller derivation height ($<$).

We write “$-$” when a part of the measure need not be considered because a lexicographically more significant part is smaller, as in the Anno rule, where the premise has a smaller subject: $\langle <, -, - \rangle$.

Algorithmic rules (soundness cases):

- **Var**: $\langle =, =, < \rangle$
- **Sub**: $\langle =, =, < \rangle$
- **Anno**: $\langle <, -, - \rangle$
- **$\forall$Spine $\forall\Pi$**: $\langle =, =, < \rangle$
- **$\exists$**: $\langle =, =, < \rangle$
- **$\forall$IE**: $\langle =, =, < \rangle$
- **SpineRecover**: $\langle =, -, < \rangle$
- **SpinePass**: $\langle =, -, < \rangle$
- **$\rightarrow$Spine**: $\langle =, =, < \rangle$
- **$\rightarrow$IE**: $\langle =, =, < \rangle$
- **Rec**: $\langle <, -, - \rangle$
- **Cons**: $\langle <, -, - \rangle$
- **Case**: $\langle <, -, - \rangle$

Declarative rules (completeness cases):

- **DecVar**, **DecSub**, **DecEmptySpine**, and **DeclNil** have no premises, or only auxiliary judgments as premises.
- **DeclSub**: $\langle =, =, < \rangle$
- **DeclAnno**: $\langle <, -, - \rangle$
J.7 Soundness of Typing

\begin{itemize}
  \item \texttt{Decl}/| Decl\textbar Spine/| Decl\textbar]| Decl\textbar]| Decl\textbar]\| Decl\textbar Spine: \langle =, =, < \rangle
  \item \texttt{Decl→}| Decl→E DeclRec: \langle <, −, − \rangle
  \item DeciSpineRecover: \langle =, <, − \rangle
  \item DeciSpinePass: \langle =, <, − \rangle
  \item \texttt{Decl→}| Decl+| Decl×| DeclCase DeclCons: \langle <, −, − \rangle
\end{itemize}

\textbf{Definition 8 (Eagerness).}

A derivation \( D \) whose conclusion is \( J \) is eager if:

(i) \( J = \Gamma \vdash e \Leftarrow A \ p \vdash \Delta \)
  if \( \Gamma \vdash A \ p \text{ type and } A = [\Gamma]A \)
  implies that
  every subderivation of \( D \) is eager.

(ii) \( J = \Gamma \vdash e \Rightarrow A \ p \vdash \Delta \)
  if \( A = [\Delta]A \)
  and every subderivation of \( D \) is eager.

(iii) \( J = \Gamma \vdash s : A \ p \gg B \ q \vdash \Delta \)
  if \( \Gamma \vdash A \ p \text{ type and } A = [\Gamma]A \)
  implies that
  \( B = [\Delta]B \)
  and every subderivation of \( D \) is eager.

(iv) \( J = \Gamma \vdash s : A \ p \gg B \ [q] \vdash \Delta \)
  if \( \Gamma \vdash A \ p \text{ type and } A = [\Gamma]A \)
  implies that
  \( B = [\Delta]B \)
  and every subderivation of \( D \) is eager.

(v) \( J = \Gamma \vdash \Pi :: \tilde{A} \ q \Leftarrow C \ p \vdash \Delta \)
  if \( \Gamma \vdash \tilde{A} \ q \text{ types and } [\Gamma]\tilde{A} = \tilde{A} \) and \( \Gamma \vdash C \ p \text{ type and } C = [\Gamma]C \)
  implies that
  every subderivation of \( D \) is eager.

(vi) \( J = \Gamma / P \vdash \Pi :: \tilde{A} ! \Leftarrow C \ p \vdash \Delta \)
  if \( \Gamma \vdash \tilde{A} ! \text{ types and } \Gamma \vdash P \text{ prop and } [\Gamma]\tilde{A} = \tilde{A} \) and \( \Gamma \vdash C \ p \text{ type and } C = [\Gamma]C \)
  implies that
  every subderivation of \( D \) is eager.

\textbf{Theorem 8 (Eagerness of Types). Go to proof}

(i) If \( D \) derives \( \Gamma \vdash e \Leftarrow A \ p \vdash \Delta \) and \( \Gamma \vdash A \ p \text{ type and } A = [\Gamma]A \) then \( D \) is eager.

(ii) If \( D \) derives \( \Gamma \vdash e \Rightarrow A \ p \vdash \Delta \) then \( D \) is eager.

(iii) If \( D \) derives \( \Gamma \vdash s : A \ p \gg B \ q \vdash \Delta \) and \( \Gamma \vdash A \ p \text{ type and } A = [\Gamma]A \) then \( D \) is eager.

(iv) If \( D \) derives \( \Gamma \vdash s : A \ p \gg B \ [q] \vdash \Delta \) and \( \Gamma \vdash A \ p \text{ type and } A = [\Gamma]A \) then \( D \) is eager.

(v) If \( D \) derives \( \Gamma \vdash \Pi :: \tilde{A} \ q \Leftarrow C \ p \vdash \Delta \) and \( \Gamma \vdash \tilde{A} \ q \text{ types and } [\Gamma]\tilde{A} = \tilde{A} \) and \( \Gamma \vdash C \ p \text{ type} \) then \( D \) is eager.
(vi) If $D$ derives $\Gamma / P \vdash \Pi : \bar{A} ! \iff \bar{C} \vdash \Delta$ and $\Gamma \vdash P$ prop and $\text{FEV}(P) = \emptyset$ and $[\Gamma]P = P$
and $\Gamma \vdash \bar{A} !$ types and $\Gamma \vdash \bar{C} P$ type
then $D$ is eager.

**Theorem 9** (Soundness of Algorithmic Typing). Go to proof

Given $\Delta \rightarrow \Omega$:

(i) If $\Gamma \vdash e \iff A \vdash \Delta$ and $\Gamma \vdash A$ p type and $A = [\Gamma]A$ then $[\Omega]\Delta \vdash [\Omega]e \iff [\Omega]A$ p.

(ii) If $\Gamma \vdash e \Rightarrow A \vdash \Delta$ then $[\Omega]\Delta \vdash [\Omega]e \Rightarrow [\Omega]A$ p.

(iii) If $\Gamma \vdash s : A \Rightarrow B : q \vdash \Delta$ and $\Gamma \vdash A$ p type and $A = [\Gamma]A$ then $[\Omega]\Delta \vdash [\Omega]s : [\Omega]A \Rightarrow [\Omega]B$ q.

(iv) If $\Gamma \vdash s : A \Rightarrow B [\eta] \vdash \Delta$ and $\Gamma \vdash A$ p type and $A = [\Gamma]A$ then $[\Omega]\Delta \vdash [\Omega]s : [\Omega]A \Rightarrow [\Omega]B [\eta]$.

(v) If $\Gamma \vdash \Pi : \bar{A} q \iff \bar{C} \vdash \Delta$ and $\Gamma \vdash \bar{A} A$ types and $[\Gamma][\bar{A}] = \bar{\bar{A}}$ and $\Gamma \vdash \bar{C}$ p type
then $\Gamma \vdash [\Omega]\Delta / [\Omega]\Pi : [\Omega]\bar{A} q [\Omega] \bar{C}$.

(vi) If $\Gamma / P \vdash \Pi : \bar{A} ! \iff \bar{C} \vdash \Delta$ and $\Gamma \vdash P$ prop and $\text{FEV}(P) = \emptyset$ and $[\Gamma]P = P$
and $\Gamma \vdash \bar{A} !$ types and $\Gamma \vdash \bar{C}$ p type
then $[\Omega]\Delta / [\Omega]P \vdash [\Omega]\Pi : [\Omega]A ! \iff [\Omega]C$ p.

**K Completeness**

**K.1 Completeness of Auxiliary Judgments**

**Lemma 92** (Completeness of Instantiation). Go to proof

Given $\Gamma \rightarrow \Omega$ and $\text{dom}(\Gamma) = \text{dom}(\Omega)$ and $\Gamma \vdash \tau : \kappa$ and $\tau = [\Gamma]\tau$ and $\bar{\kappa} \in \text{unsolved}(\Gamma)$ and $\bar{\kappa} \notin \text{FV}(\tau)$:

If $[\Omega][\bar{\kappa}] = [\Omega]\tau$
then there are $\Delta, \Omega'$ such that $\Omega \rightarrow [\Omega]'$ and $\Delta \rightarrow [\Omega]'$ and $\text{dom}(\Delta) = \text{dom}(\Omega')$ and $\Gamma \vdash \bar{\kappa} := \tau : \kappa \rightarrow \Delta$.

**Lemma 93** (Completeness of Checkeq). Go to proof

Given $\Gamma \rightarrow \Omega$ and $\text{dom}(\Gamma) = \text{dom}(\Omega)$
and $\Gamma \vdash \sigma : \kappa$ and $\Gamma \vdash \tau : \kappa$
and $[\Omega]\sigma = [\Omega]\tau$
then $\Gamma \vdash [\Gamma]\sigma \equiv [\Gamma]\tau : \kappa \rightarrow \Delta$
where $\Delta \rightarrow [\Omega]'$ and $\text{dom}(\Delta) = \text{dom}(\Omega')$ and $\Omega \rightarrow [\Omega]'$.

**Lemma 94** (Completeness of Elimeq). Go to proof

If $[\Gamma]\sigma = \sigma$ and $[\Gamma]t = t$ and $\Gamma \vdash \sigma : \kappa$ and $\Gamma \vdash t : \kappa$ and $\text{FEV}(\sigma) \cup \text{FEV}(t) = \emptyset$ then:

1. If $\text{mgu}(\sigma, t) = \emptyset$
then $\Gamma \vdash \sigma \equiv t : \kappa \rightarrow \Delta$
where $\Delta$ has the form $\alpha_1 = t_1, \ldots, \alpha_n = t_n$
and for all $\omega$ such that $\Gamma \vdash \omega : \kappa$, it is the case that $[\Gamma, \Delta] \omega = \emptyset([\Gamma] \omega)$.

2. If $\text{mgu}(\sigma, t) = \perp$ (that is, no most general unifier exists) then $\Gamma / \sigma \equiv t : \kappa \rightarrow \perp$.

**Lemma 95** (Substitution Upgrade). Go to proof

If $\Delta$ has the form $\alpha_1 = t_1, \ldots, \alpha_n = t_n$
and, for all $\omega$ such that $\Gamma \vdash \omega : \kappa$, it is the case that $[\Gamma, \Delta] \omega = \emptyset([\Gamma] \omega)$,
then:

1. If $\Gamma \vdash A$ type then $[\Gamma, \Delta]A = \emptyset([\Gamma]A)$.

2. If $\Gamma \rightarrow \Omega$ then $[\Omega]\Gamma = \emptyset([\Omega] \Gamma)$.

3. If $\Gamma \rightarrow \Omega$ then $[\Omega, \Delta][\Gamma, \Delta] = \emptyset([\Omega] \Gamma)$.
Lemma 96 (Completeness of Propequiv).\footnote{Go to proof}
Given $\Gamma \rightarrow \Omega$ and $\Gamma \vdash P$ prop and $\Gamma \vdash Q$ prop and $[\Omega]P = [\Omega]Q$
then $\Gamma \vdash [\Gamma]P \equiv [\Gamma]Q \vdash \Delta$
where $\Delta \rightarrow \Omega'$ and $\Omega \rightarrow \Omega'$.

Lemma 97 (Completeness of Checkprop).\footnote{Go to proof}
If $\Gamma \rightarrow \Omega$ and $\text{dom}(\Gamma) = \text{dom}(\Omega)$
and $\Gamma \vdash P$ prop
and $[\Omega]P = P$
and $[\Omega]G \vdash [\Omega]P$ true
then $\Gamma \vdash P$ true $\vdash \Delta$
where $\Delta \rightarrow \Omega'$ and $\Omega \rightarrow \Omega'$ and $\text{dom}(\Delta) = \text{dom}(\Omega')$.

K.2 Completeness of Equivalence and Subtyping

Lemma 98 (Completeness of Equiv).\footnote{Go to proof}
If $\Gamma \rightarrow \Omega$ and $\Gamma \vdash A$ type and $\Gamma \vdash B$ type
and $[\Omega]A = [\Omega]B$
then there exist $\Delta$ and $\Omega'$ such that $\Delta \rightarrow \Omega'$ and $\Omega \rightarrow \Omega'$ and $\Gamma \vdash [\Gamma]A \equiv [\Gamma]B \vdash \Delta$.

Theorem 10 (Completeness of Subtyping).\footnote{Go to proof}
If $\Gamma \rightarrow \Omega$ and $\text{dom}(\Gamma) = \text{dom}(\Omega)$ and $\Gamma \vdash A$ type and $\Gamma \vdash B$ type
and $[\Omega]G \vdash [\Omega]A \leq^P [\Omega]B$
then there exist $\Delta$ and $\Omega'$ such that $\Delta \rightarrow \Omega'$
and $\text{dom}(\Delta) = \text{dom}(\Omega')$
and $\Omega \rightarrow \Omega'$
and $\Gamma \vdash [\Gamma]A <;^P [\Gamma]B \vdash \Delta$.

K.3 Completeness of Typing

Lemma 99 (Variable Decomposition).\footnote{Go to proof}
If $\Pi \vdash^\var \Pi'$, then

1. if $\Pi \vdash^\var \Pi''$ then $\Pi'' = \Pi'$.
2. if $\Pi \vdash^\var \Pi''$ then there exists $\Pi''$ such that $\Pi''' \vdash^\var \Pi''$ and $\Pi'' \vdash^\var \Pi'$,
3. if $\Pi \vdash\Pi_L \parallel \Pi_R$ then $\Pi_L \vdash^\var \Pi'$ and $\Pi_R \vdash^\var \Pi'$,
4. if $\Pi \vdash^\var \Pi[I] \parallel \Pi[I]$ then $\Pi' = \Pi[I]$.

Lemma 100 (Pattern Decomposition and Substitution).\footnote{Go to proof}

1. If $\Pi \vdash^\var \Pi'$ then $[\Omega]\Pi \vdash^\var [\Omega]\Pi'$.
2. If $\Pi \vdash^\var \Pi'$ then $[\Omega]\Pi \vdash^\var [\Omega]\Pi'$.
3. If $\Pi \vdash^\var \Pi'$ then $[\Omega]\Pi \vdash^\var [\Omega]\Pi'$.
4. If $\Pi \vdash^\var \Pi[I] \parallel \Pi[I]$ then $[\Omega]\Pi \vdash^\var [\Omega]\Pi[I] \parallel [\Omega]\Pi[I]$.\footnote{Go to proof}
5. If $\Pi \vdash^\var \Pi[I] \parallel \Pi[I]$ then $[\Omega]\Pi \vdash^\var [\Omega]\Pi[I] \parallel [\Omega]\Pi[I]$.\footnote{Go to proof}
6. If $[\Omega]\Pi \vdash^\var \Pi'$ then there is $\Pi''$ such that $[\Omega]\Pi'' = \Pi'$ and $\Pi \vdash^\var \Pi''$.\footnote{Go to proof}
K.3 Completeness of Typing

7. If $\Omega \parallel \Pi \Downarrow \Pi'$ then there is $\Pi''$ such that $\Omega \parallel \Pi'' = \Pi'$ and $\Pi \Downarrow \Pi''$.

8. If $\Omega \parallel \Pi \Downarrow \Pi'$ then there is $\Pi''$ such that $\Omega \parallel \Pi'' = \Pi'$ and $\Pi \Downarrow \Pi''$.

9. If $\Omega \parallel \Pi \Downarrow \Pi_1 \parallel \Pi_2'$ then there are $\Pi_1$ and $\Pi_2$ such that $\Omega \parallel \Pi_1 = \Pi_1'$ and $\Omega \parallel \Pi_2 = \Pi_2'$ and $\Pi \Downarrow \Pi_1 \parallel \Pi_2$.

10. If $\Omega \parallel \Pi \Downarrow \Pi_1 \parallel \Pi_2'$ then there are $\Pi_1$ and $\Pi_2$ such that $\Omega \parallel \Pi_1 = \Pi_1'$ and $\Omega \parallel \Pi_2 = \Pi_2'$ and $\Pi \Downarrow \Pi_1 \parallel \Pi_2$.

Lemma 101 (Pattern Decomposition Functionality).

1. If $\Pi \Downarrow \Pi'$ and $\Pi \Downarrow \Pi''$ then $\Pi' = \Pi''$.
2. If $\Pi \Downarrow \Pi'$ and $\Pi \Downarrow \Pi''$ then $\Pi' = \Pi''$.
3. If $\Pi \Downarrow \Pi'$ and $\Pi \Downarrow \Pi''$ then $\Pi' = \Pi''$.
4. If $\Pi \Downarrow \Pi_1$ then $\Pi_1 = \Pi_1'$ and $\Pi_2 = \Pi_2'$.
5. If $\Pi \Downarrow \Pi_1 \parallel \Pi_2$ then $\Pi_1 = \Pi_1'$ and $\Pi_2 = \Pi_2'$.

Lemma 102 (Decidability of Variable Removal).

For all $\Pi$, either there exists a $\Pi'$ such that $\Pi \Downarrow \Pi'$ or there does not.

Lemma 103 (Variable Inversion).

1. If $\Pi \Downarrow \Pi'$ and $\Gamma \vdash \Pi$ covers $\Lambda, \bar{\Lambda} q$ then $\Gamma \vdash \Pi'$ covers $\bar{\Lambda} q$.
2. If $\Pi \Downarrow \Pi'$ and $\Gamma \vdash \Pi$ covers $\Lambda, \bar{\Lambda} q$ then $\Gamma \vdash \Pi'$ covers $\bar{\Lambda} q$.

Theorem 11 (Completeness of Match Coverage).

1. If $\Gamma \vdash \bar{\Lambda} q$ types and $[\Gamma] \bar{\Lambda} = \bar{\Lambda}$ and (for all $\Omega$ such that $\Gamma \rightarrow \Omega$, we have $\Omega \Gamma \vdash [\Omega] \Pi$ covers $[\Omega] \bar{\Lambda} q$) then $\Gamma \vdash \Pi$ covers $\bar{\Lambda} q$.
2. If $[\Gamma] \bar{\Lambda} = \bar{\Lambda}$ and $[\Gamma] \bar{\Lambda} !$ types and (for all $\Omega$ such that $\Gamma \rightarrow \Omega$, we have $\Omega \Gamma / \Omega \Pi$ covers $[\Omega] \bar{\Lambda} !$) then $\Gamma \vdash \Pi$ covers $\bar{\Lambda} !$.

Theorem 12 (Completeness of Algorithmic Typing).

(i) If $\Gamma \vdash \bar{\Lambda} q$ type and $[\Gamma] \bar{\Lambda} \vdash [\Omega] e \leftrightarrow [\Omega] \bar{\Lambda} q$ and $p' \subseteq p$ then there exist $\Delta$ and $\Omega'$ such that $\Delta \rightarrow \Omega'$ and $\text{dom}(\Delta) = \text{dom}(\Omega')$ and $\Omega \rightarrow \Omega'$ and $\Gamma \vdash e \leftrightarrow [\Gamma] \bar{\Lambda} p' \rightarrow \Delta$.

(ii) If $\Gamma \vdash \bar{\Lambda} q$ type and $[\Gamma] \bar{\Lambda} \vdash [\Omega] e \Rightarrow \bar{\Lambda} p$ then there exist $\Delta, \Omega', \bar{A}'$, and $p' \subseteq p$ such that $\Delta \rightarrow \Omega'$ and $\text{dom}(\Delta) = \text{dom}(\Omega')$ and $\Omega \rightarrow \Omega'$ and $\Gamma \vdash e \Rightarrow \bar{A}' p' \rightarrow \Delta$ and $\bar{A}' = [\Delta] \bar{A}'$ and $\bar{A} = [\Omega'] \bar{A}'$.

(iii) If $\Gamma \vdash \bar{\Lambda} q$ type and $[\Gamma] \bar{\Lambda} \vdash [\Omega] s : [\Omega] \bar{\Lambda} p \Rightarrow B q$ and $p' \subseteq p$ then there exist $\Delta, \Omega', B'$, and $q' \subseteq q$ such that $\Delta \rightarrow \Omega'$ and $\text{dom}(\Delta) = \text{dom}(\Omega')$ and $\Omega \rightarrow \Omega'$ and $\Gamma \vdash s : [\Gamma] \bar{\Lambda} p' \Rightarrow B' q' \rightarrow \Delta$ and $B' = [\Delta] B'$ and $B = [\Omega'] B'$.

(iv) If $\Gamma \vdash \bar{\Lambda} q$ type and $[\Gamma] \bar{\Lambda} \vdash [\Omega] s : [\Omega] \bar{\Lambda} p \Rightarrow B q$ and $p' \subseteq p$ then there exist $\Delta, \Omega', B'$, and $q' \subseteq q$ such that $\Delta \rightarrow \Omega'$ and $\text{dom}(\Delta) = \text{dom}(\Omega')$ and $\Omega \rightarrow \Omega'$ and $\Gamma \vdash s : [\Gamma] \bar{\Lambda} p' \Rightarrow B' q' \rightarrow \Delta$ and $B' = [\Delta] B'$ and $B = [\Omega'] B'$.
(v) If \( \Gamma \vdash \vec{A} \) ! types and \( \Gamma \vdash C \) p type and \( [\Omega] \Gamma \vdash [\Omega] \Pi :: [\Omega] \vec{A} \ q \Leftarrow [\Omega] C \ p \) and \( p' \sqsubseteq p \) then there exist \( \Delta, \Omega' \), and \( C \) such that \( \Delta \rightarrow \Omega' \) and \( \text{dom}(\Delta) = \text{dom}(\Omega') \) and \( \Omega \rightarrow \Omega' \) and \( \Gamma \vdash \Pi :: [\Gamma] \vec{A} \ q \Leftarrow [\Gamma] C \ p' \rightarrow \Delta \).

(vi) If \( \Gamma \vdash \vec{A} \) ! types and \( \Gamma \vdash P \) prop and \( \text{FEV}(P) = 0 \) and \( \Gamma \vdash C \) p type and \( [\Omega] \Gamma \vdash [\Omega] \Pi :: [\Omega] \vec{A} \ ! \Leftarrow [\Omega] C \ p \) and \( p' \sqsubseteq p \) then there exist \( \Delta, \Omega' \), and \( C \) such that \( \Delta \rightarrow \Omega' \) and \( \text{dom}(\Delta) = \text{dom}(\Omega') \) and \( \Omega \rightarrow \Omega' \) and \( \Gamma \vdash \Pi :: [\Gamma] \vec{A} \ ! \Leftarrow [\Gamma] C \ p' \rightarrow \Delta \).
Proofs

In the rest of this document, we prove the results stated above, with the same sectioning.

A’ Properties of the Declarative System

Lemma 1 (Declarative Well-foundedness).
The inductive definition of the following judgments is well-founded:

(i) synthesis $\Gamma \vdash e \Rightarrow B \ p$
(ii) checking $\Gamma \vdash e \Leftarrow A \ p$
(iii) checking, equality elimination $\Gamma / P \vdash e \Leftarrow C \ p$
(iv) ordinary spine $\Gamma \vdash s : A \ p \gg B \ q$
(v) recovery spine $\Gamma \vdash s : A \ p \gg B \ [q]$
(vi) pattern matching $\Gamma \vdash \Pi :: \vec{A} ! \Leftarrow C \ p$
(vii) pattern matching, equality elimination $\Gamma / P \vdash \Pi :: \vec{A} ! \Leftarrow C \ p$

Proof. Let $|e|$ be the size of the expression $e$. Let $|s|$ be the size of the spine $s$. Let $|\Pi|$ be the size of the branch list $\Pi$. Let $\#\text{large}(A)$ be the number of “large” connectives $\forall, \exists, \top, \bot, \land$ in $A$.

First, stratify judgments by the size of the term (expression, spine, or branches), and say that a judgment is at $n$ if it types a term of size $n$. Order the main judgment forms as follows:

\[
\begin{align*}
\text{synthesis judgment at } n & < \text{ checking judgments at } n \\
& < \text{ ordinary spine judgment at } n \\
& < \text{ recovery spine judgment at } n \\
& < \text{ match judgments at } n \\
& < \text{ synthesis judgment at } n + 1 \\
& \vdots
\end{align*}
\]

Within the checking judgment forms at $n$, we compare types lexicographically, first by the number of large connectives, and then by the ordinary size. Within the match judgment forms at $n$, we compare using a lexicographic order of, first, $\#\text{large}(\vec{A})$; second, the judgment form, considering the match judgment to be smaller than the matchelim judgment; third, the size of $\vec{A}$. These criteria order the judgments as follows:

\[
\begin{align*}
\text{synthesis judgment at } n & < \text{ (checking judgment at } n \text{ with } \#\text{large}(A) = 1 \\
& < \text{ checkelim judgment at } n \text{ with } \#\text{large}(A) = 1 \\
& < \text{ checking judgment at } n \text{ with } \#\text{large}(A) = 2 \\
& < \text{ checkelim judgment at } n \text{ with } \#\text{large}(A) = 2 \\
& < \text{ checkelim judgment at } n \text{ with } \#\text{large}(A) = 2 \\
& \vdots
\end{align*}
\]

\[
\begin{align*}
& < \text{ (match judgment at } n \text{ with } \#\text{large}(\vec{A}) = 1 \text{ and } \vec{A} \text{ of size } 1 \\
& < \text{ match judgment at } n \text{ with } \#\text{large}(\vec{A}) = 1 \text{ and } \vec{A} \text{ of size } 2 \\
& < \text{ matchelim judgment at } n \text{ with } \#\text{large}(\vec{A}) = 1 \\
& < \text{ match judgment at } n \text{ with } \#\text{large}(\vec{A}) = 2 \text{ and } \vec{A} \text{ of size } 1 \\
& < \text{ match judgment at } n \text{ with } \#\text{large}(\vec{A}) = 2 \text{ and } \vec{A} \text{ of size } 2 \\
& < \text{ matchelim judgment at } n \text{ with } \#\text{large}(\vec{A}) = 2 \\
& < \text{ ...}
\end{align*}
\]
The class of ordinary spine judgments at 1 need not be refined, because the only ordinary spine rule applicable to a spine of size 1 is `DeclEmptySpine` which has no premises; rules `DeclSpine`, `Decl→Spine`, and `Decl→Spine` are restricted to non-empty spines and can only apply to larger terms.

Similarly, the class of match judgments at 1 need not be refined, because only `DeclMatchEmpty` is applicable.

Note that we distinguish the “checkelim” form `Ψ / P ⊢ e ⇐ A p` of the checking judgment. We also define the size of an expression `e` to consider all types in annotations to be of the same size, that is,

\[ |e : A| = |e| + 1 \]

Thus, `|θ(e)| = |e|`, even when `e` has annotations. This is used for `DeclCheckUnify`, see below.

We assume that coverage, which does not depend on any other typing judgments, is well-founded. We likewise assume that subtyping, `Ψ ⊢ A type`, `Ψ ⊢ τ : κ`, and `Ψ ⊢ P prop` are well-founded.

We now show that, for each class of judgments, every judgment in that class depends only on smaller judgments.

- **Synthesis judgments**
  
  **Claim:** For all `n`, synthesis at `n` depends only on judgments at `n − 1` or less.
  
  **Proof.** Rule `DeclVar` has no premises.
  Rule `DeclAnno` depends on a premise at a strictly smaller term.
  Rule `Decl→E` depends on (1) a synthesis premise at a strictly smaller term, and (2) a recovery spine judgment at a strictly smaller term.

- **Checking judgments**
  
  **Claim:** For all `n ≥ 1`, the checking judgment over terms of size `n` with type of size `m` depends only on
  
  (1) synthesis judgments at size `n` or smaller, and
  (2) checking judgments at size `n − 1` or smaller, and
  (3) checking judgments at size `n` with fewer large connectives, and
  (4) checkelim judgments at size `n` with fewer large connectives, and
  (5) match judgments at size `n − 1` or smaller.
  
  **Proof.** Rule `DeclSub` depends on a synthesis judgment of size `n`. (1)
  Rule `DeclI` has no premises.
  Rule `DeclV` depends on a checking judgment at `n` with fewer large connectives. (3)
  Rule `DeclI` depends on a checking judgment at `n` with fewer large connectives. (3)
  Rule `Decl∧I` depends on a checking judgment at `n` with fewer large connectives. (3)
  Rule `Decl→I` depends on a checkelim judgment at `n` with fewer large connectives. (4)
  Rules `Decl→I`, `DeclRec`, `Decl+I`, `Decl×I`, and `DeclCons` depend on checking judgments at size `< n`. (2)
  Rule `DeclNil` depends only on an auxiliary judgment.
  Rule `DeclCase` depends on:
  
  - a synthesis judgment at size `n` (1),
  - a match judgment at size `< n` (5), and
  - a coverage judgment.

- **Checkelim judgments**
  
  **Claim:** For all `n ≥ 1`, the checkelim judgment `Ψ / P ⊢ e ⇐ A p` over terms of size `n` depends only on checking judgments at size `n`, with a type `A'` such that `#large(A') = #large(A)`.
  
  **Proof.** Rule `DeclCheck⊥` has no nontrivial premises.
  Rule `DeclCheckUnify` depends on a checking judgment: Since `|θ(e)| = |e|`, this checking judgment is at `n`. Since the mgu θ is over monotypes, `#large(θ(A)) = #large(A)`. 


Proof of **Lemma 1** (Declarative Well-foundedness).

- **Ordinary spine judgments**
  
  An ordinary spine judgment at 1 depends on no other judgments: the only spine of size 1 is the empty spine, so only \texttt{DeclEmptySpine} applies, and it has no premises.

  **Claim:** For all \( n \geq 2 \), the ordinary spine judgment \( \Psi \vdash s : A \triangleright C q \) over spines of size \( n \) depends only on

  (a) checking judgments at size \( n - 1 \) or smaller, and

  (b) ordinary spine judgments at size \( n - 1 \) or smaller, and

  (c) ordinary spine judgments at size \( n \) with strictly smaller \#large(\( A \)).

  **Proof.** Rule \texttt{Decl\trianglerightSpine} depends on an ordinary spine judgment of size \( n \), with a type that has fewer large connectives. (c)

  Rule \texttt{Decl\triangleright\trianglerightSpine} depends on an ordinary spine judgment of size \( n \), with a type that has fewer large connectives. (c)

  Rule \texttt{DeclEmptySpine} has no premises.

  Rule \texttt{Decl\rightarrowSpine} depends on a checking judgment of size \( n - 1 \) or smaller (a) and an ordinary spine judgment of size \( n - 1 \) or smaller (b).

- **Recovery spine judgments**

  **Claim:** For all \( n \), the recovery spine judgment at \( n \) depends only on ordinary spine judgments at \( n \).

  **Proof.** Rules \texttt{DeclSpineRecover} and \texttt{DeclSpinePass} depend only on ordinary spine judgments at \( n \).

- **Match judgments**

  **Claim:** For all \( n \geq 1 \), the match judgment \( \Psi \vdash \Pi :: \vec{A} ! \iff C p \) over \( \Pi \) of size \( n \) depends only on

  (a) checking judgments at size \( n - 1 \) or smaller, and

  (b) match judgments at size \( n - 1 \) or smaller, and

  (c) match judgments at size \( n \) with smaller \( \vec{A} \), and

  (d) matchelim judgments at size \( n \) with fewer large connectives in \( \vec{A} \).

  **Proof.** Rule \texttt{DeclMatchEmpty} has no premises.

  Rule \texttt{DeclMatchSeq} depends on match judgments at \( n - 1 \) or smaller (b).

  Rule \texttt{DeclMatchBase} depends on a checking judgment at \( n - 1 \) or smaller (a).

  Rules \texttt{DeclMatchUnit}, \texttt{DeclMatch\times}, \texttt{DeclMatch+k}, \texttt{DeclMatchNeg}, and \texttt{DeclMatchWild} depend on match judgments at \( n - 1 \) or smaller (b).

  Rule \texttt{DeclMatch∃} depends on a match judgment at size \( n \) with smaller \( \vec{A} \) (c).

  Rule \texttt{DeclMatch∧} depends on an matchelim judgment at \( n \) with fewer large connectives in \( \vec{A} \). (d)

- **Matchelim judgments**

  **Claim:** For all \( n \geq 1 \), the matchelim judgment \( \Psi / \Pi \vdash P :: \vec{A} ! \iff C p \) over \( \Psi \) of size \( n \) depends only on match judgments with the same number of large connectives in \( \vec{A} \).

  **Proof.** Rule \texttt{DeclMatch⊥} has no nontrivial premises.

  Rule \texttt{DeclMatchUnify} depends on a match judgment with the same number of large connectives (similar to \texttt{DeclCheckUnify} considered above).

**Lemma 2** (Declarative Weakening).

(i) If \( \Psi_0, \Psi_1 \vdash t : \kappa \) then \( \Psi_0, \Psi, \Psi_1 \vdash t : \kappa \).

(ii) If \( \Psi_0, \Psi_1 \vdash P \text{ prop} \) then \( \Psi_0, \Psi, \Psi_1 \vdash P \text{ prop} \).

(iii) If \( \Psi_0, \Psi_1 \vdash P \text{ true} \) then \( \Psi_0, \Psi, \Psi_1 \vdash P \text{ true} \).

(iv) If \( \Psi_0, \Psi_1 \vdash A \text{ type} \) then \( \Psi_0, \Psi, \Psi_1 \vdash A \text{ type} \).
Proof. By induction on the derivation.

Lemma 3 (Declarative Term Substitution). Suppose $\Psi \vdash t : \kappa$. Then:

1. If $\Psi_0, \alpha : \kappa, \Psi_1 \vdash t' : \kappa$ then $\Psi_0, [t/\alpha] \Psi_1 \vdash [t/\alpha] t' : \kappa$.

2. If $\Psi_0, \alpha : \kappa, \Psi_1 \vdash P$ prop then $\Psi_0, [t/\alpha] \Psi_1 \vdash [t/\alpha] P$ prop.

3. If $\Psi_0, \alpha : \kappa, \Psi_1 \vdash A$ type then $\Psi_0, [t/\alpha] \Psi_1 \vdash [t/\alpha] A$ type.

4. If $\Psi_0, \alpha : \kappa, \Psi_1 \vdash A \leq^P B$ then $\Psi_0, [t/\alpha] \Psi_1 \vdash [t/\alpha] A \leq^P [t/\alpha] B$.

5. If $\Psi_0, \alpha : \kappa, \Psi_1 \vdash P$ true then $\Psi_0, [t/\alpha] \Psi_1 \vdash [t/\alpha] P$ true.

Proof. By induction on the derivation of the substitutee.

Lemma 4 (Reflexivity of Declarative Subtyping).

Given $\Psi \vdash A$ type, we have that $\Psi \vdash A \leq^P A$.

Proof. By induction on $A$, writing $p$ for the sign of the subtyping judgment.

Our induction metric is the number of quantifiers on the outside of $A$, plus one if the polarity of $A$ and the subtyping judgment do not match up (that is, if $\text{neg}(A)$ and $P = +$, or $\text{pos}(A)$ and $P = -$).

- **Case nonpos($A$), nonneg($A$):**
  By rule $\leq\text{Refl}P$.

- **Case $A = \exists b : \kappa. B$ and $P = +$:**
  $\Psi, b : \kappa \vdash B \leq^+ B$
  $\Psi, b : \kappa \vdash b : \kappa$
  $\Psi, b : \kappa \vdash \exists b : \kappa. B$
  By rule $\leq\exists R$
  $\Psi \vdash \exists b : \kappa. B \leq^+ \exists b : \kappa. B$
  By rule $\leq\exists L$

- **Case $A = \exists b : \kappa. B$ and $P = -:**$
  $\Psi \vdash \exists b : \kappa. B \leq^- \exists b : \kappa. B$
  By i.h. (polarities match)

- **Case $A = \forall b : \kappa. B$ and $P = +$:**
  $\Psi \vdash \forall b : \kappa. B \leq^+ \forall b : \kappa. B$
  By i.h. (polarities match)

- **Case $A = \forall b : \kappa. B$ and $P = -:**$
  $\Psi, b : \kappa \vdash B \leq^- B$
  $\Psi, b : \kappa \vdash b : \kappa$
  $\Psi, b : \kappa \vdash \forall b : \kappa. B \leq^- \forall b : \kappa. B$
  By rule $\leq\forall L$

- **Case $A \vdash \exists \alpha : \kappa. A \leq^+ B$ then $\Psi, \alpha : \kappa \vdash A \leq^+ B$.**

- **Case $A \vdash \forall \beta : \kappa. B$ then $\Psi, \beta : \kappa \vdash A \leq^- B$.**

Lemma 5 (Subtyping Inversion).

$\Psi \vdash \exists \alpha : \kappa. A \leq^+ B$ then $\Psi, \alpha : \kappa \vdash A \leq^+ B$.
Proof. By a routine induction on the subtyping derivations.

Lemma 6 (Subtyping Polarity Flip).

• If \(\text{nonpos}(A)\) and \(\text{nonpos}(B)\) and \(\Gamma \vdash A \leq^+ B\) then \(\Gamma \vdash A \leq^- B\) by a derivation of the same or smaller size.

• If \(\text{nonneg}(A)\) and \(\text{nonneg}(B)\) and \(\Gamma \vdash A \leq^- B\) then \(\Gamma \vdash A \leq^+ B\) by a derivation of the same or smaller size.

• If \(\text{nonpos}(A)\) and \(\text{nonneg}(A)\) and \(\text{nonpos}(B)\) and \(\text{nonneg}(B)\) and \(\Gamma \vdash A \leq^P B\) then \(A = B\).

Proof. By a routine induction on the subtyping derivations.

Lemma 7 (Transitivity of Declarative Subtyping).

Given \(\Gamma \vdash A \text{ type}\) and \(\Gamma \vdash B \text{ type}\) and \(\Gamma \vdash C \text{ type}\):

(i) If \(D_1 : \Gamma \vdash A \leq^P B\) and \(D_2 : \Gamma \vdash B \leq^P C\) then \(\Gamma \vdash A \leq^P C\).

Proof. By lexicographic induction on (1) the sum of head quantifiers in \(A\), \(B\), and \(C\), and (2) the size of the derivation.

We begin by case analysis on the shape of \(B\), and the polarity of subtyping:

• Case \(B = \forall \beta : \kappa_2 . B'\), polarity = \(-\):
  We case-analyze \(D_1\):

  \[
  \begin{array}{c}
  \frac{
  \begin{array}{c}
  \Gamma \vdash \tau : \kappa_1 \\
  \Gamma \vdash \lfloor \tau/\alpha \rfloor A' \leq^- B \\
  \end{array} 
  }{
  \Gamma \vdash \forall \alpha : \kappa_1 . A' \leq^- B \quad \text{\#VL}
  }
  \end{array}
  \]

  \[
  \begin{array}{c}
  \begin{array}{c}
  \Gamma \vdash \tau : \kappa_1 \\
  \Gamma \vdash \lfloor \tau/\alpha \rfloor A' \leq^- B \\
  \Gamma \vdash B \leq^- C \\
  \end{array} 
  \begin{array}{c}
  \Gamma \vdash (\lfloor \tau/\alpha \rfloor A') \leq^- C \\
  \Gamma \vdash A \leq^- C 
  \end{array} 
  \end{array}
  \]

  By i.h. (\(A\) lost a quantifier)

  \[
  \begin{array}{c}
  \frac{
  \begin{array}{c}
  \Gamma \vdash \tau : \kappa_2 \\
  \Gamma \vdash \lfloor \tau/\beta \rfloor B' \leq^- C \\
  \end{array} 
  }{
  \Gamma \vdash \forall \beta : \kappa_2 . B' \leq^- \#VR
  }
  \end{array}
  \]

We case-analyze \(D_2\):

* Case \(\begin{array}{c}
\frac{
\begin{array}{c}
\Gamma \vdash \tau : \kappa_2 \\
\Gamma \vdash \lfloor \tau/\beta \rfloor B' \leq^- C \\
\end{array}
\end{array} 
\frac{
\begin{array}{c}
\Gamma \vdash \forall \beta : \kappa_2 . B' \leq^- C \\
\end{array}
\text{\#VL}
\end{array}\]

  By Lemma 5 (Subtyping Inversion) on \(D_1\)

  \[
  \begin{array}{c}
  \begin{array}{c}
  \Gamma, \beta : \kappa_2 \vdash A \leq^- B' \\
  \end{array} 
  \begin{array}{c}
  \Gamma \vdash \tau : \kappa_2 \\
  \Gamma \vdash \lfloor \tau/\beta \rfloor B' \leq^- C \\
  \end{array} 
  \begin{array}{c}
  \Gamma \vdash A \leq^- (\lfloor \tau/\beta \rfloor B') \\
  \Gamma \vdash A \leq^- C \\
  \end{array} 
  \end{array}
  \]

  By rule \(\leq^\text{\#VR}\)

Proof of Lemma 7 (Transitivity of Declarative Subtyping) lem:declarative-transitivity
* Case \( \Psi, c : \kappa_3 \vdash B \leq C' \)
  \( \Psi \vdash B \leq \forall c : \kappa_3. C' \) \( \leq \forall R \)

  \( \Psi \vdash A \leq B \) \( \text{Given} \)
  \( \Psi, c : \kappa_3 \vdash A \leq B \) \( \text{By Lemma 2 (Declarative Weakening)} \)
  \( \Psi, c : \kappa_3 \vdash B \leq C' \) \( \text{Subderivation} \)
  \( \Psi, c : \kappa_3 \vdash A \leq C' \) \( \text{By i.h. (C lost a quantifier)} \)
  \( \Psi \vdash B \leq \forall c : \kappa_3. C' \) \( \text{By } \leq \forall R \)

- Case nonpos(B), polarity = +:
  Now we case-analyze \( D_1 \):

  - Case \( \Psi, \alpha : \tau \vdash A' \leq^+ B \)
    \( \Psi \vdash \exists \alpha : \kappa_1. A' \leq^+ B \) \( \leq \exists L \)

    \( \Psi, \alpha : \tau \vdash A' \leq^+ B \) \( \text{Subderivation} \)
    \( \Psi, \alpha : \tau \vdash B \leq^+ C \) \( \text{By Lemma 2 (Declarative Weakening)} (D_2) \)
    \( \Psi, \alpha : \tau \vdash A' \leq^+ C \) \( \text{By i.h. (A lost a quantifier)} \)
    \( \Psi \vdash \exists \alpha : \kappa_1. A' \leq^+ C \) \( \text{By } \leq \exists L \)

  - Case \( \Psi \vdash A \leq^+ B \) \( \text{nonpos}(A) \) \( \text{nonpos}(B) \)
    \( \Psi \vdash A \leq^+ B \) \( \leq \)

  Now we case-analyze \( D_2 \):

  * Case \( \Psi, \tau : \kappa_3 \vdash B \leq^+ [\tau/c]C' \)
    \( \Psi \vdash B \leq^+ \exists c : \kappa_3. C' \) \( \leq \exists R \)

    \( \Psi \vdash A \leq^+ B \) \( \text{Given} \)
    \( \Psi \vdash \tau : \kappa_3 \) \( \text{Subderivation of } D_2 \)
    \( \Psi \vdash B \leq^+ [\tau/c]C' \) \( \text{Subderivation of } D_2 \)
    \( \Psi \vdash A \leq^+ [\tau/c]C' \) \( \text{By i.h. (C lost a quantifier)} \)
    \( \Psi \vdash A \leq^+ \exists c : \kappa_3. C' \) \( \text{By } \leq \exists R \)

  * Case \( \Psi \vdash B \leq^+ C \) \( \text{nonpos}(B) \) \( \text{nonpos}(C) \)
    \( \Psi \vdash B \leq^+ C \) \( \leq \)

    \( \Psi \vdash A \leq^+ B \) \( \text{Subderivation of } D_1 \)
    \( \Psi \vdash B \leq^+ C \) \( \text{Subderivation of } D_2 \)
    \( \Psi \vdash A \leq^+ C \) \( \text{By } \leq \)
• Case $B = \exists \beta : \kappa_2. B'$, polarity $= +$:

Now we case-analyze $D_2$:

1. **Case**

   \[
   \frac{
   \psi \vdash \tau : \kappa_3 \quad \psi \vdash B \leq^+ [\tau/\alpha] C' \\
   }{
   \psi \vdash B \leq^+ \exists \alpha : \kappa_3. C' 
   }_{\leq \exists \forall R}
   \]

   Subderivation of $D_2$

   \[
   \frac{
   \psi \vdash B \leq^+ [\tau/\alpha] C' \\
   }{
   \psi \vdash_A \leq^+ [\tau/\alpha] C' 
   }_{\leq \exists \forall R}
   \]

   Subderivation of $D_2$

   \[
   \psi \vdash A \leq^+ B 
   \]

   Given

   \[
   \psi \vdash A \leq^+ [\tau/\alpha] C' 
   \]

   By i.h. (C lost a quantifier)

   \[
   \psi \vdash A \leq^+ C 
   \]

   By rule $\leq \exists \forall R$

2. **Case**

   \[
   \frac{
   \psi, \beta : \kappa_2 \vdash B' \leq^+ C \\
   }{
   \psi \vdash \exists \beta : \kappa_2. B' \leq^+ C 
   }_{\leq \exists \forall L}
   \]

   Subderivation of $D_2$

   \[
   \frac{
   \psi \vdash \tau : \kappa_2 
   }{
   \psi \vdash A \leq^+ [\tau/\beta] B' 
   }_{\leq \exists \forall R}
   \]

   Subderivation of $D_1$

   \[
   \frac{
   \psi \vdash [\tau/\beta] B' \leq^+ C \\
   }{
   \psi \vdash_A \leq^+ [\tau/\beta] B' 
   }_{\leq \exists \forall R}
   \]

   By Lemma 3 (Declarative Term Substitution)

   \[
   \frac{
   \psi \vdash A \leq^+ C \\
   }{
   \psi \vdash \exists \alpha : \kappa_1. A' \leq^+ C 
   }_{\leq \exists \forall L}
   \]

   By i.h. (B lost a quantifier)

   By Lemma 2 (Declarative Weakening)

   \[
   \frac{
   \psi, \alpha : \kappa_1 \vdash A \leq^+ B 
   }{
   \psi \vdash \exists \alpha : \kappa_1. A' \leq^+ B 
   }_{\leq \exists \forall L}
   \]

   By i.h. (A lost a quantifier)

\[
\]

\[
\]

• Case $\text{nonneg}(B)$, polarity $= -$:

We case-analyze $D_2$:

1. **Case**

   \[
   \frac{
   \psi, c : \kappa_3 \vdash B \leq^+ C' \\
   }{
   \psi \vdash B \leq^+ \exists c : \kappa_3. C' 
   }_{\leq \forall \exists R}
   \]

   Subderivation of $D_2$

   \[
   \frac{
   \psi, c : \kappa_3 \vdash A \leq^+ B 
   }{
   \psi \vdash A \leq^+ \forall c : \kappa_3. C' 
   }_{\leq \forall \exists R}
   \]

   By Lemma 2 (Declarative Weakening)

   \[
   \frac{
   \psi, c : \kappa_3 \vdash A \leq^+ C' 
   }{
   \psi \vdash A \leq^+ \forall c : \kappa_3. C' 
   }_{\leq \forall \exists R}
   \]

   By Lemma 2 (Declarative Weakening)

   By i.h. (C lost a quantifier)
Proof of Lemma 7 (Transitivity of Declarative Subtyping). 

\[ \Gamma \vdash A \; \text{type} \quad \text{and} \quad \Gamma \vdash B \; \text{type} \quad \text{then} \quad \Gamma \vdash [\tau / \alpha]A \; \text{type}. \]

\[ \text{Lemma 8 (Substitution—Well-formedness).} \]

(i) If \( \Gamma \vdash A \; \text{type} \) and \( \Gamma \vdash \tau \; \text{type} \) then \( \Gamma \vdash [\tau / \alpha]A \; \text{type} \).

(ii) If \( \Gamma \vdash \text{prop} \) and \( \Gamma \vdash \tau \; \text{type} \) then \( \Gamma \vdash [\tau / \alpha]\text{prop} \).

Moreover, if \( \text{prop} = \top \) and \( \text{FEV}(\Gamma|\text{prop}) = 0 \) then \( \text{FEV}(\Gamma|[\tau / \alpha]\text{prop}) = 0 \).

**Proof.** By induction on the derivations of \( \Gamma \vdash A \; \text{type} \) and \( \Gamma \vdash \text{prop} \). 

Lemma 9 (Uvar Preservation).

If \( \Delta \rightarrow \Omega \) then:

(i) If \( (\alpha : \kappa) \in \Omega \) then \( (\alpha : \kappa) \in [\Omega]\Delta \).

(ii) If \( (x : A \; \text{p}) \in \Omega \) then \( (x : [\Omega]A \; \text{p}) \in [\Omega]\Delta \).

**Proof.** By induction on \( \Omega \), following the definition of context application (Figure 13).

Lemma 10 (Sorting Implies Typing). If \( \Gamma \vdash t : \ast \) then \( \Gamma \vdash t \; \text{type} \).

**Proof.** By induction on the given derivation. All cases are straightforward.

Lemma 11 (Right-Hand Substitution for Sorting). If \( \Gamma \vdash t : \kappa \) then \( \Gamma \vdash [\Gamma]t : \kappa \).

**Proof.** By induction on \( |\Gamma|t| \) (the size of \( t \) under \( \Gamma \)).
Proof of Lemma 11 (Right-Hand Substitution for Sorting)

Cases UnitSort: Here \( t = 1 \), so applying \( \Gamma \) to \( t \) does not change it: \( t = [\Gamma]1 \). Since \( \Gamma \vdash t : \kappa \), we have \( \Gamma \vdash [\Gamma]t : \kappa \), which was to be shown.

Case VarSort: If \( t \) is an existential variable \( \exists \alpha \), then \( \Gamma = \Gamma_0[\exists \alpha] \), so applying \( \Gamma \) to \( t \) does not change it, and we proceed as in the UnitSort case above.

If \( t \) is a universal variable \( \alpha \) and \( \Gamma \) has no equation for it, then proceed as in the UnitSort case.

Otherwise, \( t = \alpha \) and \( \{\alpha = \tau\} \in \Gamma \):

\[
\Gamma = (\Gamma_L, \alpha : \kappa, \Gamma_M, \alpha = \tau, \Gamma_R)
\]

By the implicit assumption that \( \Gamma \) is well-formed, \( \Gamma_L, \alpha : \kappa, \Gamma_M \vdash \tau : \kappa \). By Lemma 34 (Suffix Weakening), \( \Gamma \vdash \tau : \kappa \). Since \( |\Gamma|_\tau < |\Gamma|_\alpha \), we can apply the i.h., giving

\( \Gamma \vdash [\Gamma]\tau : \kappa \)

By the definition of substitution, \( [\Gamma]\tau = [\Gamma]\alpha \), so we have \( \Gamma \vdash [\Gamma]\alpha : \kappa \).

Case SolvedVarSort: In this case \( t = \exists \alpha \) and \( \Gamma = (\Gamma_L, \exists \alpha = \tau, \Gamma_R) \). Thus \( [\Gamma]t = [\Gamma]\exists \alpha \). We assume contexts are well-formed, so all free variables in \( \tau \) are declared in \( \Gamma_L \). Consequently, \( |\Gamma|_\exists \alpha \), which is less than \( |\Gamma|_\alpha \). We can therefore apply the i.h. to \( \tau \), yielding \( \Gamma \vdash [\Gamma]\tau : \kappa \). By the definition of substitution, \( [\Gamma]\tau = [\Gamma]\exists \alpha \), so we have \( \Gamma \vdash [\Gamma]\exists \alpha : \kappa \).

Case BinSort: In this case \( t = t_1 \oplus t_2 \). By i.h., \( \Gamma \vdash [\Gamma]t_1 : \kappa \) and \( \Gamma \vdash [\Gamma]t_2 : \kappa \). By BinSort, \( \Gamma \vdash ([\Gamma]t_1) \oplus ([\Gamma]t_2) : \kappa \). By the definition of substitution is \( \Gamma \vdash [\Gamma](t_1 \oplus t_2) : \kappa \). □

Lemma 12 (Right-Hand Substitution for Propositions). If \( \Gamma \vdash P \) prop then \( \Gamma \vdash [\Gamma]P \) prop.

Proof. Use inversion (EqProp), apply Lemma 11 (Right-Hand Substitution for Sorting) to each premise, and apply EqProp again. □

Lemma 13 (Right-Hand Substitution for Typing). If \( \Gamma \vdash A \) type then \( \Gamma \vdash [\Gamma]A \) type.

Proof. By induction on \( |\Gamma|_A \) (the size of \( A \) under \( \Gamma \)). Several cases correspond to cases in the proof of Lemma 11 (Right-Hand Substitution for Sorting):

- the case for UnitWF is like the case for UnitSort
- the case for SolvedVarSort is like the cases for VarWF and SolvedVarWF
- the case for VarSort is like the case for VarWF, but in the last subcase, apply Lemma 10 (Sorting Implies Typing) to move from a sorting judgment to a typing judgment.
- the case for BinWF is like the case for BinSort

Now, the new cases:

Case ForallWF: In this case \( A = \forall \alpha : \kappa.A_0 \). By i.h., \( \Gamma, \alpha : \kappa \vdash [\Gamma, \alpha : \kappa]A_0 \) type. By the definition of substitution, \( [\Gamma, \alpha : \kappa]A_0 = [\Gamma]A_0 \), so by ForallWF \( \Gamma \vdash \forall \alpha. [\Gamma]A_0 \) type, which by the definition of substitution is \( \Gamma \vdash [\Gamma](\forall \alpha. A_0) \) type.

Case ExistsWF: Similar to the ForallWF case.

Case ImpliesWF, WithWF: Use the i.h. and Lemma 12 (Right-Hand Substitution for Propositions), then apply ImpliesWF or WithWF. □

Lemma 14 (Substitution for Sorting). If \( \Omega \vdash t : \kappa \) then \( [\Omega]\Omega \vdash [\Omega]t : \kappa \).

Proof. By induction on \( |\Omega|_t \) (the size of \( t \) under \( \Omega \)).
Proof of Lemma 14 (Substitution for Sorting)

- Case $u : \kappa \in \Omega$
  \[ \Omega \vdash u : \kappa \]

We have a complete context $\Omega$, so $u$ cannot be an existential variable: it must be some universal variable $\alpha$.

If $\Omega$ lacks an equation for $\alpha$, use Lemma 9 (Uvar Preservation) and apply rule UvarSort.

Otherwise, $(\alpha = \tau \in \Omega$, so we need to show $\Omega \vdash [\Omega] \tau : \kappa$. By the implicit assumption that $\Omega$ is well-formed, plus Lemma 34 (Suffix Weakening), $\Omega \vdash \tau : \kappa$. By Lemma 11 (Right-Hand Substitution for Sorting), $\Omega \vdash [\Omega] \tau : \kappa$.

- Case $\delta : \kappa = \tau \in \Omega$
  \[ \Omega \vdash \delta : \kappa \]

Subderivation

\[
\begin{align*}
\delta : \kappa & = \tau \in \Omega & \text{Decomposing } \Omega \\
\Omega & = (\Omega_L, \delta : \kappa = \tau, \Omega_R) & \text{By implicit assumption that } \Omega \text{ is well-formed} \\
\Omega_L, \delta : \kappa & = \tau, \Omega_R \vdash \tau : \kappa & \text{By Lemma 34 (Suffix Weakening)} \\
\Omega_L, \delta : \kappa & = \tau, \Omega_R & \text{By Lemma 11 (Right-Hand Substitution for Sorting)} \\
\end{align*}
\]

\[ [\Omega] \Omega \vdash [\Omega] \delta : \kappa \]

<table>
<thead>
<tr>
<th>Case</th>
<th>$\Omega \vdash 1 : \star$</th>
</tr>
</thead>
</table>

Since $1 = [\Omega] 1$, applying UnitSort gives the result.

- Case $\Omega \vdash \tau_1 : \star$ $\Omega \vdash \tau_2 : \star$
  \[ \Omega \vdash \tau_1 \oplus \tau_2 : \star \]

By i.h. on each premise, rule BinSort and the definition of substitution.

- Case $\Omega \vdash \text{zero} : \mathbb{N}$
  \[ \text{ZeroSort} \]

Since zero $= [\Omega] \text{zero}$, applying ZeroSort gives the result.

- Case $\Omega \vdash t : \mathbb{N}$
  \[ \text{SuccSort} \]

By i.h., rule SuccSort and the definition of substitution.

Lemma 15 (Substitution for Prop Well-Formedness).

If $\Omega \vdash P \text{ prop}$ then $[\Omega] \Omega \vdash [\Omega] P \text{ prop}$.

Proof. Only one rule derives this judgment form:

- Case $\Omega \vdash t : \mathbb{N}$ $\Omega \vdash t' : \mathbb{N}$
  \[ \Omega \vdash t = t' \text{ prop} \]

EqProp
Proof of Lemma 15 (Substitution for Prop Well-Formedness)

\[ \Omega \vdash t : N \] Subderivation
\[ [\Omega], \Omega \vdash [\Omega]t : N \] By Lemma 14 (Substitution for Sorting)
\[ \Omega \vdash t' : N \] Subderivation
\[ [\Omega], \Omega \vdash [\Omega]t' : N \] By Lemma 14 (Substitution for Sorting)
\[ [\Omega], \Omega \vdash (\Omega)t = (\Omega)t' \] \(\text{prop}\) By \(\text{EqProp}\)
\[ [\Omega], \Omega \vdash [\Omega](t = t') \] By def. of subst.

\[ \square \]

Lemma 16 (Substitution for Type Well-Formedness). If \( \Omega \vdash A \) type then \([\Omega], \Omega \vdash [\Omega]A \) type.

Proof. By induction on \( [\Omega] + A \).

Several cases correspond to those in the proof of Lemma 14 (Substitution for Sorting):

- the UnitWF case is like the UnitSort case (using DeclUnitWF instead of UnitSort);
- the VarWF case is like the VarSort case (using DeclUvarWF instead of UvarSort);
- the SolvedVarWF case is like the SolvedVarSort case.

However, uses of Lemma 11 (Right-Hand Substitution for Sorting) are replaced by uses of Lemma 13 (Right-Hand Substitution for Typing).

Now, the new cases:

- Case \( \Omega, \alpha : \kappa \vdash A_0 \) type \( \forall\alpha : \kappa. A_0 \) type \(\text{ForallWF}\)

\[ \Omega, \alpha : \kappa \vdash A_0 : \kappa' \] Subderivation
\[ [\Omega], \alpha : \kappa \vdash [\Omega]A_0 : \kappa' \] By i.h.
\[ [\Omega], \alpha : \kappa \vdash [\Omega]A_0 : \kappa' \] By definition of completion
\[ [\Omega], \Omega \vdash \forall\alpha : \kappa. [\Omega]A_0 : \kappa' \] By DeclAllWF
\[ [\Omega], \Omega \vdash [\Omega]\forall\alpha : \kappa. [\Omega]A_0 : \kappa' \] By def. of subst.

- Case \(\text{ExistsWF}\): Similar to the \(\text{ForallWF}\) case, using DeclExistsWF instead of DeclAllWF.

- Case \( \Omega \vdash A_1 \) type \( \Omega \vdash A_2 \) type \( \Omega \vdash A_1 \oplus A_2 \) type \(\text{BinWF}\)

By i.h. on each premise, rule DeclBinWF and the definition of substitution.

- Case \(\text{VecWF}\): Similar to the \(\text{BinWF}\) case.

- Case \( \Omega \vdash P \) prop \( \Omega \vdash A_0 \) type \( \Omega \vdash P \supset A_0 \) type \(\text{ImpliesWF}\)

\[ \Omega \vdash P \] Subderivation
\[ [\Omega], [\Omega] \vdash [\Omega]P \] By Lemma 15 (Substitution for Prop Well-Formedness)
\[ \Omega \vdash A_0 \] Subderivation
\[ [\Omega], [\Omega] \vdash [\Omega]A_0 \] By i.h.
\[ [\Omega], [\Omega] \vdash (\Omega)P \supset (\Omega)A_0 \] \(\text{prop}\) By DeclImpliesWF
\[ [\Omega], [\Omega] \vdash [\Omega]P \supset [\Omega]A_0 \] By def. of subst.
Proof of Lemma 16 (Substitution for Type Well-Formedness)

**Lemma 17** (Substitution Stability).

If \((\Omega, \Omega_Z)\) is well-formed and \(\Omega \vdash A\) type then \([\Omega]A = [\Omega, \Omega_Z]A\).

**Proof.** By induction on \(\Omega_Z\).

Since \(\Omega_Z\) is soft, either (1) \(\Omega_Z = \cdot\) (and the result is immediate) or (2) \(\Omega_Z = (\Omega', \alpha : \kappa = t)\). However, according to the grammar for complete contexts such as \(\Omega_Z\), (2) is impossible.

By i.h., \([\Omega]A = [\Omega, \Omega']A\). Use the fact that \(\Omega \vdash A\) type implies \(\text{FV}(A) \cap \text{dom}(\Omega_Z) = \emptyset\).

**Lemma 18** (Equal Domains).

If \(\Omega_1 \vdash A\) type and \(\text{dom}(\Omega_1) = \text{dom}(\Omega_2)\) then \(\Omega_2 \vdash A\) type.

**Proof.** By induction on the given derivation.

\[\text{C'} \quad \text{Properties of Extension}\]

**Lemma 19** (Declaration Preservation). If \(\Gamma \rightarrow \Delta\) and \(u\) is declared in \(\Gamma\), then \(u\) is declared in \(\Delta\).

**Proof.** By induction on the derivation of \(\Gamma \rightarrow \Delta\).

- **Case** \(\cdot \rightarrow \text{Id}\)
  
  This case is impossible, since by hypothesis \(u\) is declared in \(\Gamma\).

- **Case** \(\Gamma \rightarrow \Delta, x : A \rightarrow \Delta, x : A'\)
  
  \(- \text{Case } u = x: \text{Immediate.}\)
  
  \(- \text{Case } u \neq x: \text{Since } u\text{ is declared in } (\Gamma, x : A), \text{it is declared in } \Gamma. \text{By i.h., } u\text{ is declared in } \Delta, \text{and therefore declared in } (\Delta, x : A').\)

- **Case** \(\Gamma \rightarrow \Delta\)
  
  \(\Gamma, \alpha : \kappa \rightarrow \Delta, \alpha : \kappa\)

  Similar to the \(\rightarrow \text{Var}\) case.

- **Case** \(\Gamma \rightarrow \Delta\)
  
  \(\Gamma, \alpha : \kappa \rightarrow \Delta, \alpha : \kappa\)

  Similar to the \(\rightarrow \text{Var}\) case.

- **Case** \(\Gamma \rightarrow \Delta, [\Delta]t = [\Delta]t'\)
  
  \(\Gamma, \alpha : \kappa \rightarrow \Delta, \alpha : \kappa = t'\)

  Similar to the \(\rightarrow \text{Var}\) case.
Proof of Lemma 19 (Declaration Preservation)

• Case \(\Gamma \rightarrow \Delta\) \([\Delta]t = [\Delta]t'\) \(\rightarrow\)Eqn

It is given that \(u\) is declared in \((\Gamma, \alpha = t)\). Since \(\alpha = t\) is not a declaration, \(u\) is declared in \(\Gamma\). By i.h., \(u\) is declared in \(\Delta\), and therefore declared in \((\Delta, \alpha = t')\).

• Case \(\Gamma \rightarrow \Delta\) \(\Gamma, \alpha \rightarrow \Delta, \alpha \rightarrow t'\) \(\rightarrow\)Marker

Similar to the \(\rightarrow\)Eqn case.

• Case \(\Gamma \rightarrow \Delta\) \(\Gamma, \beta : \kappa' \rightarrow \Delta, \beta : \kappa = t\) \(\rightarrow\)Solve

Similar to the \(\rightarrow\)Var case.

• Case \(\Gamma \rightarrow \Delta\) \(\Gamma \rightarrow \Delta, \alpha : \kappa \rightarrow t\) \(\rightarrow\)Add

It is given that \(u\) is declared in \(\Gamma\). By i.h., \(u\) is declared in \(\Delta\), and therefore declared in \((\Delta, \alpha : \kappa)\).

• Case \(\Gamma \rightarrow \Delta\) \(\Gamma \rightarrow \Delta, \alpha : \kappa \rightarrow t\) \(\rightarrow\)AddSolved

Similar to the \(\rightarrow\)Add case. \(\square\)

Lemma 20 (Declaration Order Preservation). If \(\Gamma \rightarrow \Delta\) and \(u\) is declared to the left of \(v\) in \(\Gamma\), then \(u\) is declared to the left of \(v\) in \(\Delta\).

Proof. By induction on the derivation of \(\Gamma \rightarrow \Delta\).

• Case \(\vdash \rightarrow\)Id

This case is impossible, since by hypothesis \(u\) and \(v\) are declared in \(\Gamma\).

• Case \(\Gamma \rightarrow \Delta\) \([\Delta]A = [\Delta]A'\) \(\rightarrow\)Var

Consider whether \(v = x\):

– Case \(v = x\):

  It is given that \(u\) is declared to the left of \(v\) in \((\Gamma, x : A)\), so \(u\) is declared in \(\Gamma\).

  By Lemma 19 (Declaration Preservation), \(u\) is declared in \(\Delta\). Therefore \(u\) is declared to the left of \(v\) in \((\Delta, x : A')\).

– Case \(v \neq x\):

  Here, \(v\) is declared in \(\Gamma\). By i.h., \(u\) is declared to the left of \(v\) in \(\Delta\).

  Therefore \(u\) is declared to the left of \(v\) in \((\Delta, x : A')\).

• Case \(\Gamma \rightarrow \Delta\) \(\Gamma, \alpha : \kappa \rightarrow \Delta, \alpha : \kappa\) \(\rightarrow\)Uvar

Similar to the \(\rightarrow\)Var case.
Proof of Lemma 20 (Declaration Order Preservation).

• Case \( \Gamma \rightarrow \Delta \)
  \( \Gamma, \hat{\alpha} : \kappa \rightarrow \Delta, \hat{\alpha} : \kappa \rightarrow \text{Unsolved} \)
  Similar to the \( \rightarrow \text{Var} \) case.

• Case \( \Gamma \rightarrow \Delta \)
  \( |\Delta|t = |\Delta|t' \)
  \( \Gamma, \hat{\alpha} : t \rightarrow \Delta, \hat{\alpha} : t \rightarrow t' \rightarrow \text{Solved} \)
  Similar to the \( \rightarrow \text{Var} \) case.

• Case \( \Gamma \rightarrow \Delta \)
  \( \Gamma, \hat{\beta} : \kappa' \rightarrow \Delta, \hat{\beta} : \kappa' \rightarrow t \rightarrow \text{Solved} \)
  Similar to the \( \rightarrow \text{Var} \) case.

• Case \( \Gamma \rightarrow \Delta \)
  \( |\Delta|t = |\Delta|t' \)
  \( \Gamma, \alpha = t \rightarrow \Delta, \alpha = t' \rightarrow \text{Eqn} \)
  The equation \( \hat{\alpha} = t \) does not declare any variables, so \( u \) and \( v \) must be declared in \( \Gamma \).
  By i.h., \( u \) is declared to the left of \( v \) in \( \Delta \).
  Therefore \( u \) is declared to the left of \( v \) in \( \Delta, \hat{\alpha} : \kappa \rightarrow t' \).

• Case \( \Gamma \rightarrow \Delta \)
  \( \Gamma, \rightarrow \Delta, \rightarrow \text{Marker} \)
  Similar to the \( \rightarrow \text{Eqn} \) case.

• Case \( \Gamma \rightarrow \Delta \)
  \( \Gamma \rightarrow \Delta, \hat{\alpha} : \kappa \rightarrow \text{Add} \)
  By i.h., \( u \) is declared to the left of \( v \) in \( \Delta \).
  Therefore \( u \) is declared to the left of \( v \) in \( \Delta, \hat{\alpha} : \kappa \rightarrow \).

• Case \( \Gamma \rightarrow \Delta \)
  \( \Gamma \rightarrow \Delta, \hat{\alpha} : \kappa \rightarrow \text{AddSolved} \)
  Similar to the \( \rightarrow \text{Add} \) case.

Lemma 21 (Reverse Declaration Order Preservation). If \( \Gamma \rightarrow \Delta \) and \( u \) and \( v \) are both declared in \( \Gamma \) and \( u \) is declared to the left of \( v \) in \( \Delta \), then \( u \) is declared to the left of \( v \) in \( \Gamma \).

Proof. It is given that \( u \) and \( v \) are declared in \( \Gamma \). Either \( u \) is declared to the left of \( v \) in \( \Gamma \), or \( v \) is declared to the left of \( u \). Suppose the latter (for a contradiction). By Lemma 20 (Declaration Order Preservation), \( v \) is declared to the left of \( u \) in \( \Delta \). But we know that \( u \) is declared to the left of \( v \) in \( \Delta \): contradiction. Therefore \( u \) is declared to the left of \( v \) in \( \Gamma \).

Lemma 22 (Extension Inversion).

(i) If \( \mathcal{D} :: \Gamma_0, \alpha : \kappa, \Gamma_1 \rightarrow \Delta \)
    then there exist unique \( \Delta_0 \) and \( \Delta_1 \)
    such that \( \Delta = (\Delta_0, \alpha : \kappa, \Delta_1) \) and \( \mathcal{D}' :: \Gamma_0 \rightarrow \Delta_0 \)
    where \( \mathcal{D}' < \mathcal{D} \).
    Moreover, if \( \Gamma_1 \) is soft, then \( \Delta_1 \) is soft.
(ii) If \( D \vdash \Gamma_0, \alpha, \Gamma_1 \rightarrow \Delta \)
then there exist unique \( \Delta_0 \) and \( \Delta_1 \)
such that \( \Delta = (\Delta_0, \alpha, \Delta_1) \) and \( D' \vdash \Gamma_0 \rightarrow \Delta_0 \) where \( D' < D \).

Moreover, if \( \Gamma_1 \) is soft, then \( \Delta_1 \) is soft.
Moreover, if \( \text{dom}(\Gamma_0, \alpha, \Gamma_1) = \text{dom}(\Delta) \) then \( \text{dom}(\Gamma_0) = \text{dom}(\Delta_0) \).

(iii) If \( D \vdash \Gamma_0, \alpha = \tau, \Gamma_1 \rightarrow \Delta \)
then there exist unique \( \Delta_0, \tau', \) and \( \Delta_1 \)
such that \( \Delta = (\Delta_0, \alpha = \tau', \Delta_1) \) and \( D' \vdash \Gamma_0 \rightarrow \Delta_0 \) and \( [\Delta_0] \tau = [\Delta_0] \tau' \) where \( D' < D \).

(iv) If \( D \vdash \Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1 \rightarrow \Delta \)
then there exist unique \( \Delta_0, \tau', \) and \( \Delta_1 \)
such that \( \Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1) \) and \( D' \vdash \Gamma_0 \rightarrow \Delta_0 \) and \( [\Delta_0] \tau = [\Delta_0] \tau' \) where \( D' < D \).

Moreover, if \( \Gamma_1 \) is soft, then \( \Delta_1 \) is soft.
Moreover, if \( \text{dom}(\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1) = \text{dom}(\Delta) \) then \( \text{dom}(\Gamma_0) = \text{dom}(\Delta_0) \).

(vi) If \( D \vdash \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \rightarrow \Delta \) then either

- there exist unique \( \Delta_0, \tau', \) and \( \Delta_1 \)
such that \( \Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1) \) and \( D' \vdash \Gamma_0 \rightarrow \Delta_0 \) where \( D' < D \),
or
- there exist unique \( \Delta_0 \) and \( \Delta_1 \)
such that \( \Delta = (\Delta_0, \hat{\alpha} : \kappa, \Delta_1) \) and \( D' \vdash \Gamma_0 \rightarrow \Delta_0 \) where \( D' < D \).

Proof.  In each part, we proceed by induction on the derivation of \( \Gamma_0, \ldots, \Gamma_1 \rightarrow \Delta \).
Note that in each part, the \( \Box \) case is impossible.
Throughout this proof, we shadow \( \Delta \) so that it refers to the largest proper prefix of \( \Delta \) in the statement of the lemma. For example, in the \( \Box \) case of part (i), we really have \( \Delta = (\Delta_{00}, x : A') \), but we call \( \Delta_{00} \) "\( \Delta \)".

(i) We have \( \Gamma_0, \alpha : \kappa, \Gamma_1 \rightarrow \Delta \).

\[
\begin{array}{c}
\text{Case } \Gamma \rightarrow \Delta \\
\hline
\text{left} \\
\hline
\text{right} \quad \vdash \Delta \quad \vdash \Delta A = [\Delta] A' \\
\hline
\vdash \Delta x : A \\
\hline
\end{array}
\]

\[
\begin{array}{c}
(\Gamma, x : A) = (\Gamma_0, \alpha : \kappa, \Gamma_1) \\
\hline
\vdash \Delta x : A \\
\hline
(\Gamma, x : A) = (\Gamma_0, \alpha : \kappa, \Gamma_1', x : A) \\
\hline
\end{array}
\]

Given
Since the last element must be equal
By transitivity
By injectivity of syntax

\[
\begin{array}{c}
\Gamma \rightarrow \Delta \\
\hline
\vdash \Delta \\
\hline
\end{array}
\]

Subderivation
By equality
By i.h.

\[
\begin{array}{c}
\vdash \Delta_0 \quad \vdash \Delta_{00} \\
\hline
\vdash \Delta_{00} \\
\end{array}
\]

"" 
By congruence
Since \( \Gamma_1', x : A \) is not soft
Proof of Lemma 22 (Extension Inversion) lem:extension-inversion

- Case $\Gamma \to \Delta$

\[
\Gamma, \beta : \kappa' \to \Delta, \beta : \kappa
\]

$\Gamma_0, \alpha : \kappa, \Gamma_1$

There are two cases:

- Case $\alpha : \kappa = \beta : \kappa'$:

\[
\frac{(\Gamma, \alpha : \kappa) = (\Gamma_0, \alpha : \kappa, \Gamma_1)}{\text{where } \Gamma_0 = \Gamma \text{ and } \Gamma_1 = \cdot}
\]

\[
\frac{(\Delta, \alpha : \kappa) = (\Delta_0, \alpha : \kappa, \Delta_1)}{\text{where } \Delta_0 = \Delta \text{ and } \Delta_1 = \cdot}
\]

- Case $\alpha \neq \beta$:

\[
\frac{(\Gamma, \beta : \kappa') = (\Gamma_0, \alpha : \kappa, \Gamma_1)}{\text{Given}}
\]

\[
\frac{(\Gamma, \alpha : \kappa) = (\Gamma_0, \alpha : \kappa, \Gamma_1)}{\text{Since the last element must be equal}}
\]

\[
\frac{\Gamma = (\Gamma_0, \alpha : \kappa, \Gamma_1')}{\text{By injectivity of syntax}}
\]

\[
\frac{\Gamma \to \Delta}{\text{Subderivation}}
\]

\[
\frac{\Gamma_0, \alpha : \kappa, \Gamma_1' \to \Delta}{\text{By equality}}
\]

\[
\frac{\Delta = (\Delta_0, \alpha : \kappa, \Delta_1)}{\text{By i.h.}}
\]

\[
\frac{\Gamma_0 \to \Delta_0}{''}
\]

- Case $\alpha : \kappa = \beta : \kappa'$:

\[
\frac{\Gamma \to \Delta}{\text{Subderivation}}
\]

\[
\frac{\Gamma_0, \alpha : \kappa, \Gamma_1' \to \Delta}{\text{By equality}}
\]

\[
\frac{\Delta = (\Delta_0, \alpha : \kappa, \Delta_1)}{\text{By i.h.}}
\]

\[
\frac{\Gamma_0 \to \Delta_0}{''}
\]

- Case $\alpha \neq \beta$:

\[
\frac{(\Gamma, \beta : \kappa') = (\Gamma_0, \alpha : \kappa, \Gamma_1)}{\text{Given}}
\]

\[
\frac{(\Gamma, \alpha : \kappa) = (\Gamma_0, \alpha : \kappa, \Gamma_1)}{\text{Since the last element must be equal}}
\]

\[
\frac{\Gamma = (\Gamma_0, \alpha : \kappa, \Gamma_1')}{\text{By injectivity of syntax}}
\]

\[
\frac{\Gamma \to \Delta}{\text{Subderivation}}
\]

\[
\frac{\Gamma_0, \alpha : \kappa, \Gamma_1' \to \Delta}{\text{By equality}}
\]

\[
\frac{\Delta = (\Delta_0, \alpha : \kappa, \Delta_1)}{\text{By i.h.}}
\]

\[
\frac{\Gamma_0 \to \Delta_0}{''}
\]

- Case $\alpha : \kappa = \beta : \kappa'$:

\[
\frac{(\Gamma, \alpha : \kappa) = (\Gamma_0, \alpha : \kappa, \Gamma_1)}{\text{Given}}
\]

\[
\frac{(\Gamma, \beta : \kappa') = (\Gamma_0, \alpha : \kappa, \Gamma_1)}{\text{By injection of syntax}}
\]

\[
\frac{\Gamma = (\Gamma_0, \alpha : \kappa, \Gamma_1')}{\text{Subderivation}}
\]

\[
\frac{\Gamma_0, \alpha : \kappa, \Gamma_1' \to \Delta}{\text{By equality}}
\]

\[
\frac{\Delta = (\Delta_0, \alpha : \kappa, \Delta_1)}{\text{By i.h.}}
\]

\[
\frac{\Gamma_0 \to \Delta_0}{''}
\]

- Case $\alpha \neq \beta$:

\[
\frac{(\Delta, \beta : \kappa') = (\Delta_0, \alpha : \kappa, \Delta_1)}{\text{By congruence}}
\]

\[
\frac{\text{if } \Gamma_1 \text{ soft then } \Delta_1 \text{ soft}}{\text{Since } \Gamma_1 \text{ is soft}}
\]

- Case $\Gamma \to \Delta$

\[
\frac{\Gamma, \hat{\alpha} : \kappa' \to \Delta, \hat{\alpha} : \kappa}{\text{Unsolved}}
\]

\[
\frac{(\Gamma, \hat{\alpha} : \kappa') = (\Gamma_0, \alpha : \kappa, \Gamma_1)}{\text{Given}}
\]

\[
\frac{(\Gamma, \alpha : \kappa) = (\Gamma_0, \alpha : \kappa, \Gamma_1)}{\text{Since the last element must be equal}}
\]

\[
\frac{\Gamma = (\Gamma_0, \alpha : \kappa, \Gamma_1')}{\text{By injectivity of syntax}}
\]

\[
\frac{\Gamma \to \Delta}{\text{Subderivation}}
\]

\[
\frac{\Gamma_0, \alpha : \kappa, \Gamma_1' \to \Delta}{\text{By equality}}
\]

\[
\frac{\Delta = (\Delta_0, \alpha : \kappa, \Delta_1)}{\text{By i.h.}}
\]

\[
\frac{\Gamma_0 \to \Delta_0}{''}
\]

- Case $\alpha : \kappa = \beta : \kappa'$:

\[
\frac{\Gamma \to \Delta}{\text{Solved}}
\]

\[
\frac{[\Delta]t = [\Delta]t'}{\text{Solved}}
\]

\[
\frac{\Gamma, \alpha : \kappa, \Gamma_1}{\text{Solved}}
\]

- Case $\alpha \neq \beta$:

\[
\frac{\Gamma \to \Delta}{\text{Solved}}
\]

\[
\frac{[\Delta]t = [\Delta]t'}{\text{Solved}}
\]

\[
\frac{\Gamma, \alpha : \kappa, \Gamma_1}{\text{Solved}}
\]
Similar to the unsolved case.

- **Case** \[ \Gamma \rightarrow \Delta \] \[ [\Delta] t = [\Delta] t' \]  
  \[ (\Gamma, \beta = t) = (\Gamma_0, \alpha : \kappa, \Gamma_1) \]  
  Given  
  \[ = (\Gamma_0, \alpha : \kappa, \Gamma_1', \beta = t) \]  
  Since the last element must be equal  
  \[ \Gamma = (\Gamma_0, \alpha : \kappa, \Gamma_1') \]  
  By injectivity of syntax  

  \[ \Gamma \rightarrow \Delta \]  
  Subderivation  
  \[ \Gamma_0, \alpha : \kappa, \Gamma_1' \rightarrow \Delta \]  
  By equality  
  \[ \Delta = (\Delta_0, \alpha : \kappa, \Delta_1) \]  
  By i.h.  
  \[ \Gamma_0 \rightarrow \Delta_0 \]  
  ”  
  if \( \Gamma_1' \) soft then \( \Delta_1 \) soft  
  ”  

  \[ (\Delta, \beta = t') = (\Delta_0, \alpha : \kappa, \Delta_1, \beta = t') \]  
  By congruence  

- **Case** \[ \Gamma \rightarrow \Delta \]  
  \[ (\Gamma, \triangleright_\alpha) = (\Gamma_0, \alpha : \kappa, \Gamma_1) \]  
  Given  
  \[ = (\Gamma_0, \alpha : \kappa, \Gamma_1', \triangleright_\alpha) \]  
  Since the last element must be equal  
  \[ \Gamma = (\Gamma_0, \alpha : \kappa, \Gamma_1') \]  
  By injectivity of syntax  

  \[ \Gamma \rightarrow \Delta \]  
  Subderivation  
  \[ \Gamma_0, \alpha : \kappa, \Gamma_1' \rightarrow \Delta \]  
  By equality  
  \[ \Delta = (\Delta_0, \alpha : \kappa, \Delta_1) \]  
  By i.h.  
  \[ \Gamma_0 \rightarrow \Delta_0 \]  
  ”  
  if \( \Gamma_1' \) soft then \( \Delta_1 \) soft  
  ”  

  \[ \Delta, \triangleright_\alpha = (\Delta_0, \alpha : \kappa, \Delta_1, \triangleright_\alpha) \]  
  By congruence  

- **Case** \[ \Gamma \rightarrow \Delta \]  
  \[ \Delta = (\Delta_0, \alpha : \kappa, \Delta_1) \]  
  By i.h.  
  \[ \Gamma_0 \rightarrow \Delta_0 \]  
  ”  
  if \( \Gamma_1 \) soft then \( \Delta_1 \) soft  
  ”  

  \[ \Delta_1, \triangleright_\alpha : \kappa' = (\Delta_0, \alpha : \kappa, \Delta_1, \triangleright_\alpha : \kappa') \]  
  By congruence of equality

Suppose \( \Gamma_1 \) soft.

- \( \Delta_1 \) soft  
  By i.h.
- \( \Delta_1, \triangleright_\alpha : \kappa' \) soft  
  By definition of softness

- if \( \Gamma_1 \) soft then \( \Delta_1, \triangleright_\alpha : \kappa' \) soft  
  Implication introduction
Proof of Lemma 22 (Extension Inversion) lem:extension-inversion

\[ \Delta = (\Delta_0, \alpha : \kappa, \Delta_1) \]

By i.h.

\[ \Gamma_0 \rightarrow \Delta_0 \]

"'

\[ \text{if } \Gamma_1 \text{ soft then } \Delta_1 \text{ soft } \]

"'

\[ (\Delta, \hat{\alpha} : \kappa' = t) = (\Delta_0, \alpha : \kappa, \Delta_1, \hat{\alpha} : \kappa' = t) \]

By congruence of equality

Suppose \( \Gamma_1 \) soft.

\[ \Delta_1 \text{ soft} \]

By i.h.

\[ (\Delta_1, \hat{\alpha} : \kappa' = t) \text{ soft} \]

By definition of softness

\[ \text{if } \Gamma_1 \text{ soft then } \Delta_1, \hat{\alpha} : \kappa' = t \text{ soft} \]

Implication introduction

\[ \Gamma \rightarrow \Delta \]

\[ \Gamma, \hat{\beta} : \kappa' \rightarrow \Delta, \hat{\beta} : \kappa' = t \rightarrow \text{AddSolved} \]

\[ (\Gamma, \hat{\beta} : \kappa') = (\Gamma_0, \alpha : \kappa, \Gamma) \]

By definition of softness

\[ (\Gamma_0, \alpha : \kappa, \Gamma, \hat{\beta} : \kappa') \]

Since the final elements are equal

\[ \Gamma = (\Gamma_0, \alpha : \kappa, \Gamma, \hat{\beta} : \kappa') \]

By injectivity of context syntax

\[ \Gamma \rightarrow \Delta \]

Subderivation

\[ \Gamma_0, \alpha : \kappa, \Gamma_1' \rightarrow \Delta \]

By equality

\[ \Delta = (\Delta_0, \alpha : \kappa, \Delta_1) \]

By i.h.

\[ \Gamma_0 \rightarrow \Delta_0 \]

"'

\[ \text{if } \Gamma_1' \text{ soft then } \Delta_1 \text{ soft } \]

"'

\[ \Delta, \hat{\beta} : \kappa' = \Delta_0, \alpha : \kappa, \Delta_1, \hat{\beta} : \kappa' \]

By congruence

Suppose \( \Gamma_1', \hat{\beta} : \kappa' \) soft.

\[ \Gamma_1' \text{ soft} \]

By definition of softness

\[ \Delta_1 \text{ soft} \]

Using i.h.

\[ \Delta_1, \hat{\beta} : \kappa' = t \text{ soft} \]

By definition of softness

\[ \text{if } \Gamma_1', \hat{\beta} : \kappa' \text{ soft then } \Delta_1, \hat{\beta} : \kappa' = t \text{ soft} \]

Implication intro

(ii) We have \( \Gamma_0, \uparrow_u, \Gamma_1 \rightarrow \Delta \). This part is similar to part (i) above, except for “if \( \text{dom}(\Gamma_0, \uparrow_u, \Gamma_1) = \text{dom}(\Delta) \) then \( \text{dom}(\Gamma_0) = \text{dom}(\Delta_0) \)”, which follows by i.h. in most cases. In the \[\text{Marker}\] case, either we have \( \ldots, \uparrow_u, \) where \( u' = u \)—in which case the i.h. gives us what we need—or we have a matching \( \uparrow_u \). In this latter case, we have \( \Gamma_1 = \cdot. \) We know that \( \text{dom}(\Gamma_0, \uparrow_u, \Gamma_1) = \text{dom}(\Delta) \) and \( \Delta = (\Delta_0, \uparrow_u). \) Since \( \Gamma_1 = \cdot, \) we have \( \text{dom}(\Gamma_0, \uparrow_u) = \text{dom}(\Delta_0, \uparrow_u). \) Therefore \( \text{dom}(\Gamma_0) = \text{dom}(\Delta_0). \)

(iii) We have \( \Gamma_0, \alpha = \tau, \Gamma_1 \rightarrow \Delta \).

\[ \Gamma \rightarrow \Delta \]

\[ \Gamma, \hat{\beta} : \kappa' \rightarrow \Delta, \hat{\beta} : \kappa' \rightarrow \text{Uvar} \]

\[ \Gamma_0, \alpha = \tau, \Gamma_1 \]
(Γ₀, α = τ, Γ₁) = (Γ, β : κ')
        Given
(Γ₀, α = τ, Γ₁') = (Γ₀, α = τ, Γ₁')
        By injectivity of context syntax
Γ = (Γ₀, α = τ, Γ₁')

Δ = (Δ₀, α = τ', Δ₁)
        By i.h.

[Δ₀]τ = [Δ₀]τ'

Γ₀' → Δ₀

(Δ, β : κ') = (Δ₀, α = τ', Δ₁, β : κ')
        By congruence of equality

• Case


Γ → Δ
Γ, x : A → Δ, x : A'

Γ₀, α = τ, Γ₁' → Var

Similar to the →Uvar case.

• Case

Γ → Δ
Γ, ▶α → Δ, ▶α →Marker

Γ₀, ▶α → Marker

Similar to the →Uvar case.

• Case

Γ → Δ
Γ, ▼α : κ' → Δ, ▼α : κ' →Unsolved

Γ₀, ▼α : κ' → Unsolved

Similar to the →Uvar case.

• Case

Γ → Δ
[Δ]t = [Δ]t' →Solved
Γ₀, ▼α : κ' = t → Δ, ▼α : κ' = t

Γ₀, ▼α : κ' = t → Solved

Similar to the →Uvar case.

• Case

Γ → Δ
Γ₀, ▼β : κ' → Δ, ▼β : κ' = t →Solve
Γ₀, ▼β : κ' = t → Solve

Similar to the →Uvar case.

• Case

Γ → Δ
Γ₀, ▼β = t → Δ, ▼β = t' →Eqn
Γ₀, ▼β = t → Eqn

There are two cases:

– Case α = β:

  τ = t and Γ₁ = · and Γ₀ = Γ
  By injectivity of syntax

  Γ₀ → Δ₀
  Subderivation (Γ₀ = Γ and let Δ₀ = Δ)

  (Δ, α = τ') = (Δ₀, α = t', Δ₁)
  where Δ₁ = ·

  [Δ₀]t = [Δ₀]t'
  By premise [Δ]t = [Δ]t'

– Case α ≠ β:
Proof of Lemma 22 \textbf{(Extension Inversion)} \lem:extension-inversion

\[(\Gamma_0, \alpha = \tau, \Gamma_1) = (\Gamma, \beta = t)\]  
Given

\[= (\Gamma_0,\alpha = \tau, \Gamma'_1,\beta = t)\]  
Since the final elements must be equal

\[\Gamma = (\Gamma_0,\alpha = \tau, \Gamma'_1)\]  
By injectivity of context syntax

\[\Delta = (\Delta_0, \alpha = \tau', \Delta_1)\]  
By i.h.

\[\llbracket \Delta_0 \rrbracket \tau = \llbracket \Delta_0 \rrbracket \tau'\]  
""

\[\Gamma_0 \longrightarrow \Delta_0\]  
""

\[\llbracket \Delta, \beta = t' \rrbracket = (\Delta_0, \alpha = \tau', \Delta_1, \beta = t')\]  
By congruence of equality

\[\llbracket \Delta, \alpha : k' \rrbracket = (\Delta_0, \alpha = \tau', \Delta_1, \alpha : k')\]  
By congruence of equality

\[\Rightarrow \quad \text{Case } \Gamma \longrightarrow \Delta\]

\[\Delta = (\Delta_0, \alpha = \tau', \Delta_1)\]  
By i.h.

\[\llbracket \Delta_0 \rrbracket \tau = \llbracket \Delta_0 \rrbracket \tau'\]  
""

\[\Gamma_0 \longrightarrow \Delta_0\]  
""

\[\llbracket \Delta, \alpha : k' = t \rrbracket = (\Delta_0, \alpha = \tau', \Delta_1, \alpha : k' = t)\]  
By congruence of equality

(iv) We have \(\Gamma_0, \alpha : k = \tau, \Gamma_1 \longrightarrow \Delta\).

\[\llbracket \Delta_0 \rrbracket A = \llbracket \Delta_0 \rrbracket A' \quad \text{by } \longrightarrow \text{Uvar}\]

\[\llbracket \Delta, \beta : \kappa = \tau, \Gamma_1 \rrbracket = (\Gamma, \beta : \kappa')\]  
Given

\[= (\Gamma_0, \alpha : k = \tau, \Gamma'_1, \beta : \kappa')\]  
Since the final elements must be equal

\[\Gamma = (\Gamma_0, \alpha : k = \tau, \Gamma'_1)\]  
By injectivity of context syntax

\[\Delta = (\Delta_0, \alpha : k = \tau', \Delta_1)\]  
By i.h.

\[\llbracket \Delta_0 \rrbracket \tau = \llbracket \Delta_0 \rrbracket \tau'\]  
""

\[\Gamma_0 \longrightarrow \Delta_0\]  
""

\[\llbracket \Delta, \beta : \kappa' \rrbracket = (\Delta_0, \alpha : k = \tau', \Delta_1, \beta : \kappa')\]  
By congruence of equality

\[\llbracket \Delta, x : A = \Delta, x : A' \rrbracket \quad \text{by } \longrightarrow \text{Var}\]

Similar to the \longrightarrow \text{Uvar} case.
• Case \[ \Gamma \rightarrow \Delta \]
\[ \Gamma, \beta \rightarrow \Delta, \beta \rightarrow \text{Marker} \]

Similar to the UVar case.

• Case \[ \Gamma \rightarrow \Delta \]
\[ \Gamma, \beta : \kappa' \rightarrow \Delta, \beta : \kappa \rightarrow \text{Unsolved} \]

Similar to the UVar case.

• Case \[ \Gamma \rightarrow \Delta \]
\[ [\Delta]t = [\Delta]t' \rightarrow \text{Solved} \]
\[ \Gamma, \beta : \kappa' \rightarrow \Delta, \beta : \kappa = t' \rightarrow \text{Eqn} \]
\[ \Gamma, \beta = t \rightarrow \Delta, \beta = t' \rightarrow \text{Eqn} \]
\[ \Gamma, \beta \rightarrow \Delta, \beta = t' \rightarrow \text{Add} \]

There are two cases.

- Case \( \hat{\alpha} = \hat{\beta} \):

  \[ \kappa = \kappa \text{ and } t = \tau \text{ and } \Gamma_1 = \cdot \text{ and } \Gamma = \Gamma_0 \]

  By injectivity of syntax

  \[ (\Delta, \hat{\beta} : \kappa' = t') = (\Delta_0, \hat{\beta} : \kappa = \tau', \Delta_1) \]

  where \( \tau' = t' \) and \( \Delta_1 = \cdot \) and \( \Delta = \Delta_0 \)

  From subderivation \( \Gamma \rightarrow \Delta \)

  From premise \( [\Delta]t = [\Delta]t' \) and \( x \)

- Case \( \hat{\alpha} \neq \hat{\beta} \):

  \( (\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1) = (\Gamma_0, \hat{\beta} : \kappa = t) \)

  Given

  Since the final elements must be equal

  \( \Gamma = (\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1') \)

  By injectivity of context syntax

  \[ \Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1) \]

  By i.h.

  \[ [\Delta_0] \tau = [\Delta_0] \tau' \]

  ""

  \[ \Gamma_0 \rightarrow \Delta_0 \]

  ""

  \[ (\Delta, \hat{\beta} : \kappa' = t') = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1, \hat{\beta} : \kappa' = t') \]

  By congruence of equality

  \( \Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1 \)

  \( (\Gamma_0, \hat{\beta} : \kappa = t) \)

  Given

  Since the final elements must be equal

  \( \Gamma = (\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1') \)

  By injectivity of context syntax

  \[ \Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1) \]

  By i.h.

  \[ [\Delta_0] \tau = [\Delta_0] \tau' \]

  ""

  \[ \Gamma_0 \rightarrow \Delta_0 \]

  ""

  \[ (\Delta, \hat{\beta} = t') = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1, \beta = t') \]

  By congruence of equality

  \( \Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1 \)

  \( \Gamma_0, \hat{\beta} = t \)

  Given

  Since the final elements must be equal

  \( \Gamma = (\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1') \)

  By injectivity of context syntax

  \[ \Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1) \]

  By i.h.

  \[ [\Delta_0] \tau = [\Delta_0] \tau' \]

  ""

  \[ \Gamma_0 \rightarrow \Delta_0 \]

  ""

  \[ (\Delta, \beta = t') = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1, \beta = t') \]

  By congruence of equality
Proof of Lemma 22 (Extension Inversion)

• Case

\[
\Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1)
\]

By i.h.

\[
\rightarrow [\Delta_0] = [\Delta_0]^{\tau'}
\]

""

\[
\Gamma_0 \rightarrow \Delta_0
\]

""

\[
(\Delta, \hat{\beta} : \kappa') = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1, \hat{\beta} : \kappa')
\]

By congruence of equality

• Case

\[
\Gamma \rightarrow \Delta
\]

\[
\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1
\]

\[
\rightarrow \Delta_0, \hat{\beta} : \kappa' = t
\]

\[
\rightarrow \text{AddSolved}
\]

\[
\Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1)
\]

By i.h.

\[
[\Delta_0] = [\Delta_0]^{\tau'}
\]

""

\[
\Gamma_0 \rightarrow \Delta_0
\]

""

\[
(\Delta, \hat{\beta} : \kappa' = t) = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1, \hat{\beta} : \kappa' = t)
\]

By congruence of equality

• Case

\[
\Gamma \rightarrow \Delta
\]

\[
\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1
\]

\[
\rightarrow \Delta_0, \hat{\beta} : \kappa' = t
\]

\[
\rightarrow \text{Solve}
\]

\[
\Gamma = (\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1)
\]

Given

\[
[\Gamma_0] = [\Gamma_0]^{\tau'}
\]

Since the last elements must be equal

\[
\Gamma = (\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1'^{\prime})
\]

By injectivity of syntax

\[
\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1'^{\prime} \rightarrow \Delta
\]

Subderivation

\[
\Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1)
\]

By equality

\[
[\Delta_0] = [\Delta_0]^{\tau'}
\]

""

\[
\Gamma_0 \rightarrow \Delta_0
\]

""

\[
(\Delta, \hat{\beta} : \kappa') = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1, \hat{\beta} : \kappa')
\]

By congruence of equality

(v) We have \(\Gamma_0, \alpha : \Lambda, \Gamma_1 \rightarrow \Delta\). This proof is similar to the proof of part (i), except for the domain condition, which we handle similarly to part (ii).

(vi) We have \(\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \rightarrow \Delta\).

• Case

\[
\Gamma \rightarrow \Delta
\]

\[
\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1
\]

\[
\rightarrow \Delta_0, \hat{\beta} : \kappa' = \text{Uvar}
\]

\[
(\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1) = (\Gamma, \hat{\beta} : \kappa')
\]

Given

\[
= (\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1', \hat{\beta} : \kappa')
\]

Since the final elements must be equal

\[
\Gamma = (\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1')
\]

By injectivity of context syntax

By induction, there are two possibilities:

- \(\hat{\alpha}\) is not solved:

\[
\Delta = (\Delta_0, \hat{\alpha} : \kappa, \Delta_1)
\]

By i.h.

\[
\rightarrow \Gamma_0 \rightarrow \Delta_0
\]

""

\[
(\Delta, \hat{\beta} : \kappa') = (\Delta_0, \hat{\alpha} : \kappa, \Delta_1, \hat{\beta} : \kappa')
\]

By congruence of equality
– \( \hat{\alpha} \) is solved:

\[
\Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1)
\]

By i.h.

\[
\Gamma_0 \rightarrow \Delta_0
\]

" 

\[ (\Delta, \hat{\beta} : \kappa') = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1, \hat{\beta} : \kappa') \]

By congruence of equality

\[ \Gamma \rightarrow \Delta \]

\[ \Delta = [\Delta]A = [\Delta]A' \]

By i.h. 

\[ \Gamma, x : A \rightarrow \Delta, x : A' \]

Similiar to the \( \rightarrow \text{Var} \) case.

\[ \Gamma, \triangleright \beta \rightarrow \Delta, \triangleright \beta \]

Similiar to the \( \rightarrow \text{Var} \) case.

\[ \Gamma \rightarrow \Delta \]

\[ [\Delta]t = [\Delta]t' \]

Similar to the \( \rightarrow \text{Eqn} \) case.

\[ \Gamma, \triangleright \beta : \kappa' = t \rightarrow \Delta, \triangleright \beta : \kappa' = t' \]

Similar to the \( \rightarrow \text{Uvar} \) case.

\[ \Gamma \rightarrow \Delta \]

\[ \Gamma, \triangleright \beta : \kappa' \rightarrow \Delta, \triangleright \beta : \kappa' \]

By injectivity of context syntax

– Case \( \hat{\alpha} \neq \hat{\beta} \):

\[
(\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1) = (\Gamma, \hat{\beta} : \kappa')
\]

Given

\[
(\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1', \hat{\beta} : \kappa')
\]

Since the final elements must be equal

\[ \Gamma = (\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1') \]

By injectivity of context syntax

By induction, there are two possibilities:

* \( \hat{\alpha} \) is not solved:

\[
\Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1)
\]

By i.h.

\[ \Gamma_0 \rightarrow \Delta_0 \]

" 

\[ (\Delta, \hat{\beta} : \kappa') = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1, \hat{\beta} : \kappa') \]

By congruence of equality

* \( \hat{\alpha} \) is solved:

\[
\Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1)
\]

By i.h.

\[ \Gamma_0 \rightarrow \Delta_0 \]

" 

\[ (\Delta, \hat{\beta} : \kappa') = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1, \hat{\beta} : \kappa') \]

By congruence of equality

– Case \( \hat{\alpha} = \hat{\beta} \):
Proof of Lemma 22 (Extension Inversion) lem:extension-inversion

\[ \kappa' = \kappa \text{ and } \Gamma_0 = \Gamma \text{ and } \Gamma_1 = \cdot \] By injectivity of syntax

\[ (\Delta, \beta : \kappa') = (\Delta_0, \alpha : \kappa, \Delta_1) \] where \( \Delta_0 = \Delta \) and \( \Delta_1 = \cdot \)

\[ \Gamma_0 \rightarrow \Delta_0 \] From premise \( \Gamma \rightarrow \Delta \)

\begin{itemize}
  \item Case \[ \Gamma \rightarrow \Delta \]
      \[ \Gamma_0, \alpha : \kappa, \Gamma_1 \]
      \[ \Delta, \beta : \kappa' \rightarrow \text{Add} \]

      By induction, there are two possibilities:
      \begin{itemize}
        \item \( \alpha \) is not solved:
          \[ \Delta = (\Delta_0, \alpha : \kappa, \Delta_1) \] By i.h.
          \[ \Gamma_0 \rightarrow \Delta_0 \] "
          \[ (\Delta, \beta : \kappa') = (\Delta_0, \alpha : \kappa, \Delta_1, \beta : \kappa') \] By congruence of equality
        \item \( \alpha \) is solved:
          \[ \Delta = (\Delta_0, \alpha : \kappa = \tau', \Delta_1) \] By i.h.
          \[ \Gamma_0 \rightarrow \Delta_0 \] "
          \[ (\Delta, \beta : \kappa') = (\Delta_0, \alpha : \kappa = \tau', \Delta_1, \beta : \kappa') \] By congruence of equality
      \end{itemize}

  \item Case \[ \Gamma \rightarrow \Delta \]
      \[ \Gamma_0, \alpha : \kappa, \Gamma_1 \]
      \[ \Delta, \beta : \kappa' = t \rightarrow \text{AddSolved} \]

      By induction, there are two possibilities:
      \begin{itemize}
        \item \( \alpha \) is not solved:
          \[ \Delta = (\Delta_0, \alpha : \kappa, \Delta_1) \] By i.h.
          \[ \Gamma_0 \rightarrow \Delta_0 \] "
          \[ (\Delta, \beta : \kappa' = t) = (\Delta_0, \alpha : \kappa, \Delta_1, \beta : \kappa' = t) \] By congruence of equality
        \item \( \alpha \) is solved:
          \[ \Delta = (\Delta_0, \alpha : \kappa = \tau', \Delta_1) \] By i.h.
          \[ \Gamma_0 \rightarrow \Delta_0 \] "
          \[ (\Delta, \beta : \kappa' = t) = (\Delta_0, \alpha : \kappa = \tau', \Delta_1, \beta : \kappa' = t) \] By congruence of equality
      \end{itemize}

  \item Case \[ \Gamma \rightarrow \Delta \]
      \[ \Gamma_0, \alpha : \kappa, \Gamma_1 \]
      \[ \Delta, \beta : \kappa' = t \rightarrow \text{Solve} \]

      By induction, there are two possibilities:
      \begin{itemize}
        \item Case \( \alpha \neq \beta \):
          \[ (\Gamma_0, \alpha : \kappa, \Gamma_1) = (\Gamma, \beta : \kappa') \] Given
          Since the final elements must be equal
          \[ \Gamma = (\Gamma_0, \alpha : \kappa, \Gamma_1) \] By injectivity of context syntax
      \end{itemize}

By induction, there are two possibilities:
Proof of Lemma 23 (Deep Evar Introduction).  

(i) If \( \Gamma_0, \Gamma_1 \) is well-formed and \( \hat{\alpha} \) is not declared in \( \Gamma_0, \Gamma_1 \) then \( \Gamma_0, \Gamma_1 \xrightarrow{} \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \).

(ii) If \( \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \) is well-formed and \( \Gamma \vdash t : \kappa \) then \( \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \xrightarrow{} \Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1 \).

(iii) If \( \Gamma_0, \Gamma_1 \) is well-formed and \( \Gamma \vdash t : \kappa \) then \( \Gamma_0, \Gamma_1 \xrightarrow{} \Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1 \).

Proof.

(i) Assume that \( \Gamma_0, \Gamma_1 \) is well-formed. We proceed by induction on \( \Gamma_1 \).

- Case \( \Gamma_1 = \_ : \)

  \[
  \begin{align*}
  \Gamma_0, \hat{\alpha} : \kappa \xrightarrow{} \Gamma_0 & \quad \text{By rule \texttt{Add}} \\
  \end{align*}
  \]

- Case \( \Gamma_1 = \Gamma'_1, x : A : \)

  \[
  \begin{align*}
  \Gamma_0, \Gamma'_1, x : A & \quad \text{Given} \\
  \Gamma_0, \Gamma'_1 & \xrightarrow{} \Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1 & \quad \text{By \texttt{Var}} \\
  \end{align*}
  \]
Proof of Lemma 23 (Deep Evar Introduction)

• Case $\Gamma_1 = \Gamma'_1, \beta : \kappa'$:
  
  $\Gamma_0, \Gamma'_1, \beta : \kappa' \text{ ctx}$
  
  Given

  $\Gamma_0, \Gamma'_1 \text{ ctx}$
  
  By inversion

  $\beta \not\in \text{dom}(\Gamma_0, \Gamma'_1)$
  
  By inversion (1)

  $\alpha \not\in \text{dom}(\Gamma_0, \Gamma'_1, \beta : \kappa')$
  
  Given

  $\alpha \neq \beta$
  
  By inversion (2)

  $\Gamma_0, \alpha : \kappa, \Gamma'_1 \text{ ctx}$
  
  By i.h.

  $\Gamma_0, \Gamma'_1 \rightarrow \Gamma_0, \alpha \vdash \kappa, \Gamma'_1, \beta : \kappa'$
  
  By $\Downarrow\text{Unsolved}$

• Case $\Gamma_1 = \Gamma'_1, \hat{\beta} : \kappa'$:

  $\Gamma_0, \Gamma'_1, \hat{\beta} : \kappa' \text{ ctx}$
  
  Given

  $\Gamma_0, \Gamma'_1 \text{ ctx}$
  
  By inversion

  $\hat{\beta} \not\in \text{dom}(\Gamma_0, \Gamma'_1)$
  
  By inversion (1)

  $\alpha \not\in \text{dom}(\Gamma_0, \Gamma'_1, \hat{\beta} : \kappa')$
  
  Given

  $\alpha \neq \hat{\beta}$
  
  By inversion (2)

  $\Gamma_0, \alpha \vdash \kappa, \Gamma'_1 \text{ ctx}$
  
  By i.h.

  $\Gamma_0, \Gamma'_1 \rightarrow \Gamma_0, \alpha \vdash \kappa, \Gamma'_1, \hat{\beta} : \kappa'$
  
  By $\Downarrow\text{Unsolved}$

• Case $\Gamma_1 = (\Gamma'_1, \beta = t)$:

  $\Gamma_0, \Gamma'_1, \hat{\beta} : \kappa' = t \text{ ctx}$
  
  Given

  $\Gamma_0, \Gamma'_1 \text{ ctx}$
  
  By inversion

  $\hat{\beta} \not\in \text{dom}(\Gamma_0, \Gamma'_1)$
  
  By inversion (1)

  $\Gamma_0, \Gamma'_1 \vdash t : \kappa'$
  
  By inversion

  $\alpha \not\in \text{dom}(\Gamma_0, \Gamma'_1, \hat{\beta} : \kappa' = t)$
  
  Given

  $\alpha \neq \hat{\beta}$
  
  By inversion (2)

  $\Gamma_0, \alpha \vdash \kappa, \Gamma'_1 \text{ ctx}$
  
  By i.h.

  $\Gamma_0, \Gamma'_1 \rightarrow \Gamma_0, \alpha \vdash \kappa, \Gamma'_1, \hat{\beta} : \kappa' = t$
  
  By $\Downarrow\text{Solved}$

• Case $\Gamma_1 = (\Gamma'_1, \beta = t)$:
Proof of Lemma 23 (Deep Evar Introduction) lem:deep-existential

\[ \Gamma_0, \Gamma'_1, \beta = t \text{ ctx} \]

Given

\[ \Gamma_0, \Gamma'_1 \text{ ctx} \]

By inversion

\[ \beta \notin \text{dom}(\Gamma_0, \Gamma'_1) \]

By inversion (1)

\[ \Gamma_0, \Gamma'_1 \vdash t : N \]

By inversion

\[ \hat{\alpha} \notin \text{dom}(\Gamma_0, \Gamma'_1, \beta = t) \]

Given

\[ \hat{\alpha} \neq \beta \]

By inversion (2)

\[ \Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1 \text{ ctx} \]

By i.h.

\[ \Gamma_0, \Gamma'_1 \rightarrow \Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1 \]

By Lemma 36 (Extension Weakening (Sorts))

\[ \beta \notin \text{dom}(\Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1) \]

By (1) and (2)

\[ \Rightarrow \Gamma_0, \Gamma'_1, \beta = t \rightarrow \Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1, \beta = t \]

By Marker

Case \( \Gamma_1 = (\Gamma'_1, \triangleright) \):

\[ \Gamma_0, \Gamma'_1, \triangleright \text{ ctx} \]

Given

\[ \Gamma_0, \Gamma'_1 \text{ ctx} \]

By inversion

\[ \hat{\beta} \notin \text{dom}(\Gamma_0, \Gamma'_1) \]

By inversion (1)

\[ \hat{\alpha} \notin \text{dom}(\Gamma_0, \Gamma'_1, \triangleright) \]

Given

\[ \hat{\alpha} \neq \hat{\beta} \]

By inversion (2)

\[ \Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1 \text{ ctx} \]

By i.h.

\[ \Gamma_0, \Gamma'_1 \rightarrow \Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1 \]

By (1) and (2)

\[ \Rightarrow \Gamma_0, \Gamma'_1, \triangleright \rightarrow \Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1, \triangleright \]

By Marker

(ii) Assume \( \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \text{ ctx} \). We proceed by induction on \( \Gamma_1 \):

Case \( \Gamma_1 = \vdash \):

\[ \Gamma_0 \vdash t : \kappa \]

Given

\[ \Gamma_0, \Gamma_1 \text{ ctx} \]

Given

\[ \Gamma_0 \text{ ctx} \]

Since \( \Gamma_1 = \vdash \)

\[ \Gamma_0 \rightarrow \Gamma_0 \]

By Lemma 32 (Extension Reflexivity)

\[ \Gamma_0, \hat{\alpha} : \kappa \rightarrow \Gamma_0, \hat{\alpha} : \kappa = t \]

By rule Solve

\[ \Rightarrow \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \rightarrow \Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1 \]

Since \( \Gamma_1 = \vdash \)

Case \( \Gamma_1 = (\Gamma'_1, x : A) \):

\[ \Gamma_0 \vdash t : \kappa \]

Given

\[ \Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1, x : A \text{ ctx} \]

Given

\[ \Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1 \text{ ctx} \]

By inversion

\[ \Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1 \vdash A \text{ type} \]

By inversion

\[ x \notin \text{dom}(\Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1) \]

By inversion (1)

\[ \Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1 \rightarrow \Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1 \]

By i.h.

\[ \Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1 \vdash A \text{ type} \]

By Lemma 36 (Extension Weakening (Sorts))

since this is the same domain as (1)

\[ \Rightarrow \Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1, x : A \rightarrow \Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1, x : A \]

By rule Var

Case \( \Gamma_1 = (\Gamma'_1, \beta : \kappa') \):

\[ \]
Proof of Lemma 23 (Deep Evar Introduction)

\[ \Gamma_0 \vdash t : \kappa \]
\[ \Gamma_0, \delta : \kappa, \Gamma'_i, \beta : \kappa' \text{ ctx} \]
\[ \Gamma_0, \delta : \kappa, \Gamma'_i \text{ ctx} \]
\[ \beta \notin \text{dom}(\Gamma_0, \delta : \kappa, \Gamma'_i) \]
\[ \Gamma_0, \delta : \kappa, \Gamma'_i \rightarrow \Gamma_0, \delta : \kappa = t, \Gamma_1 \]
\[ \beta \notin \text{dom}(\Gamma_0, \delta : \kappa = t, \Gamma'_i) \]
\[ \Gamma_0, \delta : \kappa, \Gamma'_i, \beta : \kappa' \rightarrow \Gamma_0, \delta : \kappa = t, \Gamma_1, \beta : \kappa' \]

- Case \( \Gamma_1 = (\Gamma'_i, \hat{\beta} : \kappa') \):

\[ \Gamma_0 \vdash t : \kappa \]
\[ \Gamma_0, \delta : \kappa, \Gamma'_i, \hat{\beta} : \kappa' \text{ ctx} \]
\[ \Gamma_0, \delta : \kappa, \Gamma'_i \text{ ctx} \]
\[ \hat{\beta} \notin \text{dom}(\Gamma_0, \delta : \kappa, \Gamma'_i) \]
\[ \Gamma_0, \delta : \kappa, \Gamma'_i \rightarrow \Gamma_0, \delta : \kappa = t, \Gamma_1 \]
\[ \hat{\beta} \notin \text{dom}(\Gamma_0, \delta : \kappa = t, \Gamma'_i) \]
\[ \Gamma_0, \delta : \kappa, \Gamma'_i, \hat{\beta} : \kappa' \rightarrow \Gamma_0, \delta : \kappa = t, \Gamma_1, \hat{\beta} : \kappa' \]

- Case \( \Gamma_1 = (\Gamma'_i, \hat{\beta} : \kappa' = t') \):

\[ \Gamma_0 \vdash t' : \kappa \]
\[ \Gamma_0, \delta : \kappa, \Gamma'_i, \hat{\beta} : \kappa' = t' \text{ ctx} \]
\[ \Gamma_0, \delta : \kappa, \Gamma'_i \text{ ctx} \]
\[ \hat{\beta} \notin \text{dom}(\Gamma_0, \delta : \kappa, \Gamma'_i) \]
\[ \Gamma_0, \delta : \kappa, \Gamma'_i \rightarrow \Gamma_0, \delta : \kappa = t, \Gamma_1 \]
\[ \hat{\beta} \notin \text{dom}(\Gamma_0, \delta : \kappa = t, \Gamma'_i) \]
\[ \Gamma_0, \delta : \kappa = t, \Gamma_1 \vdash t' : \kappa' \]

By Lemma 36 (Extension Weakening (Sorts))

- Case \( \Gamma_1 = (\Gamma'_i, \beta = t') \):

\[ \Gamma_0 \vdash t' : \kappa \]
\[ \Gamma_0, \delta : \kappa, \Gamma'_i, \beta = t' \text{ ctx} \]
\[ \Gamma_0, \delta : \kappa, \Gamma'_i \text{ ctx} \]
\[ \beta \notin \text{dom}(\Gamma_0, \delta : \kappa, \Gamma'_i) \]
\[ \Gamma_0, \delta : \kappa, \Gamma'_i \rightarrow \Gamma_0, \delta : \kappa = t, \Gamma_1 \]
\[ \beta \notin \text{dom}(\Gamma_0, \delta : \kappa = t, \Gamma'_i) \]
\[ \Gamma_0, \delta : \kappa = t, \Gamma_1 \vdash t' : \kappa' \]

By Lemma 36 (Extension Weakening (Sorts))

- Case \( \Gamma_1 = (\Gamma'_i, \hat{\beta}) \):

(iii) Apply parts (i) and (ii) as lemmas, then Lemma 33 (Extension Transitivity). □

Lemma 26 (Parallel Admissibility).

If $\Gamma \rightarrow \Delta_L$ and $\Gamma_R \rightarrow \Delta_L, \Delta_R$ then:

(i) $\Gamma_L, \tilde{\alpha} : \kappa, \Gamma_R \rightarrow \Delta_L, \tilde{\alpha} : \kappa, \Delta_R$

(ii) If $\Delta_L \vdash \tau' : \kappa$ then $\Gamma_L, \tilde{\alpha} : \kappa, \Gamma_R \rightarrow \Delta_L, \tilde{\alpha} : \kappa = \tau', \Delta_R$.

(iii) If $\Gamma_L \vdash \tau : \kappa$ and $\Delta_L \vdash \tau'$ and type and $|\Delta_L|_{\tau} = |\Delta_L|_{\tau'}$, then $\Gamma_L, \tilde{\alpha} : \kappa = \tau, \Gamma_R \rightarrow \Delta_L, \tilde{\alpha} : \kappa = \tau', \Delta_R$.

Proof. By induction on $\Delta_R$. As always, we assume that all contexts mentioned in the statement of the lemma are well-formed. Hence, $\tilde{\alpha} \notin \text{dom}(\Gamma_L) \cup \text{dom}(\Gamma_R) \cup \text{dom}(\Delta_L) \cup \text{dom}(\Delta_R)$.

(i) We proceed by cases of $\Delta_R$. Observe that in all the extension rules, the right-hand context gets smaller, so as we enter subderivations of $\Gamma_L, \Gamma_R \rightarrow \Delta_L, \Delta_R$, the context $\Delta_R$ becomes smaller.

The only tricky part of the proof is that to apply the i.h., we need $\Gamma_L \rightarrow \Delta_L$. So we need to make sure that as we drop items from the right of $\Gamma_R$ and $\Delta_R$, we don’t go too far and start decomposing $\Gamma_L$ or $\Delta_L$. It’s easy to avoid decomposing $\Delta_L$: when $\Delta_R = \cdot$, we don’t need to apply the i.h. anyway. To avoid decomposing $\Gamma_L$, we need to reason by contradiction, using Lemma 19 (Declaration Preservation).

- **Case $\Delta_R = \cdot$**: We have $\Gamma_L \rightarrow \Delta_L$. Applying $\text{Unsolved}$ to that derivation gives the result.

- **Case $\Delta_R = (\Delta'_R, \beta)$**: We have $\beta \neq \tilde{\alpha}$ by the well-formedness assumption.

  The concluding rule of $\Gamma_L, \Gamma_R \rightarrow \Delta_L, \Delta'_R, \beta$ must have been $\text{Unsolved}$ or $\text{Add}$. In both cases, the result follows by i.h. and applying $\text{Unsolved}$ or $\text{Add}$.

  Note: In $\text{Add}$, the left-hand context doesn’t change, so we clearly maintain $\Gamma_L \rightarrow \Delta_L$. In $\text{Unsolved}$, we can correctly apply the i.h. because $\Gamma_R \neq \cdot$. Suppose, for a contradiction, that $\Gamma_R = \cdot$. Then $\Gamma_L = (\Gamma'_L, \beta)$. It was given that $\Gamma_L \rightarrow \Delta_L$, that is, $\Gamma'_L, \beta \rightarrow \Delta_L$. By Lemma 19 (Declaration Preservation), $\Delta_L$ has a declaration of $\beta$. But then $\Delta = (\Delta_L, \Delta'_R, \beta)$ is not well-formed: contradiction. Therefore $\Gamma_R \neq \cdot$.

- **Case $\Delta_R = (\Delta'_R, \kappa : \tau)$**: We have $\beta \neq \tilde{\alpha}$ by the well-formedness assumption.

  The concluding rule must have been $\text{Solved}$ or $\text{AddSolved}$. In each case, apply the i.h. and then the corresponding rule. (In $\text{Solved}$ and $\text{AddSolved}$, use Lemma 19 (Declaration Preservation) to show $\Gamma_R \neq \cdot$.)

- **Case $\Delta_R = (\Delta'_R, \alpha)$**: The concluding rule must have been $\text{Uvar}$. The result follows by i.h. and applying $\text{Uvar}$.

- **Case $\Delta_R = (\Delta'_R, \beta)$**: The concluding rule must have been $\text{Eqn}$. The result follows by i.h. and applying $\text{Eqn}$.

- **Case $\Delta_R = (\Delta'_R, \beta)$**: Similar to the previous case, with rule $\text{Marker}$.

- **Case $\Delta_R = (\Delta'_R, x : A)$**: Similar to the previous case, with rule $\text{Var}$.

(ii) Similar to part (i), except that when $\Delta_R = \cdot$, apply rule $\text{Solve}$.
(iii) Similar to part (i), except that when $\Delta_R = \cdot$, apply rule $\rightarrow Solved$ using the given equality to satisfy the second premise.

**Lemma 27** (Parallel Extension Solution).
If $\Gamma_L, \hat{\alpha} : \kappa, \Gamma_R \rightarrow \Delta_L, \hat{\alpha} : \kappa = \tau', \Delta_R$ and $\Gamma_L \vdash \tau : \kappa$ and $[\Delta_L] \tau = [\Delta_L] \tau'$
then $\Gamma_L, \hat{\alpha} : \kappa = \tau, \Gamma_R \rightarrow \Delta_L, \hat{\alpha} : \kappa = \tau', \Delta_R$.

*Proof.* By induction on $\Delta_R$.
In the case where $\Delta_R = \cdot$, we know that rule $\rightarrow Solve$ must have concluded the derivation (we can use Lemma 19 (Declaration Preservation) to get a contradiction that rules out $\rightarrow AddSolved$); then we have a subderivation $\Gamma_L \rightarrow \Delta_L$, to which we can apply $\rightarrow Solved$.

**Lemma 28** (Parallel Variable Update).
If $\Gamma_L, \hat{\alpha} : \kappa, \Gamma_R \rightarrow \Delta_L, \hat{\alpha} : \kappa = \tau_0, \Delta_R$ and $\Gamma_L \vdash \tau_1 : \kappa$ and $\Delta_L \vdash \tau_2 : \kappa$ and $[\Delta_L] \tau_0 = [\Delta_L] \tau_1 = [\Delta_L] \tau_2$
then $\Gamma_L, \hat{\alpha} : \kappa = \tau_1, \Gamma_R \rightarrow \Delta_L, \hat{\alpha} : \kappa = \tau_2, \Delta_R$.

*Proof.* By induction on $\Delta_R$. Similar to the proof of Lemma 27 (Parallel Extension Solution), but applying $\rightarrow Solved$ at the end.

**Lemma 29** (Substitution Monotonicity).

(i) If $\Gamma \rightarrow \Delta$ and $\Gamma \vdash t : \kappa$ then $[\Delta]\Gamma t = [\Delta]t$.

(ii) If $\Gamma \rightarrow \Delta$ and $\Gamma \vdash P \text{ prop}$ then $[\Delta]\Gamma P = [\Delta]P$.

(iii) If $\Gamma \rightarrow \Delta$ and $\Gamma \vdash A \text{ type}$ then $[\Delta]\Gamma A = [\Delta]A$.

*Proof.* We prove each part in turn; part (i) does not depend on parts (ii) or (iii), so we can use part (i) as a lemma in the proofs of parts (ii) and (iii).

- **Proof of Part (i):** By lexicographic induction on the derivation of $\mathcal{D} \vdash \Gamma \rightarrow \Delta$ and $\Gamma \vdash t : \kappa$. We proceed by cases on the derivation of $\Gamma \vdash t : \kappa$.

  - **Case** $\hat{\alpha} : \kappa \in \Gamma$
    
    $\Gamma \vdash \hat{\alpha} : \kappa$

    $[\Gamma]\hat{\alpha} = \hat{\alpha}$ Since $\hat{\alpha}$ is not solved in $\Gamma$

    $[\Delta]\hat{\alpha} = [\Delta]\hat{\alpha}$ Reflexivity

    $= [\Delta]\Gamma \hat{\alpha}$ By above equality

  - **Case** $(\alpha : \kappa) \in \Gamma$
    
    $\Gamma \vdash \alpha : \kappa$

    Consider whether or not there is a binding of the form $(\alpha = \tau) \in \Gamma$.

    - **Case** $(\alpha = \tau) \in \Gamma$: 

Proof of Lemma 29 (Substitution Monotonicity): lem:substitution-monotonicity
Proof of Lemma 29

(Substitution Monotonicity)

\[ \Delta = (\Delta_0, \alpha = \tau', \Delta_1) \]

By Lemma 22 (Extension Inversion) (i)

\[ D' :: \Gamma_0 \rightarrow \Delta_0 \]

\[ D' < D \]

(1) \[ [\Delta_0] \tau' = [\Delta_0] \tau \]

By i.h.

(2) \[ [\Delta_0] [\Gamma_0] \alpha = [\Delta_0, \alpha = \tau', \Delta_1] [\Gamma_0, \alpha = \tau, \Gamma_1] \alpha \]

By definition

\[ \Delta_0, \alpha = \tau', \Delta_1 \]

By definition of substitution

\[ FV([\Gamma_0] \tau) \cap dom(\Delta_1) = \emptyset \]

By (2) and (1)

\[ [\Delta_0, \alpha = \tau'] \alpha \]

By definition of substitution

\[ FV([\Delta_0] \tau) \cap dom(\Delta_1) = \emptyset \]

By definition of \( \Delta \)

\[ \star \text{ Case } (\alpha = \tau) \notin \Gamma: \]

\[ [\Gamma] \alpha = \alpha \]

By definition of substitution

\[ [\Delta] [\Gamma] \alpha = [\Delta] \alpha \]

Apply \( [\Delta] \) to both sides

– Case

\[ \Gamma_0, \Delta : \kappa, \Gamma_1 \vdash \Delta : \kappa \]

SolvedVarSort

Similar to the VarSort case.

– Case

\[ \Gamma \vdash 1 : \star \]

UnitSort

\[ [\Delta] 1 = 1 = [\Delta] [\Gamma] 1 \]

Since \( FV(1) = \emptyset \)

– Case

\[ \Gamma \vdash \tau_1 : \star \quad \Gamma \vdash \tau_2 : \star \]

BinSort

\[ [\Delta] [\Gamma] \tau_1 = [\Delta] \tau_1 \]

By i.h.

\[ [\Delta] [\Gamma] \tau_2 = [\Delta] \tau_2 \]

By i.h.

\[ [\Delta] [\Gamma] (\tau_1 \oplus \tau_2) = [\Delta] (\tau_1 \oplus \tau_2) \]

By congruence of equality

Definition of substitution

– Case

\[ \Gamma \vdash \text{zero} : \mathbb{N} \]

ZeroSort

\[ [\Delta] \text{zero} = \text{zero} = [\Delta] [\Gamma] \text{zero} \]

Since \( FV(\text{zero}) = \emptyset \)

– Case

\[ \Gamma \vdash t : \mathbb{N} \]

SuccSort

\[ [\Delta] [\Gamma] t = [\Delta] t \]

By i.h.

\[ \text{succ}([\Delta] [\Gamma] t) = \text{succ}([\Delta] t) \]

By congruence of equality

\[ [\Delta] [\Gamma] \text{succ} (t) = [\Delta] \text{succ} (t) \]

By definition of substitution
• Proof of Part (ii): We have a derivation of $\Gamma \vdash P$ prop, and will use the previous part as a lemma.

- Case $\Gamma \vdash t : N \quad \Gamma \vdash t' : N$ \hspace{1cm} $EqProp$

$$\begin{align*}
\vdash \Delta | \Gamma | t &= \Delta | t \\
\vdash \Delta | \Gamma | t' &= \Delta | t' \\
(\vdash \Delta | \Gamma | t = \Delta | \Gamma | t') &= (\vdash \Delta | t = \Delta | t') & \text{By congruence of equality} \\
\vdash \Delta | \Gamma | (t = t') &= \Delta | (t = t') & \text{Definition of substitution}
\end{align*}$$

• Proof of Part (iii): By induction on the derivation of $\Gamma \vdash A$ type, using the previous parts as lemmas.

- Case $(u : \star) \in \Gamma \hspace{1cm} \text{VarWF}$

$$\begin{align*}
\vdash \Gamma \vdash u : \star & & \text{By rule VarSort} \\
\vdash \Delta | \Gamma | u &= \Delta | u & \text{By part (i)}
\end{align*}$$

- Case $(\alpha : \star = \tau) \in \Gamma \hspace{1cm} \text{SolvedVarWF}$

$$\begin{align*}
\vdash \Gamma \vdash \alpha : \star & & \text{By rule SolvedVarSort} \\
\vdash \Delta | \Gamma | \alpha &= \Delta | \alpha & \text{By part (i)}
\end{align*}$$

- Case $\Gamma \vdash 1$ type \hspace{1cm} \text{UnitWF}

$$\begin{align*}
\vdash \Gamma \vdash 1 : \star & & \text{By rule UnitSort} \\
\vdash \Delta | \Gamma | 1 &= \Delta | 1 & \text{By part (i)}
\end{align*}$$

- Case $\Gamma \vdash A_1$ type \hspace{1cm} $\Gamma \vdash A_2$ type \hspace{1cm} $\text{BinWF}$

$$\begin{align*}
\vdash \Gamma \vdash A_1 \oplus A_2 & & \text{By i.h.} \\
\vdash \Delta | \Gamma | A_1 &= \Delta | A_1 & \text{By i.h.} \\
\vdash \Delta | \Gamma | A_2 &= \Delta | A_2 & \text{By congruence of equality} \\
\vdash \Delta | \Gamma | (A_1 \oplus A_2) &= \Delta | (A_1 \oplus A_2) & \text{Definition of substitution}
\end{align*}$$

- Case $\text{VecWF}$ \hspace{1cm} Similar to the $\text{BinWF}$ case.

- Case $\Gamma, \alpha : \kappa \vdash A_0$ type \hspace{1cm} $\forall \alpha : \kappa \vdash A_0$ type \hspace{1cm} $\text{ForallWF}$

$$\begin{align*}
\Gamma \vdash A_0 & & \text{Given} \\
\Gamma, \alpha : \kappa \rightarrow \Delta, \alpha : \kappa & & \text{By rule Uvar} \\
\vdash \Delta, \alpha : \kappa | \Gamma | A_0 &= \Delta, \alpha : \kappa | A_0 & \text{By i.h.} \\
\vdash \Delta | \Gamma | A_0 &= \Delta | A_0 & \text{By definition of substitution} \\
\vdash \forall \alpha : \kappa, \Delta | \Gamma | A_0 &= \forall \alpha : \kappa, \Delta | A_0 & \text{By congruence of equality} \\
\vdash \Delta | \Gamma | (\forall \alpha : \kappa, A_0) &= \Delta | (\forall \alpha : \kappa, A_0) & \text{By definition of substitution}
\end{align*}$$
Proof of Lemma 29 (Substitution Monotonicity).

(i) If \( \Gamma \rightarrow \Delta \) and \( \Gamma \vdash t : \kappa \) and \( \text{FEV}([\Gamma]t) = \emptyset \) then \( [\Delta][\Gamma]t = [\Gamma]t \).

(ii) If \( \Gamma \rightarrow \Delta \) and \( \Gamma \vdash P \) prop and \( \text{FEV}([\Gamma]P) = \emptyset \) then \( [\Delta][\Gamma]P = [\Gamma]P \).

(iii) If \( \Gamma \rightarrow \Delta \) and \( \Gamma \vdash A \) type and \( \text{FEV}([\Gamma]A) = \emptyset \) then \( [\Delta][\Gamma]A = [\Gamma]A \).

Proof. Each part is a separate induction, relying on the proofs of the earlier parts. In each part, the result follows by an induction on the derivation of \( \Gamma \rightarrow \Delta \).

The main observation is that \( \Delta \) adds no equations for any variable of \( t \), \( P \), and \( A \) that \( \Gamma \) does not already contain, and as a result applying \( \Delta \) as a substitution to \([\Gamma]t\) does nothing.

Lemma 24 (Soft Extension).

If \( \Gamma \rightarrow \Delta \) and \( \Gamma, \Theta \; \text{ctx} \) and \( \Theta \) is soft, then there exists \( \Omega \) such that \( \text{dom}(\Theta) = \text{dom}(\Omega) \) and \( \Gamma, \Theta \rightarrow \Delta, \Omega \).

Proof. By induction on \( \Theta \).

• Case \( \Theta = \cdot \): We have \( \Gamma \rightarrow \Delta \). Let \( \Omega = \cdot \). Then \( \Gamma, \Theta \rightarrow \Delta, \Omega \).

• Case \( \Theta = (\Theta', \hat{\kappa} : \kappa = t) \):

  \[ \Gamma, \Theta' \rightarrow \Gamma, \Omega' \] 

  By i.h.

  \[ \Gamma, \Theta', \hat{\kappa} : \kappa = t \rightarrow \Delta, \Omega', \hat{\kappa} : \kappa = t \] 

  By rule \( \text{Solved} \)

• Case \( \Theta = (\Theta', \hat{\kappa} : \kappa) \):

  If \( \kappa = * \), let \( t = 1 \); if \( \kappa = \mathbb{N} \), let \( t = 0 \).

  \[ \Gamma, \Theta' \rightarrow \Gamma, \Omega' \] 

  By i.h.

  \[ \Gamma, \Theta', \hat{\kappa} : \kappa \rightarrow \Delta, \Omega', \hat{\kappa} : \kappa = t \] 

  By rule \( \text{Solve} \)

Lemma 31 (Split Extension).

If \( \Delta \rightarrow \Omega \)
and \( \hat{\kappa} \in \text{unsolved}(\Delta) \)
and \( \Omega = \Omega_1[\hat{\kappa} : \kappa = t_1] \)
and \( \Omega \) is canonical (Definition 3)
and \( \Omega \vdash t_2 : \kappa \)
then \( \Delta \rightarrow \Omega_1[\hat{\kappa} : \kappa = t_2] \).
Proof of Lemma 31 (Split Extension) lem:split-extension

**Proof.** By induction on the derivation of $\Delta \rightarrow \Omega$. Use the fact that $\Omega_1[\hat{\alpha} : \kappa = t_1]$ and $\Omega_1[\hat{\alpha} : \kappa = t_2]$ agree on all solutions except the solution for $\hat{\alpha}$. In the $\rightarrow \text{Solve}$ case where the existential variable is $\hat{\alpha}$, use $\Omega \vdash t_2 : \kappa$. \qed

C'.1 Reflexivity and Transitivity

Lemma 32 (Extension Reflexivity).

*If* $\Gamma \text{ ctx}$ *then* $\Gamma \rightarrow \Gamma$.

**Proof.** By induction on the derivation of $\Gamma \text{ ctx}$.

- **Case** $\text{EmptyCtx}

  - $\text{ctx}$

  - $\rightarrow \cdot$ *By rule $\rightarrow \text{Id}$*

- **Case** $\Gamma \text{ ctx} \ x \notin \text{dom}(\Gamma) \quad \Gamma \vdash A$ *type

  $\Gamma, x : A \text{ ctx}$

  $\Gamma \rightarrow \Gamma$ *By i.h.

  $[\Gamma]A = [\Gamma]A$ *By reflexivity

  $\Gamma, x : A \rightarrow \Gamma, x : A$ *By rule $\rightarrow \text{Var}$*

- **Case** $\Gamma \text{ ctx} \ u : \kappa \notin \text{dom}(\Gamma)$

  $\Gamma, u : \kappa \text{ ctx}$

  $\Gamma \rightarrow \Gamma$ *By i.h.

  $\Gamma, u : \kappa \rightarrow \Gamma, u : \kappa$ *By rule $\rightarrow \text{Uvar}$ or $\rightarrow \text{Unsolved}$

- **Case** $\Gamma \text{ ctx} \ ^{\hat{\alpha}} \notin \text{dom}(\Gamma)$

  $\Gamma \vdash \tau$ *type

  $\Gamma, ^{\hat{\alpha}} = \tau \text{ ctx}$

  $\Gamma \rightarrow \Gamma$ *By i.h.

  $[\Gamma] \tau = [\Gamma] \tau$ *By reflexivity

  $\Gamma, ^{\hat{\alpha}} = \tau \rightarrow \Gamma, ^{\hat{\alpha}} = \tau$ *By rule $\rightarrow \text{Solved}$

- **Case** $\Gamma \text{ ctx} \ \alpha \in \Gamma \quad (\alpha = -) \notin \Gamma \quad \Gamma \vdash \tau$ *type

  $\Gamma, \alpha = \tau \text{ ctx}$

  $\Gamma \rightarrow \Gamma$ *By i.h.

  $[\Gamma] \tau = [\Gamma] \tau$ *By reflexivity

  $\Gamma, \alpha = \tau \rightarrow \Gamma, \alpha = \tau$ *By rule $\rightarrow \text{Eqn}$
Proof of Lemma 32 (Extension Reflexivity).

• Case \( \Gamma \text{ctx } \triangleright u \not\in \Gamma \)
  \[
  \Gamma, \triangleright u \text{ctx} \quad \text{MarkerCtx}
  \]

  \( \Gamma \rightarrow \Gamma \) By i.h.
  \( \Gamma, \triangleright u \rightarrow \Gamma, \triangleright u \) By rule \( \rightarrow \text{Marker} \)

Lemma 33 (Extension Transitivity).

If \( D :: \Gamma \rightarrow \Theta \) and \( D' :: \Theta \rightarrow \Delta \) then \( \Gamma \rightarrow \Delta \).

Proof. By induction on \( D' \).

• Case

  \[
  \Theta \rightarrow \Delta' \quad \text{Var} \\
  \Theta', x : A \rightarrow \Delta', x : A' \quad \text{Var}
  \]

  \( \Gamma = \cdot \) By inversion on \( D \)
  \( \cdot \rightarrow \cdot \) By rule \( \rightarrow \text{Id} \)
  \( \Gamma \rightarrow \Delta \) Since \( \Gamma = \Delta = \cdot \)

  \[
  \Theta' \rightarrow \Delta' \quad \text{Var} \\
  \Theta', x : A'' \rightarrow \Delta', x : A' \quad \text{Var}
  \]

  \( \Gamma = (\Gamma', x : A''') \) By inversion on \( D \)
  \( |\Theta|A''' = |\Theta|A \) By inversion on \( D \)
  \( \Gamma' \rightarrow \Theta' \) By inversion on \( D \)
  \( \Gamma' \rightarrow \Delta' \) By i.h.
  \( |\Delta'|\Theta'' = |\Delta'|\Theta'A \) By congruence of equality
  \( |\Delta'|A'' = |\Delta'|A \) By Lemma 29 (Substitution Monotonicity)
  \( \Gamma', x : A'' \rightarrow \Delta', x : A' \) By \( \rightarrow \text{Var} \)

• Case

  \[
  \Theta' \rightarrow \Delta' \quad \text{Uvar} \\
  \Theta', \alpha : \kappa \rightarrow \Delta', \alpha : \kappa
  \]

  \( \Gamma = (\Gamma', \alpha : \kappa) \) By inversion on \( D \)
  \( \Gamma' \rightarrow \Theta' \) By inversion on \( D \)
  \( \Gamma' \rightarrow \Delta' \) By i.h.
  \( \Gamma', \alpha : \kappa \rightarrow \Delta', \alpha : \kappa \) By \( \rightarrow \text{Uvar} \)

• Case

  \[
  \Theta' \rightarrow \Delta' \quad \text{Unsolved} \\
  \Theta', \alpha : \kappa \rightarrow \Delta', \alpha : \kappa
  \]

  Two rules could have concluded \( D :: \Gamma \rightarrow (\Theta', \alpha : \kappa) \):
Proof of Lemma 33 (Extension Transitivity)

\[ \text{lem:extension-transitivity} \]

- Case $\Gamma' \rightarrow \Theta'$
  
  $\Gamma', \alpha : \kappa \rightarrow \Theta', \alpha : \kappa$ \hspace{1cm} Unresolved

  $\Gamma' \rightarrow \Delta'$ \hspace{1cm} By i.h.
  $\Gamma', \alpha : \kappa \rightarrow \Delta', \alpha : \kappa$ \hspace{1cm} By rule Add

- Case $\Gamma \rightarrow \Theta'$
  
  $\Gamma' \rightarrow \Theta'$ \hspace{1cm} Add

  $\Gamma \rightarrow \Delta'$ \hspace{1cm} By i.h.
  $\Gamma \rightarrow \Delta', \alpha : \kappa$ \hspace{1cm} By rule Add

- Case $\Theta' \rightarrow \Delta'$
  
  $[\Delta']t = [\Delta']t'$ \hspace{1cm} Solved

  $\Theta', \alpha : \kappa = t \rightarrow \Delta', \alpha : \kappa = t'$ \hspace{1cm} Solved

  $\Theta', \alpha : \kappa = t' \rightarrow \Delta', \alpha : \kappa = t'$ \hspace{1cm} Solved

Two rules could have concluded $D :: \Gamma \rightarrow (\Theta', \alpha : \kappa = t)$:

- Case $\Gamma' \rightarrow \Theta'$
  
  $[\Theta']t'' = [\Theta']t$ \hspace{1cm} Solved

  $\Gamma', \alpha : \kappa = t'' \rightarrow \Theta', \alpha : \kappa = t$ \hspace{1cm} Solved

  $\Gamma' \rightarrow \Delta'$ \hspace{1cm} By i.h.
  $[\Theta']t'' = [\Theta']t$ \hspace{1cm} Premise
  $[\Delta'][\Theta']t'' = [\Delta'][\Theta']t$ \hspace{1cm} Applying $\Delta'$ to both sides
  $[\Delta']t'' = [\Delta']t$ \hspace{1cm} By Lemma 29 (Substitution Monotonicity)
  $[\Delta']t' = [\Delta']t'$ \hspace{1cm} By premise $[\Delta']t = [\Delta']t'$
  $\Gamma', \alpha : \kappa = t'' \rightarrow \Delta', \alpha : \kappa = t'$ \hspace{1cm} By rule Solved

- Case $\Gamma \rightarrow \Theta'$
  
  $\Gamma' \rightarrow \Theta'$ \hspace{1cm} Add Solved

  $\Gamma \rightarrow \Delta'$ \hspace{1cm} By i.h.
  $\Gamma \rightarrow \Delta', \alpha : \kappa = t'$ \hspace{1cm} By rule Add Solved

- Case $\Theta' \rightarrow \Delta'$
  
  $[\Delta']t = [\Delta']t'$ \hspace{1cm} Eqn

  $\Theta', \alpha = t \rightarrow \Delta', \alpha = t'$ \hspace{1cm} Eqn
\begin{proof}[Proof of Lemma 33 (Extension Transitivity)]
\begin{align*}
\Gamma &= (\Gamma', \alpha = t'') \quad \text{By inversion on } D \\
\Gamma' &\rightarrow \Theta' \quad \text{By inversion on } D \\
[\Theta']t'' &= [\Theta']t \quad \text{By inversion on } D \\
[\Delta'][\Theta']t'' &= [\Delta'][\Theta']t \quad \text{Applying } \Delta' \text{ to both sides} \\
\Gamma' &\rightarrow \Delta' \quad \text{By i.h.} \\
[\Delta']t'' &= [\Delta']t' \quad \text{By premise } [\Delta']t = [\Delta']t' \\
\Gamma', \alpha = t'' &\rightarrow \Delta', \alpha = t' \quad \text{By rule } \rightarrow \text{Eqn}
\end{align*}

\begin{itemize}
\item \textbf{Case} \\
\begin{align*}
\Theta &\rightarrow \Delta' \quad \text{By i.h.} \\
\Theta &\rightarrow \Delta', \hat{\alpha} : \kappa \quad \text{By rule } \rightarrow \text{Eqn}
\end{align*}
\end{itemize}

\begin{itemize}
\item \textbf{Case} \\
\begin{align*}
\Theta &\rightarrow \Delta' \quad \text{By i.h.} \\
\Theta &\rightarrow \Delta', \hat{\alpha} : \kappa = t \quad \text{By rule } \rightarrow \text{AddSolved}
\end{align*}
\end{itemize}

\begin{itemize}
\item \textbf{Case} \\
\begin{align*}
\Theta' &\rightarrow \Delta' \quad \text{By i.h.} \\
\Theta' &\rightarrow \Delta', \check{\alpha} : \kappa = t \quad \text{By rule } \rightarrow \text{AddSolved}
\end{align*}
\end{itemize}

\begin{itemize}
\item \textbf{Case} \\
\begin{align*}
\Theta' &\rightarrow \Delta' \quad \text{By i.h.} \\
\Theta' &\rightarrow \Delta', \check{\alpha} : \kappa \quad \text{By rule } \rightarrow \text{Uvar}
\end{align*}
\end{itemize}

\end{proof}

\section{C.2 Weakening}

\textbf{Lemma 34} (Suffix Weakening). \textit{If } \Gamma \vdash t : \kappa \text{ then } \Gamma, \Theta \vdash t : \kappa. \\
\textit{Proof.} By induction on the given derivation. All cases are straightforward.

\textbf{Lemma 35} (Suffix Weakening). \textit{If } \Gamma \vdash A \text{ type then } \Gamma, \Theta \vdash A \text{ type.} \\
\textit{Proof.} By induction on the given derivation. All cases are straightforward.

\textbf{Lemma 36} (Extension Weakening (Sorts)). \textit{If } \Gamma \vdash t : \kappa \text{ and } \Gamma \rightarrow \Delta \text{ then } \Delta \vdash t : \kappa. \\
\textit{Proof.} By a straightforward induction on \( \Gamma \vdash t : \kappa. \\
\text{In the VarSort case, use Lemma 22 (Extension Inversion) (i) or (v). In the SolvedVarSort case, use Lemma 22 (Extension Inversion) (iv). In the other cases, apply the i.h. to all subderivations, then apply the rule.} \\
\textbf{Lemma 37} (Extension Weakening (Props)). \textit{If } \Gamma \vdash P \text{ prop and } \Gamma \rightarrow \Delta \text{ then } \Delta \vdash P \text{ prop.}
Proof. By inversion on rule $\mathsf{EqProp}$ and Lemma 36 (Extension Weakening (Sorts)) twice.

Lemma 38 (Extension Weakening (Types)). If $\Gamma \vdash A : \text{type}$ and $\Gamma \rightarrow \Delta$ then $\Delta \vdash A : \text{type}$.  

Proof. By a straightforward induction on $\Gamma \vdash A : \text{type}$.

In the $\text{VarWF}$ case, use Lemma 22 (Extension Inversion) (i) or (v). In the $\text{SolvedVarWF}$ case, use Lemma 22 (Extension Inversion) (iv).

In the other cases, apply the i.h. and/or (for $\text{ImpliesWF}$ and $\text{WithWF}$) Lemma 37 (Extension Weakening (Props)) to all subderivations, then apply the rule.

C’.3 Principal Typing Properties

Lemma 39 (Principal Agreement).

(i) If $\Gamma \vdash A : \text{type}$ and $\Gamma \rightarrow \Delta$ then $[\Delta]A = [\Gamma]A$.

(ii) If $\Gamma \vdash P : \text{prop}$ and $\text{FEV}(P) = \emptyset$ and $\Gamma \rightarrow \Delta$ then $[\Delta]P = [\Gamma]P$.

Proof. By induction on the derivation of $\Gamma \rightarrow \Delta$.

Part (i):

- Case $\Gamma_0 \rightarrow \Delta_0$

  $\Gamma_0, \alpha = t \rightarrow \frac{[\Delta_0]t = [\Delta_0]t'}{\Delta}$

  If $\alpha \notin \text{FV}(A)$, then:
  
  $[\Gamma_0, \alpha = t]A = [\Gamma_0]A$ \hspace{1cm} \text{By def. of subst.}
  
  $= [\Delta_0]A$ \hspace{1cm} \text{By i.h.}
  
  $= [\Delta_0, \alpha = t']A$ \hspace{1cm} \text{By def. of subst.}

  Otherwise, $\alpha \in \text{FV}(A)$.

  $\Gamma_0 \vdash t : \text{type}$ \hspace{1cm} \text{\textit{Γ} is well-formed}

  $\Gamma_0 \vdash [\Gamma_0]t : \text{type}$ \hspace{1cm} \text{By Lemma 13 (Right-Hand Substitution for Typing)}

  Suppose, for a contradiction, that $\text{FEV}([\Gamma_0]t) \neq \emptyset$.
  
  Since $\alpha \in \text{FV}(A)$, we also have $\text{FEV}([\Gamma]A) \neq \emptyset$, a contradiction.
Proof of Lemma 39 (Principal Agreement).

lem:substitution-tpp-stable

\[ \text{FEV}([\Gamma_0] t) \neq \emptyset \]
\[ [\Gamma_0] t = [\Gamma] \alpha \]
Assumption (for contradiction)

By def. of subst.

\[ \text{FEV}(\alpha) \neq \emptyset \]
\[ \alpha \in \text{FV}(A) \]
By above equality

Above

\[ \Gamma \vdash A \text{ type} \]
\[ \text{FEV}(\alpha) \neq \emptyset \]
By a property of subst.

Given

\[ \Gamma_0 \vdash [\Delta_0] t \]
\[ \text{FEV}(\alpha) \neq \emptyset \]
By above equality

\[ A \]
\[ \alpha \in \text{FV}(A) \]
Above

\[ \Gamma_0 \vdash [\Delta_0] t \]
\[ \text{FEV}(\alpha) \neq \emptyset \]
By above equality

\[ \Gamma_0 \vdash [\Delta_0] A \]
\[ \text{By inversion} \]

\[ \Rightarrow \]
\[ \text{FEV}(\alpha) \neq \emptyset \]
By contradiction

\[ \Gamma \rightarrow [\Delta_0] t \]
\[ \text{By Lemma} \ [\text{Substitution—Well-formedness}] \ (i) \]

\[ \text{Given} \]

\[ \text{FEV}(\alpha) \neq \emptyset \]
By above equality

\[ \Gamma_0 \vdash [\Delta_0] A \]
\[ \text{By i.h.} \]

\[ \Rightarrow \]
\[ \text{FEV}(\alpha) \neq \emptyset \]
By above equality

\[ [\Delta_0] t := [\alpha] A \]
\[ \text{By def. of subst.} \]

\[ \Gamma_0 \vdash \alpha = t A \]
\[ \text{By def. of subst.} \]

\[ \Gamma \rightarrow [\Delta_0] A \]
\[ \text{By i.h.} \]

\[ \text{If} \Gamma_0 \vdash [\Delta_0] A \text{ type} \text{ then } \Gamma_0 \vdash [\Delta_0] A \text{ type} \]

Proof.

By cases of \( p \):

- Case \( p = ! \):
  \[ \Gamma \vdash A \text{ type} \]
  By inversion
  \[ \text{FEV}(\alpha) \neq \emptyset \]
  By inversion
  \[ \Gamma \vdash [\Delta_0] A \text{ type} \]
  By Lemma 13 (Right-Hand Substitution for Typing)
  \[ \Gamma \rightarrow \Gamma \]
  By Lemma 32 (Extension Reflexivity)
  \[ [\Gamma] [\Delta_0] A = [\Gamma] A \text{ type} \]
  By Lemma 29 (Substitution Monotonicity)
  \[ \text{FEV}(\alpha) \neq \emptyset \]
  By inversion
  \[ \Gamma \vdash [\Delta_0] A \text{ type} \]
  By rule \text{PrincipalWF}

- Case \( p = \emptyset \):
  \[ \Gamma \vdash A \text{ type} \]
  By inversion
  \[ \Gamma \vdash [\Delta_0] A \text{ type} \]
  By Lemma 13 (Right-Hand Substitution for Typing)
  \[ \Gamma \vdash A \emptyset \text{ type} \]
  By rule \text{NonPrincipalWF}

\[ \square \]

Lemma 40 (Right-Hand Subst. for Principal Typing). \( \text{If } \Gamma \vdash A \text{ type } \text{ then } \Gamma \vdash [\Gamma] A \text{ type} \).

Proof. By cases of \( p \):

- Case \( p = ! \):
  \[ \Gamma \vdash A \text{ type} \]
  By inversion
  \[ \text{FEV}(\alpha) \neq \emptyset \]
  By inversion
  \[ \Gamma \vdash [\Delta_0] A \text{ type} \]
  By Lemma 13 (Right-Hand Substitution for Typing)
  \[ \Gamma \rightarrow \Gamma \]
  By Lemma 32 (Extension Reflexivity)
  \[ [\Gamma] [\Delta_0] A = [\Gamma] A \text{ type} \]
  By Lemma 29 (Substitution Monotonicity)
  \[ \text{FEV}(\alpha) \neq \emptyset \]
  By inversion
  \[ \Gamma \vdash [\Delta_0] A \text{ type} \]
  By rule \text{PrincipalWF}

- Case \( p = \emptyset \):
  \[ \Gamma \vdash A \text{ type} \]
  By inversion
  \[ \Gamma \vdash [\Delta_0] A \text{ type} \]
  By Lemma 13 (Right-Hand Substitution for Typing)
  \[ \Gamma \vdash A \emptyset \text{ type} \]
  By rule \text{NonPrincipalWF}

\[ \square \]

Lemma 41 (Extension Weakening for Principal Typing). \( \text{If } \Gamma \vdash A \text{ type } \text{ and } \Gamma \rightarrow \Delta \text{ then } \Delta \vdash A \text{ type} \).

Proof. By cases of \( p \):

- Case \( p = ! \):
  \[ \Gamma \vdash A \text{ type} \]
  By inversion
  \[ \text{FEV}(\alpha) \neq \emptyset \]
  By inversion
  \[ \Gamma \vdash [\Delta_0] A \text{ type} \]
  By Lemma 13 (Right-Hand Substitution for Typing)
  \[ \Gamma \rightarrow \Gamma \]
  By Lemma 32 (Extension Reflexivity)
  \[ [\Gamma] [\Delta_0] A = [\Gamma] A \text{ type} \]
  By Lemma 29 (Substitution Monotonicity)
  \[ \text{FEV}(\alpha) \neq \emptyset \]
  By inversion
  \[ \Gamma \vdash [\Delta_0] A \text{ type} \]
  By rule \text{PrincipalWF}

- Case \( p = \emptyset \):
  \[ \Gamma \vdash A \text{ type} \]
  By inversion
  \[ \Gamma \vdash [\Delta_0] A \text{ type} \]
  By Lemma 13 (Right-Hand Substitution for Typing)
  \[ \Gamma \vdash A \emptyset \text{ type} \]
  By rule \text{NonPrincipalWF}

\[ \square \]
Proof. By cases of $p$:

- Case $p = \uparrow$:
  
  \[
  \begin{align*}
  & \Gamma \vdash A \text{ type} \quad \text{By inversion} \\
  & \Delta \vdash A \text{ type} \quad \text{By Lemma 38 [Extension Weakening (Types)]} \\
  & \Delta \vdash A \uparrow \text{ type} \quad \text{By rule NonPrincipalWF}
  \end{align*}
  \]

- Case $p = !$:
  
  \[
  \begin{align*}
  & \Gamma \vdash A \text{ type} \quad \text{By inversion} \\
  & \text{FEV}(\Gamma \vdash A) = 0 \quad \text{By inversion} \\
  & \Delta \vdash A \text{ type} \quad \text{By Lemma 38 [Extension Weakening (Types)]} \\
  & \Delta \vdash |\Delta| A \text{ type} \quad \text{By Lemma 13 [Right-Hand Substitution for Typing]} \\
  & [\Delta] \vdash [\Delta] A \text{ type} \quad \text{By Lemma 30 [Substitution Invariance]} \\
  & \text{FEV}([\Delta]|A) = 0 \quad \text{By congruence of equality} \\
  & \Delta \vdash [\Delta] A \uparrow \text{ type} \quad \text{By rule PrincipalWF}
  \end{align*}
  \]

Lemma 42 (Inversion of Principal Typing).

1. If $\Gamma \vdash (A \rightarrow B) \ p \text{ type}$ then $\Gamma \vdash A \ p \text{ type}$ and $\Gamma \vdash B \ p \text{ type}$.
2. If $\Gamma \vdash (P \supset A) \ p \text{ type}$ then $\Gamma \vdash P \text{ prop}$ and $\Gamma \vdash A \ p \text{ type}$.
3. If $\Gamma \vdash (A \land P) \ p \text{ type}$ then $\Gamma \vdash P \text{ prop}$ and $\Gamma \vdash A \ p \text{ type}$.

Proof. Proof of part 1:

We have $\Gamma \vdash A \rightarrow B \ p \text{ type}$.

- Case $p = \uparrow$:
  
  \[
  \begin{align*}
  & 1 \quad \Gamma \vdash A \rightarrow B \text{ type} \quad \text{By inversion} \\
  & \Gamma \vdash A \text{ type} \quad \text{By inversion on 1} \\
  & \Gamma \vdash B \text{ type} \quad \text{By inversion on 1} \\
  & \Gamma \vdash A \uparrow \text{ type} \quad \text{By rule NonPrincipalWF} \\
  & \Gamma \vdash B \uparrow \text{ type} \quad \text{By rule NonPrincipalWF}
  \end{align*}
  \]

- Case $p = !$:
  
  \[
  \begin{align*}
  & 1 \quad \Gamma \vdash A \rightarrow B \text{ type} \quad \text{By inversion on } \Gamma \vdash A \rightarrow B \uparrow \text{ type} \\
  & \emptyset = \text{FEV}(\Gamma \vdash (A \rightarrow B)) \\
  & = \text{FEV}(\Gamma \vdash A \rightarrow [\Gamma] B) \quad \text{By definition of substitution} \\
  & = \text{FEV}(\Gamma \vdash A) \cup \text{FEV}(\Gamma \vdash B) \quad \text{By definition of FEV(−)} \\
  & \text{FEV}(\Gamma \vdash A) = \text{FEV}(\Gamma \vdash B) = 0 \quad \text{By properties of empty sets and unions} \\
  & \Gamma \vdash A \text{ type} \quad \text{By inversion on 1} \\
  & \Gamma \vdash B \text{ type} \quad \text{By inversion on 1} \\
  & \Gamma \vdash A \uparrow \text{ type} \quad \text{By rule PrincipalWF} \\
  & \Gamma \vdash B \uparrow \text{ type} \quad \text{By rule PrincipalWF}
  \end{align*}
  \]

Part 2: We have $\Gamma \vdash P \supset A \ p \text{ type}$. Similar to Part 1.

Part 3: We have $\Gamma \vdash A \land P \ p \text{ type}$. Similar to Part 2.
\subsection*{C'.4 Instantiation Extends}

\textbf{Lemma 43} (Instantiation Extension).
\[\text{If } \Gamma \vdash \alpha := \tau : \kappa \rightarrow \Delta \text{ then } \Gamma \rightarrow \Delta.\]
\textbf{Proof.} By induction on the given derivation.

- \textbf{Case} \[\Gamma \vdash \tau : \kappa\]
  \[\Gamma, \alpha : \kappa, \Gamma \vdash \alpha := \tau : \kappa \rightarrow \Gamma, \alpha : \tau, \Gamma \]
  Follows by Lemma 23 (Deep Evar Introduction) (ii).

- \textbf{Case} \[\beta \in \text{unsolved}(\Gamma_0[\alpha : \kappa][\beta : \kappa])\]
  \[\Gamma_0[\alpha : \kappa][\beta : \kappa] \vdash \alpha := \beta : \kappa \rightarrow \Gamma_0[\alpha : \kappa][\beta : \kappa = \alpha]\]
  Follows by Lemma 23 (Deep Evar Introduction) (ii).

- \textbf{Case} \[\Gamma_0[\alpha_2 : *, \alpha_1 : *, \alpha : * = \alpha_1 \oplus \alpha_2] \vdash \alpha_1 := \tau_1 : \tau_1 \rightarrow \Theta\]
  \[\Theta \vdash \alpha_2 := [\Theta]^{\tau_2} : \tau_2 \rightarrow \Delta\]
  By reasoning similar to the \textbf{InstBin} case.

\textbf{C'.5 Equivalence Extends}

\textbf{Lemma 44} (Elimeq Extension).
\[\text{If } \Gamma / s \equiv t : \kappa \rightarrow \Delta \text{ then there exists } \Theta \text{ such that } \Gamma, \Theta \rightarrow \Delta.\]
Proof of Lemma 44 (Elimeq Extension).

Proof. By induction on the given derivation. Note that the statement restricts the output to be a (consistent) context $\Delta$.

- **Case**
  \[
  \Gamma / \alpha \equiv \alpha : \kappa \vdash \Gamma
  \]
  \[
  \text{(ElimeqUvarRef)}
  \]
  Since $\Delta = \Gamma$, applying Lemma 32 (Extension Reflexivity) suffices (let $\Theta = \varnothing$).

- **Case**
  \[
  \Gamma / \text{zero} \equiv \text{zero} : \mathbb{N} \vdash \Gamma
  \]
  \[
  \text{(ElimeqZero)}
  \]
  Similar to the ElimeqUvarRef case.

- **Case**
  \[
  \Gamma / \sigma \equiv t : \mathbb{N} \vdash \Delta
  \]
  \[
  \Gamma / \text{succ}(\sigma) \equiv \text{succ}(t) : \mathbb{N} \vdash \Delta
  \]
  \[
  \text{(ElimeqSucc)}
  \]
  Follows by i.h.

- **Case**
  \[
  \Gamma_0[\delta : \kappa] \vdash \delta := t : \kappa \vdash \Delta
  \]
  \[
  \Gamma_0[\delta : \kappa] / \delta \equiv t : \kappa \vdash \Delta
  \]
  \[
  \text{(ElimeqInstL)}
  \]
  
  \[
  \Gamma \vdash \delta := t : \kappa \vdash \Delta
  \]
  Subderivation
  
  \[
  \Gamma \rightarrow \Delta
  \]
  By Lemma 43 (Instantiation Extension)
  
  Let $\Theta = \varnothing$.
  
  \[
  \Rightarrow \Gamma, \Theta \rightarrow \Delta
  \]
  By $\Theta = \varnothing$.

- **Case**
  \[
  \alpha \notin FV([\Gamma]t) \quad (\alpha = \vDash) \notin \Gamma
  \]
  \[
  \Gamma / \alpha \equiv t : \kappa \vdash \Gamma, \alpha = t
  \]
  \[
  \text{(ElimeqUvarL)}
  \]
  Let $\Theta$ be $(\alpha = t)$.
  
  \[
  \Rightarrow \Gamma, \alpha = t \rightarrow \Gamma, \alpha = t
  \]
  By Lemma 32 (Extension Reflexivity)

- **Cases** ElimeqInstR, ElimeqUvarR
  Similar to the respective L cases.

- **Case**
  \[
  \sigma \neq t
  \]
  \[
  \Gamma / \sigma \equiv t : \kappa \vdash \bot
  \]
  \[
  \text{(ElimeqClash)}
  \]
  The statement says that the output is a (consistent) context $\Delta$, so this case is impossible. □

Lemma 45 (Elimprop Extension).

If $\Gamma / P \rightarrow \Delta$ then there exists $\Theta$ such that $\Gamma, \Theta \rightarrow \Delta$.

Proof. By induction on the given derivation. Note that the statement restricts the output to be a (consistent) context $\Delta$. 

Proof of Lemma 45 (Elimprop Extension).
Proof of Lemma 45 (Elimprop Extension).

\[ \frac{\Gamma / \sigma \equiv t : \mathbb{N} \vdash \Delta}{\Gamma / \sigma = t \vdash \Delta} \text{ ElimpropEq} \]

\[ \equiv \Gamma, \Theta \rightarrow \Delta \quad \text{Subderivation} \]

By Lemma 44 (Elimeq Extension).

Lemma 46 (Checkeq Extension).
If \( \Gamma \vdash A \equiv B \vdash \Delta \) then \( \Gamma \rightarrow \Delta \).

Proof. By induction on the given derivation.

\[ \cdot \text{ Case } \Gamma \vdash u \equiv u : \kappa \rightarrow \Gamma \text{ CheckeqVar} \]

Since \( \Delta = \Gamma \), applying Lemma 32 (Extension Reflexivity) suffices.

\[ \cdot \text{ Cases CheckeqUnit, CheckeqZero } \quad \text{Similar to the CheckeqVar case.} \]

\[ \cdot \text{ Case } \Gamma \vdash \tau_1 \equiv \tau_1' : * \rightarrow \Theta \quad \Theta \vdash [\Theta] \tau_2 \equiv [\Theta] \tau_2' : * \rightarrow \Delta \]

\[ \Gamma \vdash \tau_1 \oplus \tau_2 \equiv \tau_1' \oplus \tau_2' : * \rightarrow \Delta \quad \text{CheckeqBin} \]

\[ \equiv \Gamma \rightarrow \Theta \quad \text{By i.h.} \]

\[ \Theta \rightarrow \Delta \quad \text{By i.h.} \]

\[ \equiv \Gamma \rightarrow \Delta \quad \text{By Lemma 33 (Extension Transitivity)} \]

\[ \cdot \text{ Case } \Gamma \vdash \sigma \equiv t : \mathbb{N} \rightarrow \Delta \text{ CheckeqSucc} \]

\[ \Gamma \vdash \text{succ}(\sigma) \equiv \text{succ}(t) : \mathbb{N} \rightarrow \Delta \quad \text{Subderivation} \]

\[ \equiv \Gamma \rightarrow \Delta \quad \text{By i.h.} \]

\[ \cdot \text{ Case } \Gamma_0[\alpha] \vdash \alpha := t : \kappa \rightarrow \Delta \quad \alpha \notin \text{FV}([\Gamma_0[\alpha]]t) \text{ CheckeqInstL} \]

\[ \Gamma_0[\alpha] \vdash \alpha \equiv t : \kappa \rightarrow \Delta \quad \text{Subderivation} \]

\[ \equiv \Gamma_0[\alpha] \rightarrow \Delta \quad \text{By Lemma 43 (Instantiation Extension)} \]

\[ \cdot \text{ Case CheckeqInstR } \quad \text{Similar to the CheckeqInstL case.} \]

Lemma 47 (Checkprop Extension).
If \( \Gamma \vdash P \text{ true } \rightarrow \Delta \) then \( \Gamma \rightarrow \Delta \).

Proof. By induction on the given derivation.
**Case**  \[\Gamma \vdash \sigma \triangleq t : N \vdash \Delta\]  
\[\Gamma \vdash \sigma \triangleq t \text{ true} \vdash \Delta\]  
\[\Gamma \vdash \sigma \triangleq t : N \vdash \Delta\]  

\[\equiv \text{PropEq}\]

**Case**  
\[\Gamma \vdash \sigma_1 \triangleq \tau_1 : N \vdash \Theta\]  
\[\Theta \vdash \sigma_2 \triangleq \tau_2 : N \vdash \Delta\]  
\[\Gamma \vdash (\sigma_1 = \sigma_2) \equiv (\tau_1 = \tau_2) \vdash \Delta\]  
\[\equiv \text{PropEq}\]

\[\Gamma \vdash \sigma_1 \triangleq \tau_1 : N \vdash \Theta\]  
By Lemma 46 (Checkeq Extension)

\[\Gamma \rightarrow \Delta\]  
By Lemma 46 (Checkeq Extension)

**Lemma 48** (Prop Equivalence Extension).
If \(\Gamma \vdash P \equiv Q \vdash \Delta\) then \(\Gamma \rightarrow \Delta\).

**Proof.** By induction on the given derivation.

**Case**  
\[\Gamma \vdash \sigma_1 \triangleq \tau_1 : N \vdash \Theta\]  
\[\Theta \vdash \sigma_2 \triangleq \tau_2 : N \vdash \Delta\]  
\[\Gamma \vdash (\sigma_1 = \sigma_2) \equiv (\tau_1 = \tau_2) \vdash \Delta\]  
\[\equiv \text{PropEq}\]

\[\Gamma \vdash \Theta\]  
By Lemma 46 (Checkeq Extension)

\[\Gamma \rightarrow \Delta\]  
By Lemma 46 (Checkeq Extension)

\[\Theta \rightarrow \Delta\]  
By Lemma 46 (Checkeq Extension)

\[\equiv \text{PropEq}\]

**Lemma 49** (Equivalence Extension).
If \(\Gamma \vdash A \equiv B \vdash \Delta\) then \(\Gamma \rightarrow \Delta\).

**Proof.** By induction on the given derivation.

**Case**  
\[\Gamma \vdash \alpha \equiv \alpha \vdash \Gamma\]  
\[\equiv \text{Var}\]

Here \(\Delta = \Gamma\), so Lemma 32 (Extension Reflexivity) suffices.

**Case**  
\[\Gamma \vdash \& \equiv \& \vdash \Gamma\]  
\[\equiv \text{Exvar}\]

Similar to the \[\equiv \text{Var}\] case.

**Case**  
\[\Gamma \vdash 1 \equiv 1 \vdash \Gamma\]  
\[\equiv \text{Unit}\]

Similar to the \[\equiv \text{Var}\] case.

**Case**  
\[\Gamma \vdash A_1 \equiv B_1 \vdash \Theta\]  
\[\Theta \vdash A_2 \equiv B_2 \vdash \Delta\]  
\[\Gamma \vdash (A_1 \oplus A_2) \equiv (B_1 \oplus B_2) \vdash \Delta\]  
\[\equiv \text{Unit}\]

\[\Gamma \vdash A_1 \equiv B_1 \vdash \Theta\]  
Subderivation

\[\Gamma \rightarrow \Theta\]  
By i.h.

\[\Theta \vdash A_2 \equiv B_2 \vdash \Delta\]  
Subderivation

\[\Theta \rightarrow \Delta\]  
By i.h.

\[\equiv \Gamma \rightarrow \Delta\]  
By Lemma 33 (Extension Transitivity)

**Proof of Lemma 49** (Equivalence Extension)
• Case \(\equiv_{\text{Vec}}\) Similar to the \(\equiv_{\text{Vec}}\) case.

• Cases \(\equiv_{\lor} \equiv_{\land}\) Similar to the \(\equiv_{\lor}\) case, but with Lemma 48 (Prop Equivalence Extension) on the first premise.

• Case 
  \[\Gamma, \alpha : \kappa \vdash A_0 \equiv B \vdash \Delta, \alpha : \kappa, \Delta'\]
  \[\Gamma \vdash \forall \alpha : \kappa. A_0 \equiv \forall \alpha : \kappa. B \vdash \Delta\]
  Subderivation
  \[\Gamma, \alpha : \kappa \rightarrow \Delta, \alpha : \kappa, \Delta'\] By i.h.
  \[\Gamma \rightarrow \Delta\]
  By Lemma 22 (Extension Inversion) (i)

• Case 
  \[\Gamma \vdash ([\hat{\alpha}] \tau : \star \vdash \Delta) \quad \hat{\alpha} \notin FV([\Gamma_0[\hat{\alpha}]]\tau)\]
  \[\Gamma_0[\hat{\alpha}] \vdash \equiv \equiv \tau \vdash \Delta\]
  \[\Gamma \vdash \tau : \tau \vdash \Delta\]
  Subderivation
  \[\Gamma_0[\hat{\alpha}] \rightarrow \Delta\]
  By Lemma 43 (Instantiation Extension)

• Case \(\equiv_{\text{InstantiateR}}\) Similar to the \(\equiv_{\text{InstantiateL}}\) case.

C’.6 Subtyping Extends

Lemma 50 (Subtyping Extension). If \(\Gamma \vdash A : \tau \vdash \Delta\) then \(\Gamma \rightarrow \Delta\).

Proof. By induction on the given derivation.

• Case 
  \[\Gamma, \hat{\alpha} : \kappa, \Theta \vdash \tau : \star \vdash \Delta, \hat{\alpha} : \kappa, \Theta\]
  \[\Gamma \vdash \forall \alpha : \kappa. A : B \vdash \Delta\]
  Subderivation
  \[\Gamma, \hat{\alpha} : \kappa \rightarrow \Delta, \hat{\alpha} : \kappa, \Theta\] By i.h. (i)
  \[\Gamma \rightarrow \Delta\]
  By Lemma 22 (Extension Inversion) (ii)

• Case \(\left.<:\right>_{\forall L}\) Similar to the \(\left.<:\right>_{\forall L}\) case.

• Case 
  \[\Gamma, \alpha : \kappa \vdash A : B \vdash \Delta, \alpha : \kappa, \Theta\]
  \[\Gamma \vdash A : B \vdash \Delta\]
  \[\Gamma \vdash \forall \alpha : \kappa. A \vdash \Delta\]
  Similar to the \(\left.<:\right>_{\forall L}\) case, but using part (i) of Lemma 22 (Extension Inversion).

• Case \(\left.<:\right>_{\exists L}\) Similar to the \(\left.<:\right>_{\exists R}\) case.

• Case 
  \[\Gamma \vdash A \equiv B \vdash \Delta\]
  \[\Gamma \vdash A : B \vdash \Delta\]
  \[\Gamma \vdash A \equiv B \vdash \Delta\]
  Subderivation
  \[\Gamma \rightarrow \Delta\]
  By Lemma 49 (Equivalence Extension)
C’.7 Typing Extends

Lemma 51 (Typing Extension).

If \( \Gamma \vdash e \Leftarrow A \vdash \Delta \)
or \( \Gamma \vdash e \Rightarrow A \vdash \Delta \)
or \( \Gamma \vdash s : A \vdash \Delta \)
or \( \Gamma \vdash \Pi :: A \vdash \Delta \)
then \( \Gamma \rightarrow \Delta \).

Proof. By induction on the given derivation.

- **Match judgments:**
  - In rule \textbf{MatchEmpty}, \( \Delta = \Gamma \), so the result follows by Lemma \textbf{32} (Extension Reflexivity).
  - Rules \textbf{MatchBase}, \textbf{Match×}, \textbf{Match+}, and \textbf{MatchWild} each have a single premise in which the contexts match the conclusion (input \( \Gamma \) and output \( \Delta \)), so the result follows by i.h. For rule \textbf{MatchSeq}, Lemma \textbf{33} (Extension Transitivity) is also needed.

- **Synthesis, checking, and spine judgments:** In rules \textbf{Var}, \textbf{1I}, \textbf{EmptySpine} and \textbf{▷} the output context \( \Delta \) is exactly \( \Gamma \), so the result follows by Lemma \textbf{32} (Extension Reflexivity).
  - Case \( \forall I \): Use the i.h. and Lemma \textbf{33} (Extension Transitivity).
  - Case \( \forall \text{Spine} \): By \textbf{Add}, \( \Gamma \rightarrow \Gamma, \vdash \kappa \).
    The result follows by i.h. and Lemma \textbf{33} (Extension Transitivity).
  - Cases \( \forall I \) \( \exists \text{Spine} \): Use Lemma \textbf{47} (Checkprop Extension), the i.h., and Lemma \textbf{33} (Extension Transitivity).
  - Cases \( \text{Nil} \) \( \text{Cons} \): Using reasoning found in the \( \forall I \) and \( \exists I \) cases.
  - Case \( \exists I \):
    - By Lemma \textbf{45} (Elimprop Extension).
    - By i.h.
  - Cases \( \rightarrow I \) \( \text{Rec} \): Use the i.h. and Lemma \textbf{22} (Extension Inversion).
  - Cases \( \text{Sub} \) \( \text{Anno} \) \( \rightarrow E \) \( \rightarrow \text{Spine} \) \( +I \) \( \times I \):
    Use the i.h., and Lemma \textbf{33} (Extension Transitivity) as needed.
  - Case \( 1I \): By Lemma \textbf{23} (Deep Evar Introduction) (ii).
Lemma 52 (Context Partitioning)

By induction on the derivation of

Proof.

If

Lemma 54 (Completing Stability)

Proof. By induction on the given derivation.

C.8 Unfiled

Lemma 52 (Context Partitioning).

If \( \Delta, \triangleright_\alpha, \Theta \rightarrow \Omega, \triangleright_\alpha, \Omega_Z \) then there is a \( \Psi \) such that \( [\Omega, \triangleright_\alpha, \Omega_Z](\Delta, \triangleright_\alpha, \Theta) = [\Omega]_\Delta, \Psi \).

Proof. By induction on the given derivation.

- Case \( \triangleright_\alpha \text{Spine} \) +\( \triangleright_\alpha \times \triangleright_\alpha \)
  Use Lemma \ref{lem:deep-eval-introduction} (Deep Evar Introduction) (i) twice, Lemma \ref{lem:deep-eval-introduction} (Deep Evar Introduction) (ii), the i.h., and Lemma \ref{lem:extension-transitivity} (Extension Transitivity).

- Case \( \triangleright_\alpha \) Use Lemma \ref{lem:deep-eval-introduction} (Deep Evar Introduction) (i) twice, Lemma \ref{lem:deep-eval-introduction} (Deep Evar Introduction) (ii), the i.h. and Lemma \ref{lem:extension-inversion} (Extension Inversion) (v).

- Case \( \text{Case} \) Use the i.h. on the synthesis premise and the match premise, and then Lemma \ref{lem:extension-transitivity} (Extension Transitivity).

\[ \square \]

Proof of Lemma 54 (Completing Stability).

If \( \Gamma \rightarrow \Omega \) then \( [\Omega]_\Gamma = [\Omega] \).

Proof. By induction on the derivation of \( \Gamma \rightarrow \Omega \).

- Case \( \Delta \rightarrow \text{Id} \)
  Impossible: \( \Delta, \triangleright_\alpha, \Theta \) cannot have the form \( . \).

- Case \( \Delta \rightarrow \text{Var} \)
  We have \( \Omega_Z = (\Omega'_Z, x : A) \) and \( \Theta = (\Theta', x : A') \). By i.h., there is \( \Psi' \) such that \( [\Omega, \triangleright_\alpha, \Omega'_Z](\Delta, \triangleright_\alpha, \Theta') = [\Omega]_\Delta, \Psi' \). Then by the definition of context application, \( [\Omega, \triangleright_\alpha, \Omega'_Z, x : A](\Delta, \triangleright_\alpha, \Theta', x : A') = [\Omega]_\Delta, \Psi', x : [\Omega']_A \). Let \( \Psi = (\Psi', x : [\Omega']_A) \).

- Case \( \Delta \rightarrow \text{Uvar} \)
  Similar to the \( \Delta \rightarrow \text{Var} \) case, with \( \Psi = (\Psi', \alpha : \kappa) \).

- Cases \( \text{Case} \rightarrow \text{Unsolved} \rightarrow \text{Solved} \rightarrow \text{Solve} \rightarrow \text{Add} \rightarrow \text{AddSolved} \rightarrow \text{Marker} \)
  Broadly similar to the \( \Delta \rightarrow \text{Uvar} \) case, but the rightmost context element disappears in context application, so we let \( \Psi = \Psi' \).

\[ \square \]
• Case $\Gamma_0 \Rightarrow \Omega_0$
  $\Gamma_0, \hat{\alpha} : \kappa \Rightarrow \Omega_0, \hat{\alpha} : \kappa$ \hspace{1cm} \text{Unsolved}
  Similar to $\Rightarrow \Var$

• Case $\Gamma_0 \Rightarrow \Omega_0$
  $[\Omega_0] t = [\Omega_0] t'$
  $\Gamma_0, \hat{\alpha} : t \Rightarrow \Omega_0, \hat{\alpha} : t'$ \hspace{1cm} \text{Solved}
  Similar to $\Rightarrow \Var$

• Case $\Gamma_0 \Rightarrow \Omega_0$
  $\Gamma_0, \hat{\beta} : \kappa' \Rightarrow \Omega_0, \hat{\beta} : \kappa' = t$
  Similar to $\Rightarrow \Var$

• Case $\Gamma_0 \Rightarrow \Omega_0$
  $[\Omega_0] t' = [\Omega_0] t$
  $\Gamma_0, \alpha = t' \Rightarrow \Omega_0, \alpha = t$ \hspace{1cm} \text{Eqn}
  \begin{align*}
    \Gamma_0 \Rightarrow \Omega_0 & \quad \text{Subderivation} \\
    [\Omega_0] t' = [\Omega_0] t & \quad \text{Subderivation} \\
    [\Omega_0] \Gamma_0 = [\Omega_0] \Omega_0 & \quad \text{By i.h.} \\
    [[\Omega_0] t/\alpha]\Gamma_0 = [[\Omega_0] t/\alpha][[\Omega_0] \Omega_0] & \quad \text{By congruence of equality} \\
    [\Omega_0, \alpha = t](\Gamma_0, \alpha = t') = [\Omega_0, \alpha = t](\Omega_0, \alpha = t) & \quad \text{By definition of context substitution}
  \end{align*}

• Case $\Gamma \Rightarrow \Omega_0$
  $\Gamma \Rightarrow \Omega_0, \hat{\alpha} : \kappa$ \hspace{1cm} \text{Add}
  \begin{align*}
    \Gamma \Rightarrow \Omega_0 & \quad \text{Subderivation} \\
    [\Omega_0] \Gamma = [\Omega_0] \Omega_0 & \quad \text{By i.h.} \\
    [\Omega_0, \hat{\alpha} : \kappa] \Gamma = [\Omega_0, \hat{\alpha} : \kappa] (\Omega_0, \hat{\alpha} : \kappa) & \quad \text{By definition of context substitution}
  \end{align*}

• Case $\Gamma \Rightarrow \Omega_0$
  $\Gamma \Rightarrow \Omega_0, \hat{\alpha} : \kappa = t$ \hspace{1cm} \text{AddSolved}
  Similar to the $\Rightarrow \Add$ case.

Lemma 55 (Completing Completeness).

(i) If $\Omega \Rightarrow \Omega'$ and $\Omega \vdash t : \kappa$ then $[\Omega] t = [\Omega'] t$.

(ii) If $\Omega \Rightarrow \Omega'$ and $\Omega \vdash A$ type then $[\Omega] A = [\Omega'] A$.

(iii) If $\Omega \Rightarrow \Omega'$ then $[\Omega] \Omega = [\Omega'] \Omega'$.

Proof.
Proof of Lemma 55 (Completing Completeness).

Part (i):

By Lemma 29 (Substitution Monotonicity) (i), \( [\Omega']t = [\Omega]\Gamma t \). Now we need to show \( [\Omega'][\Omega]t = [\Omega]t \). Considered as a substitution, \( \Omega' \) is the identity everywhere except existential variables \( \hat{\alpha} \) and universal variables \( \alpha \). First, since \( \Omega \) is complete, \( [\Omega]t \) has no free existentials. Second, universal variables free in \( [\Omega]t \) have no equations in \( \Omega \) (if they had, their occurrences would have been replaced). But if \( \Omega \) has no equation for \( \alpha \), it follows from \( \alpha \rightarrow \Omega' \) and the definition of context extension in Figure 15 that \( \Omega' \) also lacks an equation, so applying \( \Omega' \) also leaves \( \alpha \) alone.

Transitivity of equality gives \( [\Omega']t = [\Omega]t \).

Part (ii):

Similar to part (i), using Lemma 29 (Substitution Monotonicity) (iii) instead of (i).

Part (iii):

By induction on \( \Delta \rightarrow \Omega' \).

Only cases \( \rightarrow Id \), \( \rightarrow Var \), \( \rightarrow Uvar \), \( \rightarrow Eqn \), \( \rightarrow Solved \), \( \rightarrow AddSolved \) and \( \rightarrow Marker \) are possible. In all of these cases, we use the i.h. and the definition of context application; in cases \( \rightarrow Var \), \( \rightarrow Eqn \) and \( \rightarrow Solved \), we also use the equality in the premise of the respective rule.

Lemma 56 (Confluence of Completeness).

If \( \Delta_1 \rightarrow \Omega \) and \( \Delta_2 \rightarrow \Omega \) then \( [\Omega]\Delta_1 = [\Omega]\Delta_2 \).

Proof.

\[
\begin{align*}
\Delta_1 \rightarrow \Omega & \quad \text{Given} \\
[\Omega]\Delta_1 = [\Omega]\Omega & \quad \text{By Lemma 54 (Completing Stability)} \\
\Delta_2 \rightarrow \Omega & \quad \text{Given} \\
[\Omega]\Delta_2 = [\Omega]\Omega & \quad \text{By Lemma 54 (Completing Stability)} \\
[\Omega]\Delta_1 = [\Omega]\Delta_2 & \quad \text{By transitivity of equality}
\end{align*}
\]

Lemma 57 (Multiple Confluence).

If \( \Delta \rightarrow \Omega \) and \( \Delta \rightarrow \Omega' \) and \( \Delta' \rightarrow \Omega' \) then \( [\Omega]\Delta = [\Omega']\Delta' \).

Proof.

\[
\begin{align*}
\Delta \rightarrow \Omega & \quad \text{Given} \\
[\Omega]\Delta = [\Omega]\Omega & \quad \text{By Lemma 54 (Completing Stability)} \\
\Omega \rightarrow \Omega' & \quad \text{Given} \\
[\Omega]\Omega = [\Omega']\Omega' & \quad \text{By Lemma 55 (Completing Completeness) (iii)} \\
= [\Omega']\Delta' & \quad \text{By Lemma 54 (Completing Stability) (\( \Delta' \rightarrow \Omega' \) given)}
\end{align*}
\]

Lemma 59 (Canonical Completion).

If \( \Gamma \rightarrow \Omega \) then there exists \( \Omega_{canon} \) such that \( \Gamma \rightarrow \Omega_{canon} \) and \( \Omega_{canon} \rightarrow \Omega \) and \( \text{dom}(\Omega_{canon}) = \text{dom}(\Gamma) \) and, for all \( \hat{\alpha} : \kappa = \tau \) and \( \alpha = \tau \) in \( \Omega_{canon} \), we have \( \text{FEV}(\tau) = \emptyset \).

Proof. By induction on \( \Omega \). In \( \Omega_{canon} \), make all solutions (for evars and uvars) canonical by applying \( \Omega \) to them, dropping declarations of existential variables that aren’t in \( \text{dom}(\Gamma) \).

Lemma 60 (Split Solutions).

If \( \Delta \rightarrow \Omega \) and \( \hat{\alpha} \in \text{unsolved}(\Delta) \) then there exists \( \Omega_1 = \Omega_1[\hat{\alpha} : \kappa = t_1] \) such that \( \Omega_1 \rightarrow \Omega \) and \( \Omega_2 = \Omega_2[\hat{\alpha} : \kappa = t_2] \) where \( \Delta \rightarrow \Omega_2 \) and \( t_2 \neq t_1 \) and \( \Omega_2 \) is canonical.

Proof. Use Lemma 59 (Canonical Completion) to get \( \Omega_{canon} \) such that \( \Delta \rightarrow \Omega_{canon} \) and \( \Omega_{canon} \rightarrow \Omega \), where for all solutions \( t \) in \( \Omega_{canon} \), we have \( \text{FEV}(t) = \emptyset \).

We have \( \Omega_{canon} = \Omega_1[\hat{\alpha} : \kappa = t_1] \), where \( \text{FEV}(t_1) = \emptyset \). Therefore \( \Omega_1[\hat{\alpha} : \kappa = t_1] \rightarrow \Omega \).

Now choose \( t_2 \) as follows:
• If $\kappa = \ast$, let $t_2 = t_1 \rightarrow t_1$.

• If $\kappa = \mathbb{N}$, let $t_2 = \text{succ}(t_1)$.

Thus, $t_2 \neq t_1$. Let $\Omega_2 = \Omega_1'[\check{\alpha}: \kappa = t_2]$.

$\Delta \rightarrow \Omega_2$ By Lemma 31 (Split Extension)

D’ Internal Properties of the Declarative System

Lemma 61 (Interpolating With and Exists).

(1) If $D :: \Psi \vdash \Pi :: \vec{A}! \iff C \land P_0$ true
then $D' :: \Psi \vdash \Pi :: \vec{A}! \iff C \land P_0$ p.

(2) If $D :: \Psi \vdash \Pi :: \vec{A}! \iff [\tau/\alpha]C_0$ p and $\Psi \vdash \tau : \kappa$
then $D' :: \Psi \vdash \Pi :: \vec{A}! \iff (\exists \alpha : \kappa. C_0)$ p.

In both cases, the height of $D'$ is one greater than the height of $D$. 
Moreover, similar properties hold for the eliminating judgment $\Psi / P \vdash \Pi :: \vec{A}! \iff C p$.

Proof. By induction on the given match derivation.
In the $\text{DecMatchBase}$ case, for part (1), apply rule $\land I$. For part (2), apply rule $\exists I$.
In the $\text{DecMatchNeg}$ case, part (1), use Lemma 2 (Declarative Weakening) (iii). In part (2), use Lemma 21 (Declarative Weakening) (i).

Lemma 62 (Case Invertibility).
If $\Psi \vdash \text{case}(e_0, \Pi) \iff C p$ then $\Psi \vdash e_0 \Rightarrow A !$ and $\Psi \vdash \Pi :: A ! \iff C p$ and $\Psi \vdash \Pi \text{ covers } A !$
where the height of each resulting derivation is strictly less than the height of the given derivation.

Proof. By induction on the given derivation.

• Case $\Psi \vdash \text{case}(e_0, \Pi) \Rightarrow A q \quad \Psi \vdash A \iff \text{join}([\text{pol}(B), \text{pol}(A)]) B$

$\Psi \vdash \text{case}(e_0, \Pi) \iff B p$

$\text{DecSub}$

Impossible, because $\Psi \vdash \text{case}(e_0, \Pi) \Rightarrow A q$ is not derivable.

• Cases $\text{Decl\forall}, \text{Decl\exists}$: Impossible: these rules have a value restriction, but a case expression is not a value.

• Case $\Psi \vdash P \text{ true} \quad \Psi \vdash \text{case}(e_0, \Pi) \iff C_0 p$

$\Psi \vdash \text{case}(e_0, \Pi) \iff C_0 \land P p$

$\text{Decl\land}$

$< n - 1 \quad \Psi \vdash e_0 \Rightarrow A !$ \quad By i.h.

$< n - 1 \quad \Psi \vdash \Pi :: A \iff C_0 p$ 

$\leq n - 1 \quad \Psi \vdash P \text{ true}$ \quad Subderivation

$< n \quad \Psi \vdash \Pi :: A \iff C_0 \land P p$ \quad By Lemma 61 (Interpolating With and Exists) (1)
Proof of Lemma 62 (Case Invertibility)

\[ \text{lem:case-invertibility} \]

88

• Case $\Psi \vdash \tau : \kappa$ $\Psi \vdash \text{case}(e_0, \Pi) \iff [\tau/\alpha]C_0 \quad \text{(Decl)}$

\[ \Rightarrow \]

$\Psi \vdash e_0 \Rightarrow A \, !$

By i.h.

$\Psi \vdash \Pi :: A \iff C_0 \, p$

""

$\Psi \vdash \Pi \text{ covers } A$

Subderivation

$\Psi \vdash \Pi :: A \iff \exists \alpha : \kappa. C_0 \, p$

By Lemma 61 (Interpolating With and Exists) (2)

The heights of the derivations are similar to those in the $\text{Decl} \land I$ case.

• Cases $\text{Decl[1]} \, \text{Decl[→]} \, \text{DeclRec} \, \text{Decl[+]} \, \text{DeclNil} \, \text{DeclCons}$

Impossible, because in these rules $e$ cannot have the form $\text{case}(e_0, \Pi)$.

• Case

\[ \text{case}(e_0, \Pi) \Rightarrow A \, ! \quad \Psi \vdash \Pi :: A \iff C \, p \quad \Psi \vdash \Pi \text{ covers } A \, ! \]

$\Psi \vdash \text{case}(e_0, \Pi) \iff C \, p$

Immediate.

E’ Miscellaneous Properties of the Algorithmic System

Lemma 63 (Well-Formed Outputs of Typing).

(Spines) If $\Gamma \vdash s : A \, q \gg C \, \! p \! \iff \! \Delta$ or $\Gamma \vdash s : A \, q \gg C \, \! [p] \! \! \iff \! \Delta$

and $\Gamma \vdash A \, q \! \text{ type}$

then $\Delta \vdash C \, p \, \text{ type}$.

(Synthesis) If $\Gamma \vdash e \Rightarrow A \, p \iff \Delta$

then $A \vdash p \, \text{ type}$.

Proof. By induction on the given derivation.

• Case $\text{Anno}$ Use Lemma 51 (Typing Extension) and Lemma 41 (Extension Weakening for Principal Typing).

• Case $\forall \text{Spine}$ We have $\Gamma \vdash (\forall \alpha : \kappa. A_0) \, q \text{ type}$.

By inversion, $\Gamma, \alpha : \kappa \vdash A_0 \, q \text{ type}$.

By properties of substitution, $\Gamma, \hat{\alpha} : \kappa \vdash [\hat{\alpha}/\alpha]A_0 \, q \text{ type}$.

Now apply the i.h.

• Case $\exists \text{Spine}$ Use Lemma 42 (Inversion of Principal Typing) (2), Lemma 47 (Checkprop Extension), and Lemma 41 (Extension Weakening for Principal Typing).

• Case $\Box \text{Spine}$ Use Lemma 42 (Inversion of Principal Typing) (1), Lemma 51 (Typing Extension), and Lemma 41 (Extension Weakening for Principal Typing).

• Case $\text{SpineRecover}$ By i.h., $\Delta \vdash C \, ! \text{ type}$.

We have as premise $\text{FEV}(C) = \emptyset$.

Therefore $\Delta \vdash C \, ! \text{ type}$.

• Case $\text{SpinePass}$ By i.h.

• Case $\text{EmptySpine}$ Immediate.

• Case $\rightarrow \text{Spine}$ Use Lemma 42 (Inversion of Principal Typing) (1), Lemma 51 (Typing Extension), and Lemma 41 (Extension Weakening for Principal Typing).

• Case $\Rightarrow \text{Spine}$ Show that $\hat{\alpha}_1 \rightarrow \hat{\alpha}_2$ is well-formed, then use the i.h.
F’ Decidability of Instantiation

Lemma 64 (Left Unsolvedness Preservation). If $\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \hat{\alpha} := \lambda: \kappa \vdash \Delta$ and $\hat{\beta} \in \text{unsolved}(\Gamma_0)$ then $\hat{\beta} \in \text{unsolved}(\Delta)$.

Proof. By induction on the given derivation.

- Case $\Gamma_0 \vdash \tau : \kappa$
  $\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \vdash \hat{\alpha} := \tau : \kappa \vdash \Gamma_0, \hat{\alpha} : \kappa \vdash \tau, \Gamma_1 \tag{InstSolve}$
  Immediate, since to the left of $\hat{\alpha}$, the contexts $\Delta$ and $\Gamma$ are the same.

- Case $\hat{\beta} \in \text{unsolved}(\Gamma'[(\hat{\alpha} : \kappa)[\hat{\beta} : \kappa]])$
  $\Gamma'[\hat{\alpha} : \kappa][\hat{\beta} : \kappa] \vdash \hat{\beta} : \kappa \vdash \Gamma'[(\hat{\alpha} : \kappa)[\hat{\beta} : \kappa]] \tag{InstReach}$
  Immediate, since to the left of $\hat{\alpha}$, the contexts $\Delta$ and $\Gamma$ are the same.

- Case $\Gamma_0, \hat{\alpha}_2 : *, \hat{\alpha}_1 : *, \hat{\alpha} : * = \hat{\alpha}_1 \oplus \hat{\alpha}_2, \Gamma_1 \vdash \hat{\alpha}_1 := \tau_1 : \star \vdash \Theta \vdash \hat{\alpha}_2 := [\Theta] \tau_2 : \star \vdash \Delta$
  $\Gamma_0, \hat{\alpha} : *, \Gamma_1 \vdash \hat{\alpha} := \tau_1 \oplus \tau_2 : \star \vdash \Delta \tag{InstBin}$
  We have $\hat{\beta} \in \text{unsolved}(\Gamma_0)$. Therefore $\hat{\beta} \in \text{unsolved}(\Gamma_0, \hat{\alpha}_2 : *)$.
  Clearly, $\hat{\alpha}_2 \in \text{unsolved}(\Gamma_0, \hat{\alpha}_2 : *)$.
  We have two subderivations:
  
  $$\Gamma_0, \hat{\alpha}_2 : *, \hat{\alpha}_1 : *, \hat{\alpha} : * = \hat{\alpha}_1 \oplus \hat{\alpha}_2, \Gamma_1 \vdash \hat{\alpha}_1 := A_1 : \star \vdash \Theta \tag{1}$$
  $$\Theta \vdash \hat{\alpha}_2 := [\Theta] A_2 : \star \vdash \Delta \tag{2}$$

  By induction on (1), $\hat{\beta} \in \text{unsolved}(\Theta)$.
  Also by induction on (1), with $\hat{\alpha}_2$ playing the role of $\hat{\beta}$, we get $\hat{\alpha}_2 \in \text{unsolved}(\Theta)$.
  Since $\hat{\beta} \in \Gamma_0$, it is declared to the left of $\hat{\alpha}_2$ in $\Gamma_0, \hat{\alpha}_2 : *$. $\hat{\alpha}_1 : *, \hat{\alpha} = \hat{\alpha}_1 \oplus \hat{\alpha}_2, \Gamma_1$.
  Hence by Lemma 20 (Declaration Order Preservation), $\hat{\beta}$ is declared to the left of $\hat{\alpha}_2$ in $\Theta$. That is, $\Theta = ([\Theta] \hat{\alpha}_2 : *, \Theta_1)$, where $\hat{\beta} \in \text{unsolved}(\Theta_0)$.
  By induction on (2), $\hat{\beta} \in \text{unsolved}(\Delta)$.

- Case $\Gamma[(\hat{\alpha} : \kappa) \vdash \hat{\alpha} := \text{zero} : \kappa \vdash \Gamma][\hat{\alpha} : \kappa = \text{zero}] \tag{InstZero}$
  Immediate, since to the left of $\hat{\alpha}$, the contexts $\Delta$ and $\Gamma$ are the same.

- Case $\Gamma[(\hat{\alpha}_1 : \kappa, \hat{\alpha} : \kappa = \text{succ}(\hat{\alpha}_1)] \vdash \hat{\alpha}_1 := t_1 : \kappa \vdash \Delta \tag{InstSucc}$
  $\Gamma[(\hat{\alpha} : \kappa) \vdash \hat{\alpha} := \text{succ}(t_1) : \kappa \vdash \Delta \tag{InstSucc}$
  We have $\hat{\beta} \in \text{unsolved}(\Gamma_0)$. Therefore $\hat{\beta} \in \text{unsolved}(\Gamma_0, \hat{\alpha}_1 : \kappa)$. By i.h., $\hat{\beta} \in \text{unsolved}(\Delta)$.

Lemma 65 (Left Free Variable Preservation). If $\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \vdash \hat{\alpha} := t : \kappa \vdash \Delta$ and $\hat{\beta} \in \text{unsolved}(\Gamma_0)$ and $\hat{\beta} \notin \text{FV}(\Gamma)$, then $\hat{\beta} \notin \text{FV}(\Delta)$.

Proof. By induction on the given instantiation derivation.
Proof of Lemma 65 (Left Free Variable Preservation)

- Case
  \[
  \Gamma_0 \vdash \tau : \kappa
  \]
  \[
  \Gamma_0, \hat{\kappa} : \kappa, \Gamma_1 \vdash \hat{\kappa} := \tau : \kappa \rightarrow \Gamma_0, \hat{\kappa} : \kappa = \tau, \Gamma_1
  \]

  We have \( \hat{\kappa} \notin \text{FV}(\Gamma[\sigma]) \). Since \( \Delta \) differs from \( \Gamma \) only in \( \hat{\kappa} \), it must be the case that \( |\Gamma|\sigma = |\Delta|\sigma \). It is given that \( \hat{\beta} \notin \text{FV}(|\Gamma|\sigma) \), so \( \hat{\beta} \notin \text{FV}(|\Delta|\sigma) \).

- Case
  \[
  \hat{\gamma} \in \text{unsolved}(\Gamma[\hat{\kappa} : \kappa][\hat{\gamma} : \kappa])
  \]
  \[
  \Gamma[\hat{\kappa} : \kappa][\hat{\gamma} : \kappa] \vdash \hat{\kappa} := \hat{\gamma} : \kappa \rightarrow \Gamma[\hat{\kappa} : \kappa][\hat{\gamma} : \kappa] \vdash \hat{\gamma} : \kappa = \hat{\kappa}
  \]

  Since \( \Delta \) differs from \( \Gamma \) only in solving \( \hat{\gamma} \) to \( \hat{\kappa} \), applying \( \Delta \) to a type will not introduce a \( \hat{\beta} \). We have \( \hat{\beta} \notin \text{FV}(\Gamma[\sigma]) \), so \( \hat{\beta} \notin \text{FV}(|\Delta|\sigma) \).

- Case
  \[
  \Gamma[\hat{\alpha}_2 : \star, \hat{\alpha}_1 : \star, \hat{\alpha} : \star = \hat{\alpha}_1 \oplus \hat{\alpha}_2] \vdash \hat{\alpha}_1 := \tau_1 : \star \rightarrow \Theta \vdash \hat{\alpha}_2 := [\Theta] \tau_2 : \star \rightarrow \Delta
  \]

  We have \( \Gamma \vdash \sigma \) type and \( \hat{\alpha} \notin \text{FV}(\Gamma[\sigma]) \) and \( \hat{\beta} \notin \text{FV}(\Gamma[\sigma]) \). By weakening, we get \( \Gamma' \vdash \sigma : \kappa' \); since \( \hat{\alpha} \notin \text{FV}(\Gamma[\sigma]) \) and \( \Gamma' \) only adds a solution for \( \hat{\alpha} \), it follows that \( |\Gamma'|\sigma = |\Gamma|\sigma \).

  Therefore \( \hat{\alpha}_1 \notin \text{FV}(\Gamma'|\sigma) \) and \( \hat{\alpha}_2 \notin \text{FV}(\Gamma'|\sigma) \) and \( \hat{\beta} \notin \text{FV}(\Gamma'|\sigma) \).

  Since we have \( \hat{\beta} \in \Gamma_0 \), we also have \( \hat{\beta} \in (\Gamma_0, \hat{\alpha}_2 : \star) \).

  By induction on the first premise, \( \hat{\beta} \notin \text{FV}(\Gamma[\Theta][\sigma]) \).

  Also by induction on the first premise, with \( \hat{\alpha}_2 \) playing the role of \( \hat{\beta} \), we have \( \hat{\alpha}_2 \notin \text{FV}(\Gamma[\Theta][\sigma]) \).

  Note that \( \hat{\alpha}_2 \in \text{unsolved}(\Gamma_0, \hat{\alpha}_2 : \star) \).

  By Lemma 64 (Left Unsolvedness Preservation), \( \hat{\alpha}_2 \in \text{unsolved}(\Theta) \).

  Therefore \( \Theta \) has the form \((\Theta_0, \hat{\alpha}_2 : \star, \Theta_1)\).

  Since \( \hat{\beta} \neq \hat{\alpha}_2 \), we know that \( \hat{\beta} \) is declared to the left of \( \hat{\alpha}_2 \) in \((\Gamma_0, \hat{\alpha}_2 : \star) \), so by Lemma 20 (Declaration Order Preservation), \( \hat{\beta} \) is declared to the left of \( \hat{\alpha}_2 \) in \( \Theta \). Hence \( \hat{\beta} \in \Theta_0 \).

  Furthermore, by Lemma 43 (Instantiation Extension), we have \( \Gamma' \rightarrow \rightarrow \Theta \).

  Then by Lemma 36 (Extension Weakening (Sorts)), we have \( \Delta \vdash \sigma : \kappa' \).

  Using induction on the second premise, \( \hat{\beta} \notin \text{FV}(\Gamma[\Delta][\sigma]) \).

- Case
  \[
  \Gamma'[\hat{\alpha} : \star] \vdash \hat{\alpha} := \text{zero} : \star \rightarrow \Gamma'[\hat{\alpha} : \star = \text{zero}] \Delta
  \]

  We have \( \hat{\alpha} \notin \text{FV}(\Gamma[\sigma]) \). Since \( \Delta \) differs from \( \Gamma \) only in \( \hat{\alpha} \), it must be the case that \( |\Gamma|\sigma = |\Delta|\sigma \). It is given that \( \hat{\beta} \notin \text{FV}(\Gamma[\sigma]) \), so \( \hat{\beta} \notin \text{FV}(\Gamma[\Delta][\sigma]) \).

- Case
  \[
  \Theta
  \]
  \[
  \Gamma'[\hat{\alpha}_1 : \star, \hat{\alpha} : \star = \text{succ}(\hat{\alpha}_1)] \vdash \hat{\alpha}_1 := t_1 : \star \rightarrow \Delta \]
  \[
  \Gamma'[\hat{\alpha} : \star] \vdash \hat{\alpha} := \text{succ}(t_1) : \star \rightarrow \Delta \]

  We have \( \hat{\alpha} \notin \text{FV}(\Gamma[\sigma]) \). Since \( \Delta \) differs from \( \Gamma \) only in \( \hat{\alpha} \), it must be the case that \( |\Gamma|\sigma = |\Delta|\sigma \). It is given that \( \hat{\beta} \notin \text{FV}(\Gamma[\sigma]) \), so \( \hat{\beta} \notin \text{FV}(\Gamma[\Delta][\sigma]) \).
Proof of Lemma 65 (Left Free Variable Preservation). By induction on the given derivation.

\[ \Gamma \vdash \sigma : \kappa \quad \text{Given} \]
\[ \Delta \vdash \sigma : \kappa' \quad \text{By weakening} \]
\[ \hat{\alpha} \notin FV([\Gamma]^{\sigma}) \quad \text{Given} \]
\[ \hat{\alpha} \notin FV([\Theta]^{\sigma}) \quad \hat{\alpha} \notin FV([\Gamma]^{\sigma}) \text{ and } \Theta \text{ only solves } \hat{\alpha} \]
\[ \Theta = (\Gamma_0, \hat{\alpha}_1 : N, \hat{\alpha} : N = \text{succ}(\hat{\alpha}_1), \Gamma_1) \quad \text{Given} \]
\[ \hat{\beta} \notin \text{unsolved}(\Gamma_0) \quad \text{Given} \]
\[ \hat{\beta} \notin \text{unsolved}(\Gamma_0, \hat{\alpha}_1 : N) \quad \hat{\alpha}_1 \text{ fresh} \]
\[ \hat{\beta} \notin FV([\Gamma]^{\sigma}) \quad \text{Given} \]
\[ \hat{\beta} \notin FV([\Theta]^{\sigma}) \quad \hat{\alpha}_1 \text{ fresh} \]
\[ \not\models \hat{\beta} \notin FV([\Delta]^{\sigma}) \quad \text{By i.h.} \]

Lemma 66 (Instantiation Size Preservation). If \( \Gamma_0, \hat{\alpha}, \Gamma_1 \vdash [\cdot] : \kappa \quad \text{and} \quad \Gamma \vdash s : \kappa' \text{ and } \hat{\alpha} \notin FV([\Gamma]^{s}), \) then \( ||[\Gamma]^{s}|| = ||[\Delta]^{s}||, \) where \( |C| \) is the plain size of the term \( C. \)

Proof. By induction on the given derivation.

- **Case**

  \[ \Gamma_0 \vdash \tau : \kappa \]

  \[ \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \vdash \hat{\alpha} := \tau : \kappa \quad \text{InstSolve} \]

  Since \( \Delta \) differs from \( \Gamma \) only in solving \( \hat{\alpha} \), and we know \( \hat{\alpha} \notin FV([\Gamma]^{\sigma}) \), we have \( |\Delta|^{\sigma} = |\Gamma|^{\sigma}; \) therefore \( ||\Delta|^{\sigma} = ||\Gamma|^{\sigma}||. \)

- **Case**

  \[ \Gamma'([\hat{\alpha} : N]) \vdash \hat{\alpha} := \text{zero} : N \quad \text{InstZero} \]

  Similar to the InstSolve case.

- **Case**

  \[ \Gamma'([\hat{\alpha} : \kappa]) \vdash \hat{\alpha} := \hat{\beta} : \kappa \quad \text{InstReach} \]

  Here, \( \Delta \) differs from \( \Gamma \) only in solving \( \hat{\beta} \) to \( \hat{\alpha} \). However, \( \hat{\alpha} \) has the same size as \( \hat{\beta} \), so even if \( \hat{\beta} \notin FV([\Gamma]^{\sigma}) \), we have \( ||\Delta|^{\sigma} = |\Gamma|^{\sigma}||. \)

- **Case**

  \[ \Gamma' \vdash \hat{\alpha}_2 : *, \hat{\alpha}_1 : *, \hat{\alpha} : * = \hat{\alpha}_1 \oplus \hat{\alpha}_2 \]

  \[ \vdash \hat{\alpha}_1 := \tau_1 : * \quad \Theta \vdash \hat{\alpha}_2 := [\Theta]^{\tau_2} : * \quad \text{InstBin} \]

  We have \( \Gamma \vdash \sigma : \kappa' \) and \( \hat{\alpha} \notin FV([\Gamma]^{\sigma}). \)

  Since \( \hat{\alpha}_1, \hat{\alpha}_2 \notin \text{dom}(\Gamma) \), we have \( \hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2 \notin FV([\Gamma]^{\sigma}) \).

  By Lemma 23 (Deep Evar Introduction), \( \Gamma[\hat{\alpha} : *] \rightarrow \Gamma' \).

  By Lemma 36 (Extension Weakening (Sorts)), \( \Gamma' \vdash \sigma : \kappa' \).

  Since \( \hat{\alpha} \notin FV(\sigma) \), it follows that \( |\Gamma'|^{\sigma} = |\Gamma|^{\sigma} \), and so \( ||\Gamma'|^{\sigma}|| = |||\Gamma|^{\sigma}||. \)

  By induction on the first premise, \( ||\Gamma'|^{\sigma}|| = ||\Theta|^{\sigma}||. \)

  By Lemma 20 (Declaration Order Preservation), since \( \hat{\alpha}_2 \) is declared to the left of \( \hat{\alpha}_1 \) in \( \Gamma' \), we have...
Proof of Lemma 66 (Instantiation Size Preservation). Let \( \Gamma \vdash \tau : \kappa \) and \( \Gamma \vdash t : \kappa \) such that \( \Gamma t = t \) and \( \hat{\alpha} \notin \text{FV}(t) \), then:

(1) Either there exists \( \Delta \) such that \( \Gamma_0[\hat{\alpha} : \kappa'] \vdash \hat{\alpha} := t : \kappa \vdash \Delta \), or not.

Proof. By induction on the derivation of \( \Gamma \vdash t : \kappa \).

\[ \begin{array}{ll}
\text{Case} & (u : \kappa) \in \Gamma \\
\hline
\Gamma_L, \hat{\alpha} : \kappa', \Gamma_R \vdash u : \kappa & \text{VarSort}
\end{array} \]

If \( \kappa \neq \kappa' \), no rule matches and no derivation exists.

Otherwise:

- If \( (u : \kappa) \in \Gamma_L \), we can apply rule \( \text{InstSolve} \).
- If \( u \) is some unsolved existential variable \( \hat{\beta} \) and \( (\hat{\beta} : \kappa) \in \Gamma_R \), then we can apply rule \( \text{InstReach} \).
- Otherwise, \( u \) is declared in \( \Gamma_R \) and is a universal variable; no rule matches and no derivation exists.

\[ \begin{array}{ll}
\text{Case} & (\hat{\beta} : \kappa = \tau) \in \Gamma \\
\hline
\Gamma \vdash \hat{\beta} : \kappa & \text{SolvedVarSort}
\end{array} \]

By inversion, \( (\hat{\beta} : \kappa = \tau) \in \Gamma \), but \( |\Gamma|\hat{\beta} = \hat{\beta} \) is given, so this case is impossible.

\[ \begin{array}{ll}
\text{Case} & \text{UnitSort} \\
\hline
\text{If } \kappa' = \star, \text{ then apply rule } \text{InstSolve}. \text{ Otherwise, no rule matches and no derivation exists.}
\end{array} \]

\[ \begin{array}{ll}
\text{Case} & \Gamma \vdash \tau_1 : \star \\
\hline
\Gamma_L, \hat{\alpha} : \kappa', \Gamma_R \vdash \tau_1 + \tau_2 : \star & \text{BinSort}
\end{array} \]
If $\kappa' \neq \ast$, then no rule matches and no derivation exists. Otherwise:

- Given, $[\Gamma]([\tau_1 \cup \tau_2] = [\tau_1 \cup \tau_2]$ and $\delta \notin \text{FV}(\Gamma([\tau_1 \cup \tau_2])$.
- If $\Gamma_1 \vdash \tau_1 \cup \tau_2 : \ast$, then we have a derivation by $\text{InstSucc}$.
- If not, the only other rule whose conclusion matches $\tau_1 \cup \tau_2$ is $\text{InstBin}$.

First, consider whether $\Gamma_L \cup \hat{\delta}_2 : \ast, \hat{\delta}_1 : \ast, \hat{\delta} : \hat{\delta}_1 \cup \hat{\delta}_2, \Gamma_R \vdash \hat{\delta}_1 : t : \ast \vdash \Theta$ is decidable.

By definition of substitution, $[\Gamma]([\tau_1 \cup \tau_2] = ([\Gamma] \tau_1) \cup ([\Gamma] \tau_2)$. Since $[\Gamma]([\tau_1 \cup \tau_2] = \tau_1 \cup \tau_2$, we have $[\Gamma] \tau_1 = \tau_1$ and $[\Gamma] \tau_2 = \tau_2$.

By weakening, $\Gamma_1, \hat{\delta}_2 : \ast, \hat{\delta}_1 : \ast, \hat{\delta} : \hat{\delta}_1 \cup \hat{\delta}_2, \Gamma_R \vdash \hat{\delta}_1 : \ast \vdash \tau_1 \cup \tau_2 : \ast$.

Since $\Gamma \vdash \tau_1 : \ast$ and $\Gamma \vdash \tau_2 : \ast$, we have $\hat{\delta}_1, \hat{\delta}_2 \notin \text{FV}([\tau_1 \cup \tau_2])$.

Since $\hat{\delta} \notin \text{FV}(t) \supseteq \text{FV}(\tau_1 \cup \tau_2)$, it follows that $[\Gamma] \tau_1 = \tau_1$.

By i.h., either there exists $\Theta$ s.t. $\Gamma_1, \hat{\delta}_2 : \ast, \hat{\delta}_1 : \ast, \hat{\delta} : \hat{\delta}_1 \cup \hat{\delta}_2, \Gamma_R \vdash \hat{\delta}_1 : \ast \vdash \tau_1 : \ast \vdash \Theta$, or not.

If not, then no derivation by $\text{InstBin}$ exists.

Otherwise, there exists such a $\Theta$. By Lemma 64 (Left Unsolvedness Preservation), we have $\hat{\delta}_2 \in \text{unsolved}(\Theta)$.

By Lemma 65 (Left Free Variable Preservation), we know that $\hat{\delta}_2 \notin \text{FV}(\Theta \tau_2)$.

Substitution is idempotent, so $\Theta \Theta \tau_2 = \Theta \tau_2$.

By i.h., either there exists $\Delta$ such that $\Theta \vdash \hat{\delta}_2 : \ast \vdash \Theta \tau_2 : \kappa \vdash \Delta$, or not.

If not, then no derivation by $\text{InstBin}$ exists.

Otherwise, there exists such a $\Delta$. By rule $\text{InstBin}$ we have $\Gamma \vdash \hat{\delta} : t : \kappa \vdash \Delta$

- **Case**

  $\Gamma \vdash \text{zero} : N$

  If $\kappa' \neq N$, then no rule matches and no derivation exists. Otherwise, apply rule $\text{InstSucc}$.

- **Case**

  $\Gamma \vdash t_0 : N$

  $\Gamma \vdash \text{succ}(t_0) : N$

  If $\kappa' \neq N$, then no rule matches and no derivation exists. Otherwise:

  If $\Gamma_1 \vdash \text{succ}(t_0) : N$, then we have a derivation by $\text{InstSucc}$.

  If not, the only other rule whose conclusion matches $\text{succ}(t_0)$ is $\text{Inst Succ}$.

  The remainder of this case is similar to the $\text{BinSort}$ case, but shorter.

G’ Separation

**Lemma 68** (Transitivity of Separation).

If $\Gamma_1 \ast \Gamma_R \rightarrow \Theta_1 \ast \Theta_R$ and $\Theta_1 \ast \Theta_R \rightarrow \Delta_1 \ast \Delta_R$,

then $\Gamma_1 \ast \Gamma_R \rightarrow \Delta_1 \ast \Delta_R$.

**Proof.**

\[
\begin{align*}
(\Gamma_1 \ast \Gamma_R) & \rightarrow (\Theta_1 \ast \Theta_R) \quad \text{By Definition}\, [5] \\
(\Gamma_1, \Gamma_R) & \rightarrow (\Theta_1, \Theta_R) \quad \text{By Definition}\, [5] \\
\Gamma_L & \subseteq \Theta_L \land \Gamma_R \subseteq \Theta_R \\
(\Theta_1 \ast \Theta_R) & \rightarrow (\Delta_1 \ast \Delta_R) \quad \text{By Definition}\, [5] \\
(\Theta_1, \Theta_R) & \rightarrow (\Delta_1, \Delta_R) \quad \text{By Definition}\, [5] \\
\Theta_L & \subseteq \Delta_L \land \Theta_R \subseteq \Delta_R
\end{align*}
\]

\[
\begin{align*}
(\Gamma_1, \Gamma_R) & \rightarrow (\Delta_1, \Delta_R) \quad \text{By Lemma}\, [33] \text{ (Extension Transitivity)} \\
\Gamma_L & \subseteq \Delta_L \land \Gamma_R \subseteq \Delta_R \\
\Gamma_1 \ast \Gamma_R & \rightarrow (\Delta_1 \ast \Delta_R) \quad \text{By Definition}\, [5]
\end{align*}
\]
Lemma 70 (Separation for Auxiliary Judgments).

If \( H \) has the form \( \alpha : \kappa \) or \( \alpha \triangleright \alpha \) or \( \alpha \triangleright \) or \( \alpha : \lambda \) and \( (\Gamma_L \ast (\Gamma_R, H)) \vdash_\alpha (\Delta_L \ast \Delta_R) \)
then \((\Gamma_L \ast \Gamma_R) \vdash_\alpha (\Delta_L \ast \Delta_R)\) where \( \Delta_R = (\Delta_0, H, \Theta) \).

Proof. By induction on \( \Delta_R \).

If \( \Delta_R = (\ldots, H) \), we have \((\Gamma_L \ast \Gamma_R, H) \vdash_\alpha (\Delta_L \ast (\Delta, H))\), and inversion on \( \vdash_\alpha \text{Uvar} \) (if \( H \) is \( (\alpha : \kappa) \), or the corresponding rule for other forms) gives the result (with \( \Theta = \ldots \)).

Otherwise, proceed into the subderivation of \((\Gamma_L, \Gamma_R, \alpha : \kappa) \vdash (\Delta_L, \Delta_R)\), with \( \Delta_R = (\Delta'_R, \Delta') \) where \( \Delta' \) is a single declaration. Use the i.h. on \( \Delta'_R \), producing some \( \Theta' \). Finally, let \( \Theta = (\Theta', \Delta') \). \( \square \)

Lemma 70 (Separation for Auxiliary Judgments).

(i) If \( \Gamma_L \ast \Gamma_R \vdash \sigma \triangleleft \tau : \kappa \vdash \Delta \)
and \( \text{FEV}(\sigma) \cup \text{FEV}(\tau) \subseteq \text{dom}(\Gamma_R) \)
then \( \Delta = (\Delta_L \ast \Delta_R) \) and \((\Gamma_L \ast \Gamma_R) \vdash_\alpha (\Delta_L \ast \Delta_R)\).

(ii) If \( \Gamma_L \ast \Gamma_R \vdash P \triangleright \Delta \)
and \( \text{FEV}(P) \subseteq \text{dom}(\Gamma_R) \)
then \( \Delta = (\Delta_L \ast \Delta_R) \) and \((\Gamma_L \ast \Gamma_R) \vdash_\alpha (\Delta_L \ast \Delta_R)\).

(iii) If \( \Gamma_L \ast \Gamma_R / \sigma \triangleright \tau : \kappa \vdash \Delta \)
and \( \text{FEV}(\sigma) \cup \text{FEV}(\tau) = \emptyset \)
then \( \Delta = (\Delta_L \ast (\Delta_R, \Theta)) \) and \((\Gamma_L \ast (\Gamma_R, \Theta)) \vdash_\alpha (\Delta_L \ast \Delta_R)\).

(iv) If \( \Gamma_L \ast \Gamma_R / P \triangleright \Delta \)
and \( \text{FEV}(P) = \emptyset \)
then \( \Delta = (\Delta_L \ast (\Delta_R, \Theta)) \) and \((\Gamma_L \ast (\Gamma_R, \Theta)) \vdash_\alpha (\Delta_L \ast \Delta_R)\).

(v) If \( \Gamma_L \ast \Gamma_R \vdash \alpha := \tau : \kappa \vdash \Delta \)
and \( \text{FEV}(\tau) \cup \{\alpha\} \subseteq \text{dom}(\Gamma_R) \)
then \( \Delta = (\Delta_L \ast \Delta_R) \) and \((\Gamma_L \ast \Gamma_R) \vdash_\alpha (\Delta_L \ast \Delta_R)\).

(vi) If \( \Gamma_L \ast \Gamma_R \vdash \Gamma \equiv R \vdash \Delta \)
and \( \text{FEV}(\Gamma) \cup \text{FEV}(R) \subseteq \text{dom}(\Gamma_R) \)
then \( \Delta = (\Delta_L \ast \Delta_R) \) and \((\Gamma_L \ast \Gamma_R) \vdash_\alpha (\Delta_L \ast \Delta_R)\).

Proof. Part (i): By induction on the derivation of the given checkeq judgment. Cases \text{CheckeqVar}, \text{CheckeqUnit}
and \text{CheckeqZero} are immediate (\( \Delta_L = \Gamma_L \) and \( \Delta_R = \Gamma_R \)). For case \text{CheckeqSucc}, apply the i.h. For cases \text{CheckeqInstL} and \text{CheckeqInstR} use the i.h. (v). For case \text{CheckeqBin} use reasoning similar to that in the \( \Delta \)
case of Lemma 72 (Separation—Main) (transitivity of separation, and applying \( \Theta \) in the second premise).

Part (ii), elimprop: Use the i.h. (i).

Part (iii), elimeq: Cases \text{ElimeqUvarRef}, \text{ElimeqUnit} and \text{CheckeqZero} are immediate (\( \Delta_L = \Gamma_L \) and \( \Delta_R = \Gamma_R \)). Cases \text{ElimeqUvarR}, \text{ElimeqUvarL}, \text{ElimeqUvarR lx} and \text{ElimeqUvarL lx} are impossible (we have \( \Delta \), not \( \bot \)). For case \text{ElimeqSucc}, apply the i.h. The case for \text{ElimeqBin} is similar to the case \text{CheckeqBin} in part (i). For cases \text{ElimeqUvarR} and \text{ElimeqUvarL} \( \Delta = (\Gamma_L, \Gamma_R, \alpha = \tau) \) which, since \( \text{FEV}(\tau) \subseteq \text{dom}(\Gamma_R) \), ensures that \((\Gamma_L \ast (\Gamma_R, \alpha = \tau)) \vdash_\alpha (\Delta_L \ast (\Delta_R, \alpha = \tau)) \).

Part (iv), elimprop: Use the i.h. (iii).

Part (v), instjudg:

Case InstSolve Here, \( \Gamma = (\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1) \) and \( \Delta = (\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1) \). We have \( \hat{\alpha} \in \text{dom}(\Gamma_R) \), so the declaration \( \hat{\alpha} : \kappa \) is in \( \Gamma_R \). Since \( \text{FEV}(\tau) \subseteq \text{dom}(\Gamma_R) \), the context \( \Delta \) maintains the separation.
Proof of **Lemma 70** (Separation for Auxiliary Judgments). If \( \Gamma \vdash A \triangleq P B \vdash \Delta \) and \( \text{FEV}(A) \subseteq \text{dom}(\Gamma_R) \) and \( \text{FEV}(B) \subseteq \text{dom}(\Gamma_R) \) then \( \Delta = (\Delta_L \ast \Delta_R) \) and \( (\Gamma_L \ast \Gamma_R) \overset{\ast}{\rightarrow} (\Delta_L \ast \Delta_R) \) and \( \text{FEV}(C) \subseteq \text{dom}(\Delta_R) \).

**Proof.** By induction on the given derivation. In the \( \triangleq \text{Equiv} \) case, use Lemma 70 (Separation for Auxiliary Judgments) (vii). Otherwise, the reasoning needed follows that used in the proof of Lemma 72 (Separation—Main).

**Lemma 72** (Separation—Main).

(Spines) If \( \Gamma_L \ast \Gamma_R \vdash s : A \triangleright\triangleright C q \vdash \Delta \) or \( \Gamma_L \ast \Gamma_R \vdash s : A \triangleright C [q] \vdash \Delta \) and \( \text{FEV}(A) \subseteq \text{dom}(\Gamma_R) \) then \( \Delta = (\Delta_L \ast \Delta_R) \) and \( (\Gamma_L \ast \Gamma_R) \overset{\ast}{\rightarrow} (\Delta_L \ast \Delta_R) \).

(Checking) If \( \Gamma_L \ast \Gamma_R \vdash e \triangleleft C p \vdash \Delta \) and \( \text{FEV}(C) \subseteq \text{dom}(\Gamma_R) \) then \( \Delta = (\Delta_L \ast \Delta_R) \) and \( (\Gamma_L \ast \Gamma_R) \overset{\ast}{\rightarrow} (\Delta_L \ast \Delta_R) \).

(Synthesis) If \( \Gamma_L \ast \Gamma_R \vdash e \rightarrow A_p \vdash \Delta \) then \( \Delta = (\Delta_L \ast \Delta_R) \) and \( (\Gamma_L \ast \Gamma_R) \overset{\ast}{\rightarrow} (\Delta_L \ast \Delta_R) \).

(Match) If \( \Gamma_L \ast \Gamma_R \vdash \Pi : A q \triangleleft C p \vdash \Delta \) and \( \text{FEV}(A) = \emptyset \) and \( \text{FEV}(C) \subseteq \text{dom}(\Gamma_R) \) then \( \Delta = (\Delta_L \ast \Delta_R) \) and \( (\Gamma_L \ast \Gamma_R) \overset{\ast}{\rightarrow} (\Delta_L \ast \Delta_R) \).
(Match Elim.) If $\Gamma_L \ast \Gamma_R / P \vdash \Pi :: \bar{A} ! \liff C \ p \vdash \Delta$
and $\text{FEV}(P) = \emptyset$
and $\text{FEV}(\bar{A}) = \emptyset$
and $\text{FEV}(C) \subseteq \text{dom}(\Gamma_R)$
then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \xrightarrow{\ast} (\Delta_L * \Delta_R)$.

Proof. By induction on the given derivation.
First, the (Match) judgment part, giving only the cases that motivate the side conditions:
- **Case** [MatchBase]. Here we use the i.h. (Checking), for which we need $\text{FEV}(C) \subseteq \text{dom}(\Gamma_R)$.
- **Case** [Match∧]. Here we use the i.h. (Match Elim.), which requires that $\text{FEV}(P) = \emptyset$, which motivates $\text{FEV}(\bar{A}) = \emptyset$.
- **Case** [MatchNeg]. In its premise, this rule appends a type $A \in \bar{A}$ to $\Gamma_R$ and claims it is principal ($z : A!$), which motivates $\text{FEV}(\bar{A} = \emptyset)$.

Similarly, (Match Elim.):
- **Case** [MatchUnity]. Here we use Lemma 70 (Separation for Auxiliary Judgments) (iii), for which we need $\text{FEV}(\sigma) \cup \text{FEV}(\tau) = \emptyset$, which motivates $\text{FEV}(P) = \emptyset$.

Now, we show the cases for the (Spine), (Checking), and (Synthesis) parts.
- **Cases** [Var]. In all of these rules, the output context is the same as the input context, so just let $\Delta_L = \Gamma_L$ and $\Delta_R = \Gamma_R$.

**Case**

$\Gamma_L \ast \Gamma_R \vdash \exists x : A \ p \gtrdot \ A \ p \cfrac{C \ p \vdash \Delta_L \ast \Gamma_R}{\Delta_L \ast \Gamma_R}$

Let $\Delta_L = \Gamma_L$ and $\Delta_R = \Gamma_R$.
We have $\text{FEV}(A) \subseteq \text{dom}(\Gamma_R)$. Since $\Delta_R = \Gamma_R$ and $C = A$, it is immediate that $\text{FEV}(C) \subseteq \text{dom}(\Delta_R)$.

**Case**

$\Gamma_L \ast \Gamma_R \vdash e \Rightarrow A \ q \vdash \Theta \vdash A \vdash \vdash B \vdash \Delta$

\[\Delta_L \ast \Gamma_R \vdash e \Leftarrow B \ p \vdash \Delta\]

By i.h., $\Theta = (\Theta_L \ast \Theta_R)$ and $(\Gamma_L \ast \Gamma_R) \xrightarrow{\ast} (\Theta_L \ast \Theta_R)$.
By Lemma 71 (Separation for Subtyping), $\Delta = (\Delta_L \ast \Delta_R)$ and $(\Theta_L \ast \Theta_R) \xrightarrow{\ast} (\Delta_L \ast \Delta_R)$.
By Lemma 68 (Transitivity of Separation), $(\Gamma_L \ast \Gamma_R) \xrightarrow{\ast} (\Delta_L \ast \Delta_R)$.

**Case**

$\Gamma \vdash \Lambda$ type

\[\Gamma \vdash (e : A) \Rightarrow (\Delta) \vdash (\Lambda \vdash \Delta)\]

By i.h.; since $\text{FEV}(A) = \emptyset$, the condition on the (Checking) part is trivial.

**Case**

$\Gamma \vdash \Lambda$ type

\[\Gamma \vdash (e : A) \Rightarrow (\Delta) \vdash (\Lambda \vdash \Delta)\]

Adding a solution with a ground type cannot destroy separation.

**Case**

$\nu \text{chk-l}
\Gamma_L, \Gamma_R, \alpha : \kappa \vdash \nu \Leftarrow A_\theta \ p \vdash \Delta, \alpha : \kappa, \Theta
\Gamma_L, \Gamma_R \vdash \nu \Leftarrow \forall \alpha : \kappa. A_\theta \ p \vdash \Delta$

Proof of Lemma 72 (Separation—Main) lem:separation-main
Proof of Lemma 72 (Separation—Main)

 FEV(∀α : κ, A₀) ⊆ dom(Γ₀)  
 FEV(Γ₀) ⊆ dom(Γ₀, α : κ)  
 {Δ, α : κ, Θ} = (Δ₀ * Δ₀)  
 (Γ₀, α : κ) → (Δ₀ * Δ₀)  
 FEV(Γ₀) ⊆ dom(Γ₀, α : κ)  
 From definition of FEV  
 By i.h.  
 By Lemma 69 (Separation Truncation)  
 ”

• Case  Γ₀, Γ₀, â : κ ⊢ e s : [â/α]A₀  
 FEV(∀α : κ, A₀) ⊆ dom(Γ₀)  
 FEV(Γ₀, â : κ) ⊆ dom(Γ₀, â : κ)  
 From definition of FEV  
 By Definition 5  
 By definition of dom(−)  
 By definition of dom(−)  
 Property of ≤  
 By i.h.  
 By Definition 5  

• Case  e not a case  Γ₀ * Γ₀ ⊢ P true  
 FEV(A₀ ∩ P) ⊆ dom(Γ₀)  
 By definition of FEV  
 By def. of FEV  
 By Lemma 70 (Separation for Auxiliary Judgments) (i)  

 FEV(P) ⊆ dom(Γ₀)  
 FEV(A₀) ⊆ dom(Γ₀)  
 Θ = (Θ₀ * Θ₀)  
 (Γ₀ * Γ₀) → (Θ₀ * Θ₀)  
 Given  
 By inversion  
 By inversion  
 Given  
 By above equation  
 By above equation  

 Δ = (Δ₀, Δ₀)  
 α not multiply declared
Proof of Lemma 72 (Separation—Main)

| Case | Similar to a section of the \[\Box\] case. |
| Case | Similar to the \[\Box\] case, with an extra use of the i.h. for the additional second premise. |
| Case | Similar to the \([\Box]\) case. |

\[
\begin{align*}
\text{FEV}(\Lambda_0) \subseteq \text{dom}(\Gamma_R) & \quad \text{Above} \\
\text{dom}(\Gamma_R) \subseteq \text{dom}(\Theta_R) & \quad \text{By Definition 5} \\
\text{FEV}(\Lambda_0) \subseteq \text{dom}(\Theta_R) & \quad \text{By previous line} \\
\text{FEV}([\Theta]\Lambda_0) \subseteq \text{dom}(\Theta_R) & \quad \text{Previous line and } (\Gamma_L \cdot \Gamma_R) \triangleleft \Theta \cdot \Theta_R \\
\Gamma_L \cdot \Gamma_R \vdash (\Lambda_0 \land \text{P}) \text{ p type} & \quad \text{Given} \\
\Gamma_L \cdot \Gamma_R \vdash \Lambda_0 \text{ p type} & \quad \text{By inversion} \\
\Theta \vdash \Lambda_0 \text{ p type} & \quad \text{By Lemma 41 (Extension Weakening for Principal Typing)} \\
\Theta \vdash [\Theta]\Lambda_0 \text{ p type} & \quad \text{By Lemma 13 (Right-Hand Substitution for Typing)} \\
\iff \Delta = (\Delta_L \cdot \Delta_R) & \quad \text{By i.h.} \\
\Theta \cdot \Theta_R \vdash (\Delta_L \cdot \Delta_R) & \quad \text{By i.h.} \\
\iff (\Gamma_L \cdot \Gamma_R) \vdash (\Delta_L \cdot \Delta_R) & \quad \text{By Lemma 68 (Transitivity of Separation)}
\end{align*}
\]

- Case Nil: Similar to a section of the \[\Box\] case.
- Case Cons: Similar to the \[\Box\] case, with an extra use of the i.h. for the additional second premise.
- Case \([\Box]\): Similar to the \([\Box]\) case.
- Case Spine: Similar to the \[\Box\] case.

\[
\begin{align*}
\gamma \text{ chk-I} & \quad \Gamma_L \cdot (\Gamma_R, \Box_p) / \text{P} \vdash \Theta \vdash \gamma \iff [\Theta]\Lambda_0 \vdash \Box_p, \Delta' \\
\Gamma_L \cdot \Gamma_R \vdash \gamma & \quad \text{Immediate} \\
\Gamma_L \cdot \Gamma_R \vdash \Box_p \text{ p type} & \quad \text{By Lemma 42 (Inversion of Principal Typing) (2)} \\
\Gamma_L, \Gamma_R \vdash \Lambda_0 \text{ ! type} & \quad \text{By Lemma 35 (Suffix Weakening)} \\
\Gamma_L, \Gamma_R, \Box_p, \Theta_Z \vdash \Lambda_0 \text{ ! type} & \quad \text{By Lemmas 41 and 40} \\
\Theta \vdash [\Theta]\Lambda_0 \text{ ! type} & \quad \text{By i.h.} \\
\text{FEV}(\Lambda_0) = \emptyset & \quad \text{Above and def. of FEV} \\
\text{FEV}(\Lambda_0) \subseteq \text{dom}(\Theta_R, \Theta_Z) & \quad \text{Immediate} \\
(\Delta, \Box_p, \Delta') = (\Delta_L \cdot \Delta'_R) & \quad \text{By i.h.} \\
(\Theta \cdot (\Theta_R, \Theta_Z)) \vdash \Box_p, \Delta' & \quad \text{Immediate} \\
(\Gamma_L \cdot (\Gamma_R, \Box_p)) \vdash (\Delta_L \cdot \Delta'_R) & \quad \text{By Lemma 68 (Transitivity of Separation)} \\
(\Gamma_L \cdot \Gamma_R) \vdash (\Delta_L \cdot \Delta'_R) & \quad \text{By Lemma 69 (Separation Truncation)} \\
\Delta'_R = (\Delta_R, \Box_p, \ldots) & \quad \text{By Lemma 69 (Separation Truncation)} \\
\iff \Delta = (\Delta_L, \Delta_R) & \quad \text{By Lemma 69 (Separation Truncation)}
\end{align*}
\]
Proof of Lemma 72 (Separation—Main) \( \text{lem:separation-main} \)

\[ \begin{align*}
\Gamma_L * \Gamma_R &\vdash (P \supset A_0) \text{ p type} & \text{Given} \\
\Gamma_L * \Gamma_R &\vdash P \text{ prop} & \text{By inversion} \\
\Gamma_L, \Gamma_R &\vdash \text{ p true } \vdash \Theta & \text{Subderivation} \\
\Theta &=(\Theta_L * \Theta_R) & \text{By Lemma 70 (Separation for Auxiliary Judgments) (i)} \\
(\Gamma_L * \Gamma_R) &\vdash \Theta & \text{”} \\
\Theta &\vdash e s : [\Theta]A_0 \gg C \vdash \Delta & \text{Subderivation} \\
(\Delta_L \triangleright \Delta) &=(\Delta_L * \Delta'_R) & \text{By i.h.} \\
(\Theta_L * \Theta_R) &\vdash \Theta & \text{”} \\
FEV(\Theta) &\subseteq \dom(\Delta_r) & \text{”} \\
(\Gamma_L * \Gamma_R) &\vdash \Theta & \text{By Lemma 68 (Transitivity of Separation)} \\
\end{align*} \]

**Case** \( \Gamma_L, \Gamma_R, x : C \vdash v \iff C \vdash \Delta, x : C, \Theta \)

\[ \frac{\Gamma_L, \Gamma_R \vdash \text{ rec } x, v \iff C \vdash \Delta}{\text{Rec}} \]

\[ \begin{align*}
\Gamma_L * \Gamma_R &\vdash C \text{ p type} & \text{Given} \\
FEV(C) &\subseteq \dom(\Gamma_R) & \text{Given} \\
\Gamma_L * (\Gamma_R, x : C p) &\vdash C \text{ p type} & \text{By weakening and Definition 4} \\
\Gamma_L, \Gamma_R, x : C \vdash v \iff C \vdash \Delta, x : C, \Theta & \text{Subderivation} \\
(\Delta, x : C, \Theta) &=(\Delta_L, \Delta'_R) & \text{By i.h.} \\
(\Gamma_L * \Gamma_R) &\vdash \Theta & \text{”} \\
(\Gamma_L * \Gamma_R) &\vdash \Delta &=(\Delta_R, x : C, \ldots) & \text{Similar to the } \text{[\[1\text{]} \text{ case}} \\
\end{align*} \]

**Case** \( \Gamma_L, \Gamma_R, x : A p \vdash e \iff B \vdash \Delta, x : A, \Theta \)

\[ \frac{\Gamma_L, \Gamma_R \vdash \lambda x. e \iff A \vdash B \vdash \Delta}{\text{[\[2\text{]} \text{]}} \]

\[ \begin{align*}
\Gamma_L * \Gamma_R &\vdash (A \rightarrow B) \text{ p type} & \text{Given} \\
\Gamma_L * \Gamma_R &\vdash B \text{ p type} & \text{By inversion} \\
FEV(A \rightarrow B) &\subseteq \dom(\Gamma_R) & \text{By def. of FEV} \\
FEV(A) &\subseteq \dom(\Gamma_R) & \text{Given} \\
\Gamma_L * (\Gamma_R, x : A p) &\vdash B \text{ p type} & \text{By weakening and Definition 4} \\
\Gamma_L, \Gamma_R, x : A \vdash e \iff B \vdash \Delta, x : A, \Theta & \text{Subderivation} \\
(\Delta, x : A, \Theta) &=(\Delta_L, \Delta'_R) & \text{By i.h.} \\
(\Gamma_L * \Gamma_R) &\vdash \Theta & \text{”} \\
(\Gamma_L * \Gamma_R) &\vdash \Delta &=(\Delta_R, x : A, \ldots) & \text{Similar to the } \text{[\[1\text{]} \text{ case}} \\
\end{align*} \]

**Case** \( \Gamma_0[\hat{\alpha}_1 : *, \hat{\alpha}_2 : *, \hat{\alpha} : * = \hat{\alpha}_1 \rightarrow \hat{\alpha}_2], x : \hat{\alpha}_1 \vdash e_0 \iff \hat{\alpha}_2 \vdash \Delta, x : \hat{\alpha}_1, \Delta' \)

\[ \frac{\Gamma_0[\hat{\alpha} : *] \vdash \lambda x. e_0 \iff \hat{\alpha} \vdash \Delta}{\text{[\[3\text{]} \text{]}} \]

We have \( \Gamma_L * \Gamma_R = \Gamma_0[\hat{\alpha} : *] \). We also have \( \text{FEV}(\hat{\alpha}) \subseteq \dom(\Gamma_R) \). Therefore \( \hat{\alpha} \in \dom(\Gamma_R) \) and

\[ \Gamma_0[\hat{\alpha} : *] = \Gamma_L, \Gamma_2, \hat{\alpha} : *, \Gamma_3 \]
where $\Gamma_R = (\Gamma_2, \hat{\alpha} : \ast, \Gamma_3)$.

Then the input context in the premise has the following form:

$$\Gamma_0[\hat{\alpha}_1:* , \hat{\alpha}_2:* , \hat{\alpha}:* = \hat{\alpha}_1 \rightarrow \hat{\alpha}_2], x : \hat{\alpha}_1 = \Gamma_L, \Gamma_2, \hat{\alpha}_1:* , \hat{\alpha}_2:* , \hat{\alpha}:* = \hat{\alpha}_1 \rightarrow \hat{\alpha}_2, \Gamma_3, x : \hat{\alpha}_1$$

Let us separate this context at the same point as $\Gamma_0[\hat{\alpha}:* ]$, that is, after $\Gamma_L$ and before $\Gamma_2$, and call the resulting right-hand context $\Gamma'_R$. That is,

$$\Gamma_0[\hat{\alpha}_1:* , \hat{\alpha}_2:* , \hat{\alpha}:* = \hat{\alpha}_1 \rightarrow \hat{\alpha}_2], x : \hat{\alpha}_1 = \Gamma_L \ast (\Gamma_2, \hat{\alpha}_1:* , \hat{\alpha}_2:* , \hat{\alpha}:* = \hat{\alpha}_1 \rightarrow \hat{\alpha}_2, \Gamma_3, x : \hat{\alpha}_1)$$

where $\Gamma_R \subseteq \operatorname{dom}(\Gamma'_R)$

$$\Gamma_L \ast \Gamma'_R \vdash e_0 \iff \hat{\alpha}_2 \dashv \Delta, x : \hat{\alpha}_1, \Delta'$$

Subderivation

$\hat{\alpha}_2 \in \operatorname{dom}(\Gamma'_R)$

By i.h.

Similar to the $\forall I$ case

$$\Delta = (\Delta_L, \Delta_R)$$

Case

$$\Gamma \vdash e \Rightarrow A \ p \Theta \vdash s : [\Theta] A \ p \triangleright C \ [q] \dashv \Delta$$

Use the i.h. and Lemma 68 (Transitivity of Separation), with Lemma 91 (Well-formedness of Algorithmic Typing) and Lemma 13 (Right-Hand Substitution for Typing).

Case

$$\Gamma \vdash s : A ! \triangleright C \ [1] \dashv \Delta$$

Use the i.h.

Case

$$\Gamma \vdash s : A p \triangleright C q \dashv \Delta \quad ((p = f) \ or \ (q = !) \ or \ (\operatorname{FEV}(\Delta[C]) \neq \emptyset))$$

Use the i.h.

Case

$$\Gamma_L \ast \Gamma_R \vdash e \ L \vdash A_1 \ p \Theta \vdash s : [\Theta] A_2 \ p \triangleright C q \dashv \Delta$$

Use the i.h.
Proof of Lemma 72 (Separation—Main)

\( \Gamma \vdash (A_1 \rightarrow A_2) \) \( \text{p type} \)

- Given
- \( \Gamma \vdash A_1 \) \( \text{type} \)
- By inversion

\( \text{FEV}(A_1 \rightarrow A_2) \subseteq \text{dom}(\Gamma_R) \)

- Given
- \( \text{FEV}(A_1) \subseteq \text{dom}(\Gamma_R) \)
- By def. of FEV

\( \Theta = (\Theta_L, \Theta_R) \)

- By i.h.

\( (\Gamma_L \ast \Gamma_R) \xrightarrow{\Theta} (\Theta_L \ast \Theta_R) \)

- "

\( \Gamma \vdash A_2 \) \( \text{type} \)

- By inversion
- \( \Gamma \vdash (\Theta \mid A_2) \) \( \text{type} \)
- By Lemma 13 (Right-Hand Substitution for Typing)

\( \Delta = (\Delta_L, \Delta_R) \)

- By def. of FEV

\( (\Theta_L \ast \Theta_R) \xrightarrow{\Theta} (\Delta_L \ast \Delta_R) \)

- "

\( \text{FEV}(C) \subseteq \text{dom}(\Delta_R) \)

- By def. of FEV

\( (\Gamma_L \ast \Gamma_R) \xrightarrow{\Theta} (\Delta_L \ast \Delta_R) \)

- By Lemma 68 (Transitivity of Separation)

- Case

\( \Gamma \vdash e \iff A_k \) \( \text{p} \vdash \Delta \)

- \( \Gamma \vdash \text{inj}_k e \iff A_1 + A_2 \) \( \text{p} \vdash \Delta \) \( \text{−1k} \)

Use the i.h. (inverting \( \Gamma \vdash (A_1 + A_2) \) \( \text{p type} \)).

- Case

\( \Gamma \vdash e_1 \iff A_1 \) \( \text{p} \vdash \Theta \)

- \( \Gamma \vdash e_2 \iff (\Theta \mid A_2) \) \( \text{p} \vdash \Delta \)

- \( \Gamma \vdash (e_1, e_2) \iff A_1 \times A_2 \) \( \text{p} \vdash \Delta \) \( \text{×1} \)

\( \Gamma \vdash (A_1 \times A_2) \) \( \text{p type} \)

- Given
- \( \Gamma \vdash A_1 \) \( \text{p type} \)
- By inversion
- \( \Gamma \vdash e_1 \iff A_1 \) \( \text{p} \vdash \Theta \)
- Subderivation
- \( \Theta = (\Theta_L, \Theta_R) \)
- By i.h.

\( (\Gamma_L \ast \Gamma_R) \xrightarrow{\Theta} (\Theta_L \ast \Theta_R) \)

- "

\( \Gamma \vdash A_2 \) \( \text{type} \)

- By inversion
- \( \Gamma \rightarrow \Theta \)
- By Lemma 51 (Typing Extension)
- \( \Theta \vdash A_2 \) \( \text{type} \)
- By Lemma 36 (Extension Weakening (Sorts))
- \( \Theta \vdash (\Theta \mid A_2) \) \( \text{type} \)
- By Lemma 13 (Right-Hand Substitution for Typing)
- \( \Theta \vdash e_2 \iff (\Theta \mid A_2) \) \( \text{p} \vdash \Delta \)
- Subderivation

\( \Delta = (\Delta_L, \Delta_R) \)

- By i.h.

\( (\Theta_L \ast \Theta_R) \xrightarrow{\Theta} (\Delta_L \ast \Delta_R) \)

- "

\( (\Gamma_L \ast \Gamma_R) \xrightarrow{\Theta} (\Delta_L \ast \Delta_R) \)

- By Lemma 68 (Transitivity of Separation)
Proof of Lemma 72 (Separation—Main)

- **Case** \( \Gamma[\hat{\alpha}_2:*;\hat{\alpha}_1:*;\hat{\alpha}:\hat{\alpha}_1 \times \hat{\alpha}_2] \Rightarrow e_1 \iff \hat{\alpha}_1 \Rightarrow \Theta \Rightarrow e_2 \iff [\Theta] \hat{\alpha}_2 \Rightarrow \Delta \)

We have \( \Gamma_{\L} \ast \Gamma_{\R} = \Gamma_0[\hat{\alpha} : \ast] \). We also have \( \text{FEV}(\hat{\alpha}) \subseteq \text{dom}(\Gamma_{\R}) \). Therefore \( \hat{\alpha} \in \text{dom}(\Gamma_{\R}) \) and

\[
\Gamma_0[\hat{\alpha} : \ast] = (\Gamma_{\L}, \Gamma_{\R}, \hat{\alpha} : \ast, \Gamma_3)
\]

where \( \Gamma_{\R} = (\Gamma_2, \hat{\alpha} : \ast, \Gamma_3) \).

Then the input context in the premise has the following form:

\[
\Gamma_0[\hat{\alpha}_1:*;\hat{\alpha}_2:*;\hat{\alpha}:\hat{\alpha}_1 \times \hat{\alpha}_2] = (\Gamma_{\L}, \Gamma_{\R}, \hat{\alpha}_1:*;\hat{\alpha}_2:*;\hat{\alpha}:\hat{\alpha}_1 \times \hat{\alpha}_2, \Gamma_3)
\]

Let us separate this context at the same point as \( \Gamma_0[\hat{\alpha} : \ast] \), that is, after \( \Gamma_{\L} \) and before \( \Gamma_{\R} \), and call the resulting right-hand context \( \Gamma'_{\R} \):

\[
\Gamma_0[\hat{\alpha}_1:*;\hat{\alpha}_2:*;\hat{\alpha}:\hat{\alpha}_1 \times \hat{\alpha}_2] = \Gamma_{\L} \ast (\Gamma_{\L}, \hat{\alpha}_1:*;\hat{\alpha}_2:*;\hat{\alpha}:\hat{\alpha}_1 \times \hat{\alpha}_2, \Gamma_3)
\]

\( \Gamma'_{\R} \)

\[
\begin{align*}
\text{FEV}(\hat{\alpha}) & \subseteq \text{dom}(\Gamma_{\R}) & \text{Given} \\
\Gamma_{\L} \ast \Gamma'_{\R} \vdash e_1 \iff \hat{\alpha}_1 \Rightarrow \Theta & \text{Subderivation} \\
\text{FEV}(\hat{\alpha}_2) & \subseteq \text{dom}(\Gamma'_{\R}) & \hat{\alpha}_2 \in \text{dom}(\Gamma'_{\R}) \\
\Theta = (\Theta_{\L}, \Theta_{\R}) & & \text{By i.h.} \\
(\Gamma_{\L} \ast \Gamma'_{\R}) \overline{\vdash} (\Theta_{\L} \ast \Theta_{\R}) & & " \text{Subderivation} \\
\Theta \vdash e_2 \iff [\Theta] \hat{\alpha}_2 \Rightarrow \Delta & & \text{By Definition} \text{[5]} \\
\text{dom}(\Gamma'_{\R}) & \subseteq \text{dom}(\Theta_{\R}) & \text{By above} \subseteq \\
\text{FEV}(\hat{\alpha}_2) & \subseteq \text{dom}(\Theta_{\R}) & \text{By Definition} \text{[4]} \\
\text{FEV}(\hat{\alpha}_2) & \subseteq \text{dom}(\Theta_{\R}) & \text{By Definition} \text{[4]} \\
\Theta_{\L} \ast \Theta_{\R} & \overline{\vdash} (\Delta_{\L} \ast \Delta_{\R}) & \text{By i.h.} \\
\Delta = (\Delta_{\L}, \Delta_{\R}) & & " \text{Subderivation} \\
\Gamma_{\R} = (\Gamma_{\L}, \hat{\alpha} : \ast, \Gamma_3) & & \text{Above} \\
\Gamma'_{\R} = (\Gamma_{\L}, \hat{\alpha}_1:*;\hat{\alpha}_2:*;\hat{\alpha}:\hat{\alpha}_1 \times \hat{\alpha}_2, \Gamma_3) & & \text{Above}
\end{align*}
\]

By Lemma 23 (Deep Evar Introduction) (i), (ii) and the definition of separation, we can show

\[
(\Gamma_{\L} \ast (\Gamma_{\L}, \hat{\alpha}_1:*;\hat{\alpha}_2:*;\hat{\alpha}:\hat{\alpha}_1 \times \hat{\alpha}_2, \Gamma_3)) \overline{\vdash} (\Gamma_{\L} \ast (\Gamma_{\L}, \hat{\alpha}_1:*;\hat{\alpha}_2:*;\hat{\alpha}:\hat{\alpha}_1 \times \hat{\alpha}_2, \Gamma_3))
\]

\[
(\Gamma_{\L} \ast \Gamma_{\R}) \overline{\vdash} (\Gamma_{\L} \ast \Gamma'_{\R}) & \text{By above equalities} \\
(\Gamma_{\L} \ast \Gamma_{\R}) \overline{\vdash} (\Delta_{\L} \ast \Delta_{\R}) & \text{By Lemma 68 (Transitivity of Separation) twice}
\]

- **Case** \( \Gamma[\hat{\alpha}_2:*;\hat{\alpha}_1:*;\hat{\alpha}:\hat{\alpha}_1 \Rightarrow \hat{\alpha}_2] \Rightarrow e \iff [\hat{\alpha}_1 \Rightarrow \hat{\alpha}_2] \Rightarrow \Delta \)

Similar to the \( \times \hat{\alpha} \) case, but simpler.

- **Case** \( \Gamma[\hat{\alpha}_2:*;\hat{\alpha}_1:*;\hat{\alpha}:\hat{\alpha}_1 \Rightarrow \hat{\alpha}_2] \Rightarrow e \Rightarrow s_0 : (\hat{\alpha}_1 \Rightarrow \hat{\alpha}_2) \Rightarrow C \Rightarrow \Delta \)

Similar to the \( \times \hat{\alpha} \) and \( \times \hat{\alpha}_2 \) cases, except that (because we’re in the spine part of the lemma) we have to show that \( \text{FEV}(C) \subseteq \text{dom}(\Delta_{\R}) \). But we have the same C in the premise and conclusion, so we get that by applying the i.h.
H’ Decidability of Algorithmic Subtyping

H’.1 Lemmas for Decidability of Subtyping

Lemma 73 (Substitution Isn’t Large).
For all contexts $\Theta$, we have $\#\text{large}(\Theta \cdot A) = \#\text{large}(A)$.

Proof. By induction on $A$, following the definition of substitution.

Lemma 74 (Instantiation Solves).
If $\Gamma \vdash \alpha := \tau \vdash \Delta$ and $[\Gamma] \tau = \tau$ and $\alpha \notin \text{FV}([\Gamma] \tau)$ then $|\text{unsolved}(\Gamma)| = |\text{unsolved}(\Delta)| + 1$.

Proof. By induction on the given derivation.

- Case
  
  $\Gamma_L \vdash \tau : \kappa$

  $\Gamma_L, \alpha : \kappa, \Gamma_R \vdash \alpha := \tau : \kappa \vdash \Gamma_L, \alpha : \kappa = \tau, \Gamma_R$

  It is evident that $|\text{unsolved}(\Gamma_L, \alpha : \kappa, \Gamma_R)| = |\text{unsolved}(\Gamma_L, \alpha : \kappa = \tau, \Gamma_R)| + 1$.

- Case
  
  $\hat{\beta} \in \text{unsolved}(\Gamma[\alpha : \kappa][\hat{\beta} : \kappa])$

  $\Gamma[\alpha : \kappa][\hat{\beta} : \kappa] \vdash \alpha := \hat{\beta} : \kappa \vdash \Gamma[\alpha : \kappa][\hat{\beta} : \kappa = \alpha]$

  Similar to the previous case.

- Case
  
  $\Gamma_0[\alpha_2 : \star, \alpha_1 : \star, \alpha : \star = \alpha_1 \oplus \alpha_2] \vdash \alpha_1 := \tau_1 : \star \vdash \Theta \vdash \alpha_2 := [\Theta] \tau_2 : \star \vdash \Delta$

  $\Gamma_0[\alpha : \star] \vdash \alpha := \tau_1 \oplus \tau_2 : \star \vdash \Delta$

  $|\text{unsolved}(\Gamma_0[\alpha_2 : \star, \alpha_1 : \star, \alpha = \alpha_1 \oplus \alpha_2])| = |\text{unsolved}(\Gamma_0[\alpha])| + 1$ Immediate

  $|\text{unsolved}(\Gamma_0[\alpha_2 : \star, \alpha_1 : \star, \alpha = \alpha_1 \oplus \alpha_2])| = |\text{unsolved}(\Theta)| + 1$ By i.h.

  $|\text{unsolved}(\Gamma)| = |\text{unsolved}(\Theta)|$ Subtracting 1

  $= |\text{unsolved}(\Delta)| + 1$ By i.h.

- Case
  
  $\Gamma[\alpha : N] \vdash \alpha := \text{zero} : N \vdash \Gamma[\alpha : N = \text{zero}]$

  Similar to the $\text{InstSolve}$ case.

- Case
  
  $\Gamma[\alpha_1 : N, \alpha : N = \text{succ}(\alpha_1)] \vdash \alpha_1 := t_1 : N \vdash \Delta$

  $\Gamma_0[\alpha : N] \vdash \alpha := \text{succ}(t_1) : N \vdash \Delta$

  $|\text{unsolved}(\Delta)| + 1 = |\text{unsolved}(\Gamma_0[\alpha_1 : N, \alpha : N = \text{succ}(\alpha_1)])| + 1$ By i.h.

  $= |\text{unsolved}(\Gamma_0[\alpha : N])| + 1$ By definition of $\text{unsolved}(\cdot)$
Lemma 75 (Checkeq Solving). If $\Gamma \vdash s \equiv t : \kappa \vdash \Delta$ then either $\Delta = \Gamma$ or $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Gamma)|$.

Proof. By induction on the given derivation.

- **Case**
  \[
  \Gamma \vdash u \equiv u : \kappa \vdash \Delta \quad \text{(CheckeqVar)}
  \]
  Here $\Delta = \Gamma$.

- **Cases**
  - **CheckeqUnit**
  - **CheckeqZero**
    Similar to the **CheckeqVar** case.

- **Case**
  \[
  \Gamma \vdash \sigma \equiv t : \mathbb{N} \vdash \Delta \quad \text{(CheckeqSucc)}
  \]
  Follows by i.h.

- **Case**
  \[
  \Gamma_0[\check{\alpha}] \vdash \check{\alpha} := t : \kappa \vdash \Delta \quad \check{\alpha} \notin \text{FV}(t)
  \]
  \[
  \Gamma_0[\check{\alpha}] \vdash \check{\alpha} \equiv t : \kappa \vdash \Delta
  \]
  \[
  \Gamma_0[\check{\alpha}] \vdash \check{\alpha} := t : \kappa \vdash \Delta
  \]
  Subderivation
  \[
  \Gamma \vdash \check{\alpha} := t : \kappa \vdash \Delta
  \]
  \[
  \Delta = \Gamma \text{ or } |\text{unsolved}(\Delta)| = |\text{unsolved}(\Gamma)| - 1 \quad \text{By Lemma 74 (Instantiation Solves)}
  \]
  \[
  \Delta = \Gamma \text{ or } |\text{unsolved}(\Delta)| < |\text{unsolved}(\Gamma)|
  \]

- **Case**
  \[
  \Gamma[\check{\alpha} : \kappa] \vdash \check{\alpha} := t : \kappa \vdash \Delta \quad \check{\alpha} \notin \text{FV}(t)
  \]
  \[
  \Gamma[\check{\alpha} : \kappa] \vdash \check{\alpha} \equiv t : \kappa \vdash \Delta
  \]
  Similar to the **CheckeqInstL** case.

- **Case**
  \[
  \Gamma \vdash \sigma_1 \equiv \tau_1 : * \vdash \Theta \quad \Theta \vdash [\Theta] \sigma_2 \equiv [\Theta] \tau_2 : * \vdash \Delta
  \]
  \[
  \Gamma \vdash \sigma_1 + \sigma_2 \equiv \tau_1 + \tau_2 : * \vdash \Delta
  \]
  \[
  \Gamma \vdash \sigma_1 \equiv \tau_1 : * \vdash \Theta
  \]
  Subderivation
  \[
  \Theta = \Gamma \text{ or } |\text{unsolved}(\Theta)| < |\text{unsolved}(\Gamma)| \quad \text{By i.h.}
  \]
  - $\Theta = \Gamma$:
    \[
    \Theta \vdash [\Theta] \sigma_2 \equiv [\Theta] \tau_2 : * \vdash \Delta
    \]
    Subderivation
    \[
    \Gamma \vdash [\Gamma] \sigma_2 \equiv [\Gamma] \tau_2 : * \vdash \Delta
    \]
    By $\Theta = \Gamma$
    \[
    \Delta = \Gamma \text{ or } |\text{unsolved}(\Delta)| = |\text{unsolved}(\Gamma)| + 1 \quad \text{By i.h.}
    \]
  - $|\text{unsolved}(\Theta)| < |\text{unsolved}(\Gamma)|$:
    \[
    \Theta \vdash [\Theta] \sigma_2 \equiv [\Theta] \tau_2 : * \vdash \Delta
    \]
    Subderivation
    \[
    \Delta = \Theta \text{ or } |\text{unsolved}(\Delta)| < |\text{unsolved}(\Theta)| \quad \text{By i.h.}
    \]

If $\Delta = \Theta$ then substituting $\Delta$ for $\Theta$ in $|\text{unsolved}(\Theta)| < |\text{unsolved}(\Gamma)|$ gives $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Gamma)|$.

If $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Theta)|$ then transitivity of $<$ gives $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Gamma)|$. □
Lemma 76 (Prop Equiv Solving).
If $\Gamma \vdash P \equiv Q \vdash \Delta$ then either $\Delta = \Gamma$ or $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Gamma)|$.

Proof. Only one rule can derive the judgment:

- **Case**
  
  \[
  \Gamma \vdash \sigma_1 \overset{\Delta}{=} t_1 : N \vdash \Theta \quad \Theta \vdash \Pi \sigma_2 \overset{\Delta}{=} [\Theta]t_2 : N \vdash \Delta \\
  \Gamma \vdash \Pi (\sigma_1 = \sigma_2) \equiv (t_1 = t_2) \vdash \Delta
  \]

  By Lemma 75 (Checkeq Solving) on the first premise, either $\Theta = \Gamma$ or $|\text{unsolved}(\Theta)| < |\text{unsolved}(\Gamma)|$.

  In the former case, the result follows from Lemma 75 (Checkeq Solving) on the second premise.

  In the latter case, applying Lemma 75 (Checkeq Solving) to the second premise either gives $\Delta = \Theta$, and therefore $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Theta)|$, which also leads to $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Gamma)|$. \qed

Lemma 77 (Equiv Solving).
If $\Gamma \vdash A \equiv B \vdash \Delta$ then either $\Delta = \Gamma$ or $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Gamma)|$.

Proof. By induction on the given derivation.

- **Case**
  
  \[
  \Gamma \vdash \alpha \equiv \alpha \vdash \Gamma \equiv \text{Var}
  \]

  Here $\Delta = \Gamma$.

- **Cases**
  
  \[
  \equiv \text{Exvar} \equiv \text{Unit} \quad \text{Similar to the } \equiv \text{Var case.}
  \]

- **Case**
  
  \[
  \Gamma \vdash A_1 \equiv B_1 \vdash \Theta \quad \Theta \vdash \Pi A_2 \equiv [\Theta]B_2 \vdash \Delta \\
  \Gamma \vdash (A_1 + A_2) \equiv [\Theta]B_1 + [\Theta]B_2 \vdash \Delta
  \]

  By i.h., either $\Theta = \Gamma$ or $|\text{unsolved}(\Theta)| < |\text{unsolved}(\Gamma)|$.

  In the former case, apply the i.h. to the second premise. Now either $\Delta = \Theta$—and therefore $\Delta = \Gamma$—or $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Theta)|$. Since $\Theta = \Gamma$, we have $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Gamma)|$.

  In the latter case, we have $|\text{unsolved}(\Theta)| < |\text{unsolved}(\Gamma)|$. By i.h. on the second premise, either $\Delta = \Theta$, and substituting $\Delta$ for $\Theta$ gives $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Gamma)|$—or $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Theta)|$, which combined with $|\text{unsolved}(\Theta)| < |\text{unsolved}(\Gamma)|$ gives $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Gamma)|$.

- **Case**
  
  \[
  \equiv \text{Vec} \quad \text{Similar to the } \equiv \text{Var} \text{ case.}
  \]

- **Case**
  
  \[
  \Gamma, \alpha : \kappa \vdash A_0 \equiv B_0 \vdash \Delta, \alpha : \kappa, \Delta' \\
  \Gamma \vdash \forall \alpha : \kappa. A_0 \equiv \forall \alpha : \kappa. B_0 \vdash \Delta \equiv \text{Ve}
  \]

  By i.h., either $(\Delta, \alpha : \kappa, \Delta') = (\Gamma, \alpha : \kappa)$, or $|\text{unsolved}(\Delta, \alpha : \kappa, \Delta')| < |\text{unsolved}(\Gamma, \alpha : \kappa)|$.

  In the former case, Lemma 22 (Extension Inversion) (i) tells us that $\Delta' = \Gamma$. Thus, $(\Delta, \alpha : \kappa) = (\Gamma, \alpha : \kappa)$, and so $\Delta = \Gamma$.

  In the latter case, we have $|\text{unsolved}(\Delta, \alpha : \kappa, \Delta')| < |\text{unsolved}(\Gamma, \alpha : \kappa)|$, that is:

  \[
  |\text{unsolved}(\Delta)| + 0 + |\text{unsolved}(\Delta')| < |\text{unsolved}(\Gamma)| + 0
  \]

  Since $|\text{unsolved}(\Delta')|$ cannot be negative, we have $|\text{unsolved}(\Delta)| < |\text{unsolved}(\Gamma)|$. \qed
Proof of Lemma 77 (Equiv Solving) lem:equiv-solving

- Case

  \[
  \Gamma \vdash P \equiv Q \implies \Theta \vdash \Theta A_0 \equiv \Theta B_0 \vdash \Delta
  \]

  \[
  \Gamma \vdash A_0 \equiv Q \supset B_0 \vdash \Delta
  \]

  Similar to the \( \equiv \equiv \) case, but using Lemma 76 (Prop Equiv Solving) on the first premise instead of the i.h.

- Case

  \[
  \Gamma \vdash P \equiv Q \implies \Theta \vdash \Theta A_0 \equiv \Theta B_0 \vdash \Delta
  \]

  \[
  \Gamma \vdash A_0 \wedge P \equiv B_0 \wedge Q \vdash \Delta
  \]

  Similar to the \( \equiv \wedge \) case.

- Case

  \[
  \Gamma_0[\hat{\alpha}] \vdash \hat{\alpha} := \tau \vdash \tilde{\alpha} \notin \text{FV}(\tau)
  \]

  \[
  \Gamma_0[\hat{\alpha}] \vdash \hat{\alpha} \equiv \tau \vdash \Delta
  \]

  \[
  \Gamma \vdash \tilde{\alpha} \equiv \tau \vdash \Delta
  \]

  By Lemma 74 (Instantiation Solves), \( |\text{unsolved}(\Delta)| = |\text{unsolved}(\Gamma)| - 1 \).

- Case

  \[
  \Gamma_0[\hat{\alpha}] \vdash \hat{\alpha} := \tau \vdash \tilde{\alpha} \notin \text{FV}(\tau)
  \]

  \[
  \Gamma_0[\hat{\alpha}] \vdash \tau \equiv \tilde{\alpha} \vdash \Delta
  \]

  Similar to the \( \text{InstantiateL} \) case.

Proof of Lemma 78 (Decidability of Propositional Judgments).

The following judgments are decidable, with \( \Delta \) as output in (1)–(3), and \( \Delta' \) as output in (4) and (5).

We assume \( \sigma = \Gamma \sigma \) and \( t = \Gamma t \) in (1) and (4). Similarly, in the other parts we assume \( P = \Gamma P \) and (in part (3)) \( Q = \Gamma Q \).

1. \( \Gamma \vdash \sigma \equiv t : \kappa \vdash \Delta \)
2. \( \Gamma \vdash P \text{ true} \vdash \Delta \)
3. \( \Gamma \vdash P \equiv Q \vdash \Delta \)
4. \( \Gamma / \sigma \equiv t : \kappa \vdash \Delta' \)
5. \( \Gamma / P \vdash \Delta' \)

Proof. Since there is no mutual recursion between the judgments, we can prove their decidability in order, separately.

1. Decidability of \( \Gamma \vdash \sigma \equiv t : \kappa \vdash \Delta \): By induction on the sizes of \( \sigma \) and \( t \).

   - Cases \( \text{CheckeqVar} \), \( \text{CheckeqUnit} \), \( \text{CheckeqZero} \): No premises.
   - Case \( \text{CheckeqSucc} \): Both \( \sigma \) and \( t \) get smaller in the premise.
   - Cases \( \text{CheckeqInstL} \), \( \text{CheckeqInstR} \): Follows from Lemma 67 (Decidability of Instantiation).

2. Decidability of \( \Gamma \vdash P \text{ true} \vdash \Delta \): By induction on \( \sigma \) and \( t \). But we have only one rule deriving this judgment form, \( \text{CheckpropEq} \), which has the judgment in (1) as a premise, so decidability follows from part (1).

3. Decidability of \( \Gamma \vdash P \equiv Q \vdash \Delta \): By induction on \( P \) and \( Q \). But we have only one rule deriving this judgment form, \( \equiv \text{PropEq} \), which has two premises of the form (1), so decidability follows from part (1).

4. Decidability of \( \Gamma / \sigma \equiv t : \kappa \vdash \Delta' \): By lexicographic induction, first on the number of unsolved variables (both universal and existential) in \( \Gamma \), then on \( \sigma \) and \( t \). We also show that the number of unsolved variables is nonincreasing in the output context (if it exists).
Proof of Lemma 78 (Decidability of Propositional Judgments)

• Cases ElimeqUvarRef, ElimeqZero: No premises, and the output is the same as the input.
• Case ElimeqClash: The only premise is the clash judgment, which is clearly decidable. There is no output.
• Case ElimeqBin: In the first premise, we have the same $\Gamma$ but both $\sigma$ and $t$ are smaller. By i.h., the first premise is decidable; moreover, either some variables in $\Theta$ were solved, or no additional variables were solved.

If some variables in $\Theta$ were solved, the second premise is smaller than the conclusion according to our lexicographic measure, so by i.h., the second premise is decidable.

If no additional variables were solved, then $\Theta = \Gamma$. Therefore $[\Theta]\tau_2 = [\Gamma]\tau_2$. It is given that $\sigma = [\Gamma]\sigma$ and $t = [\Gamma]t$, so $[\Gamma]\tau_2 = \tau_2$. Likewise, $[\Theta]\tau_2 = [\Gamma]\tau_2 = \tau_2$, so we are making a recursive call on a strictly smaller subterm.

Regardless, $\Delta^2$ is either $\bot$, or is a $\Delta$ which has no more unsolved variables than $\Theta$, which in turn has no more unsolved variables than $\Gamma$.

• Case ElimeqBinBot: The premise is invoked on subterms, and does not yield an output context.
• Case ElimeqSucc: Both $\sigma$ and $t$ get smaller. By i.h., the output context has fewer unsolved variables, if it exists.
• Cases ElimeqInstL, ElimeqInstR: Follows from Lemma 67 (Decidability of Instantiation). Furthermore, by Lemma 74 (Instantiation Solves), instantiation solves a variable in the output.
• Cases ElimeqUvarL, ElimeqUvarR: These rules have no nontrivial premises, and $\alpha$ is solved in the output context.
• Cases ElimeqUvarL $\bot$, ElimeqUvarR $\bot$: These rules have no nontrivial premises, and produce the output context $\bot$.

(5) Decidability of $\Gamma / P \vdash \Delta^\bot$: By induction on $P$. But we have only one rule deriving this judgment form, ElimpropEq, for which decidability follows from part (4).

Lemma 79 (Decidability of Equivalence).
Given a context $\Gamma$ and types $A$, $B$ such that $\Gamma \vdash A$ type and $\Gamma \vdash B$ type and $[\Gamma]A = A$ and $[\Gamma]B = B$, it is decidable whether there exists $\Delta$ such that $\Gamma \vdash A \equiv B \vdash \Delta$.

Proof. Let the judgment $\Gamma \vdash A \equiv B \vdash \Delta$ be measured lexicographically by

(E1) $\#\text{large}(A) + \#\text{large}(B)$;

(E2) $|\text{unsolved}(\Gamma)|$, the number of unsolved existential variables in $\Gamma$;

(E3) $|A| + |B|$.

• Cases $\equiv \text{Var}$, $\equiv \text{Exvar}$, $\equiv \text{Unit}$: No premises.

• Case $\Gamma \vdash A_1 \equiv B_1 \vdash \Theta \quad \Theta \vdash [\Theta]A_2 \equiv [\Theta]B_2 \vdash \Delta$

\[
\Gamma \vdash A_1 \oplus A_2 \equiv B_1 \oplus B_2 \vdash \Delta \)

In the first premise, part (E1) either gets smaller (if $A_2$ or $B_2$ have large connectives) or stays the same. Since the first premise has the same input context, part (E2) remains the same. However, part (E3) gets smaller.

In the second premise, part (E1) either gets smaller (if $A_1$ or $B_1$ have large connectives) or stays the same.

• Case $\equiv \text{Vec}$: Similar to a special case of $\equiv$ where two of the types are monotypes.
Proof of Lemma 79 (Decidability of Equivalence) lem:equiv-decidable

- Case \( \Gamma, \alpha : \kappa \vdash A_0 \equiv B_0 \vdash \Delta, \alpha : \kappa, \Delta' \)

\[
\Gamma \vdash \forall \alpha : \kappa. A_0 \equiv \forall \alpha : \kappa. B_0 \vdash \Delta \quad \text{\[equiv]}\]

Since \( \#\text{large}(A_0) + \#\text{large}(B_0) = \#\text{large}(A) + \#\text{large}(B) - 2 \), the first part of the measure gets smaller.

- Case \( \Gamma \vdash P \equiv Q \vdash \Theta \quad \Theta \vdash [\Theta]A_0 \equiv [\Theta]B_0 \vdash \Delta \)

\[
\Gamma \vdash P \supset A_0 \equiv [\Theta]B_0 \vdash \Delta \quad \text{\[\supset]}\]

The first premise is decidable by Lemma 78 (Decidability of Propositional Judgments) (3).

For the second premise, by Lemma 73 (Substitution Isn’t Large), \( \#\text{large}([\Theta]B_0) = \#\text{large}(B_0) \). Since \( \#\text{large}(A) = \#\text{large}(A_0) + 1 \) and \( \#\text{large}(B) = \#\text{large}(B_0) + 1 \), we have

\[
\#\text{large}([\Theta]A_0) + \#\text{large}([\Theta]B_0) < \#\text{large}(A) + \#\text{large}(B)
\]

which makes the first part of the measure smaller.

- Case \( \Gamma \vdash P \equiv Q \vdash \Theta \quad \Theta \vdash [\Theta]A_0 \equiv [\Theta]B_0 \vdash \Delta \)

\[
\Gamma \vdash A_0 \land P \equiv B_0 \land Q \vdash \Delta \quad \text{\[\land]}\]

Similar to the \[\supset\] case.

- Case \( \Gamma[\alpha] \vdash \alpha := \tau : \ast \vdash \Delta \quad \alpha \notin \text{FV}(\tau) \)

\[
\Gamma[\alpha] \vdash \alpha \equiv \tau \vdash \Delta \quad \text{\[\equiv\]}
\]

Follows from Lemma 67 (Decidability of Instantiation).

- Case \[\equiv\text{InstantiateR} \]

Similar to the \[\equiv\text{InstantiateR} \] case.

\[\Box\]

H’.2 Decidability of Subtyping

Theorem 1 (Decidability of Subtyping).
Given a context \( \Gamma \) and types \( A, B \) such that \( \Gamma \vdash A \) type and \( \Gamma \vdash B \) type and \( [\Gamma]A = A \) and \( [\Gamma]B = B \), it is decidable whether there exists \( \Delta \) such that \( \Gamma \vdash A <_{P} B \vdash \Delta \).

Proof. Let the judgments be measured lexicographically by \( \#\text{large}(A) + \#\text{large}(B) \).

For each subtyping rule, we show that every premise is smaller than the conclusion, or already known to be decidable. The condition that \( [\Gamma]A = A \) and \( [\Gamma]B = B \) is easily satisfied at each inductive step, using the definition of substitution.

Now, we consider the rules deriving \( \Gamma \vdash A <_{P} B \vdash \Delta \).

- Case \( A \) not headed by \( \forall / \exists \)

\[
\text{B not headed by } \forall / \exists \quad \Gamma \vdash A \equiv B \vdash \Delta \quad \text{\[\equiv\]}
\]

In this case, we appeal to Lemma 79 (Decidability of Equivalence).

- Case \( B \) not headed by \( \forall \)

\[
\Gamma, \triangleright_{\alpha}, \delta : \kappa \vdash [\delta/\alpha]A <_{P} B \vdash \Delta, \triangleright_{\alpha}, \Theta \quad \text{\[\triangleright_{\alpha}\]}
\]

The premise has one fewer quantifier.
Proof of Theorem 1 (Decidability of Subtyping) thm:subtyping-decidable

- Case \( \Gamma, \beta : \kappa \vdash A <: \neg B \vdash \Delta, \beta : \kappa, \Theta \)
  \[
  \Gamma \vdash A <: \forall \beta : \kappa. B \vdash \Delta \quad \text{<:\forall R}
  \]
  The premise has one fewer quantifier.

- Case \( \Gamma, \alpha : \kappa \vdash A <:+ B \vdash \Delta, \alpha : \kappa, \Theta \)
  \[
  \Gamma \vdash \exists \alpha : \kappa. A <:+ B \vdash \Delta \quad \text{<:\exists L}
  \]
  The premise has one fewer quantifier.

- Case \( A \) not headed by \( \exists \)
  \[
  \Gamma, \triangledown \beta, \triangledown \beta : \kappa \vdash A <: \neg \triangledown \beta | \neg \beta B \vdash \Delta, \triangledown \beta, \Theta \)
  \[
  \Gamma \vdash A <: \triangledown \beta : \kappa. B \vdash \Delta \quad \text{<:\forall R}
  \]
  The premise has one fewer quantifier.

- Case \( \Gamma \vdash A <:+ B \vdash \Delta \)

  Consider whether \( B \) is negative.

  - Case \( \text{neg}(B) \):
    \[
    B = \forall \beta : \kappa. B' \quad \text{Definition of \text{neg}(B)}
    \]
    \[
    \Gamma, \beta : \kappa \vdash A <: \neg B' \vdash \Delta, \beta : \kappa, \Theta \quad \text{Inversion on the premise}
    \]
    There is one fewer quantifier in the subderivation.

  - Case \( \text{nonneg}(B) \):
    In this case, \( B \) is not headed by a \( \forall \).
    \[
    A = \forall \alpha : \kappa. A' \quad \text{Definition of \text{neg}(A)}
    \]
    \[
    \Gamma, \triangledown \alpha, \triangledown \alpha : \kappa \vdash [\triangledown \alpha / \alpha] A' <: \neg \triangledown \alpha B' \vdash \Delta, \triangledown \alpha, \Theta \quad \text{Inversion on the premise}
    \]
    There is one fewer quantifier in the subderivation.

- Case \( \Gamma \vdash A <:+ B \vdash \Delta \)

  Consider whether \( A \) is negative.

  - Case \( \text{neg}(A) \):
    \[
    A = \forall \alpha : \kappa. A' \quad \text{Definition of \text{neg}(A)}
    \]
    \[
    \Gamma, \beta : \kappa \vdash A <: \neg \forall \beta | \forall \beta A' \vdash \Delta, \beta : \kappa, \Theta \quad \text{Inversion on the premise}
    \]
    There is one fewer quantifier in the subderivation.

  - Case \( \text{pos}(A) \):
    \[
    B = \forall \beta : \kappa. B' \quad \text{Definition of \text{neg}(B)}
    \]
    \[
    \Gamma, \beta : \kappa \vdash A <: \neg B' \vdash \Delta, \beta : \kappa, \Theta \quad \text{Inversion on the premise}
    \]
    There is one fewer quantifier in the subderivation.

- Case \( \Gamma \vdash A <:+ B \vdash \Delta \)

  Consider whether \( B \) is positive.

  - Case \( \text{nonneg}(B) \):
    In this case, \( B \) is not headed by a \( \forall \).
    \[
    A = \forall \alpha : \kappa. A' \quad \text{Definition of \text{neg}(A)}
    \]
    \[
    \Gamma, \triangledown \alpha, \triangledown \alpha : \kappa \vdash [\triangledown \alpha / \alpha] A' <: \neg \triangledown \alpha B' \vdash \Delta, \triangledown \alpha, \Theta \quad \text{Inversion on the premise}
    \]
    There is one fewer quantifier in the subderivation.

- Case \( \Gamma \vdash A <:+ B \vdash \Delta \)

  Consider whether \( A \) is positive.

  - Case \( \text{pos}(A) \):
    \[
    B = \forall \beta : \kappa. B' \quad \text{Definition of \text{neg}(B)}
    \]
    \[
    \Gamma, \beta : \kappa \vdash A <: \neg B' \vdash \Delta, \beta : \kappa, \Theta \quad \text{Inversion on the premise}
    \]
    There is one fewer quantifier in the subderivation.

  - Case \( \text{neg}(B) \):
    \[
    B = \forall \beta : \kappa. B' \quad \text{Definition of \text{neg}(B)}
    \]
    \[
    \Gamma, \beta : \kappa \vdash A <: \neg \forall \beta | \forall \beta A' \vdash \Delta, \beta : \kappa, \Theta \quad \text{Inversion on the premise}
    \]
    There is one fewer quantifier in the subderivation.

  - Case \( \text{pos}(A) \):
    \[
    B = \forall \beta : \kappa. B' \quad \text{Definition of \text{neg}(B)}
    \]
    \[
    \Gamma, \beta : \kappa \vdash A <: \neg B' \vdash \Delta, \beta : \kappa, \Theta \quad \text{Inversion on the premise}
    \]
    There is one fewer quantifier in the subderivation.

This case is similar to the \text{\textless::\textgreater _R} case.
Proof of Theorem 1 (Decidability of Subtyping)

thm:subtyping-decidable

- Case

\[
\begin{align*}
\Gamma \vdash A <: B &\rightarrow\Delta \\
\text{nonneg}(A) &\quad \text{pos}(B) \\
\Gamma \vdash A <: B &\rightarrow\Delta
\end{align*}
\]

\[<:R\]

This case is similar to the \( <:L \) case.

\[\qed\]

H'.3 Decidability of Matching and Coverage

Lemma 80 (Decidability of Guardedness Judgment).
For any set of branches \( \Pi \), the relation \( \Pi \text{ guarded} \) is decidable.

Proof. This follows via a routine induction on \( \Pi \), counting the number of branch lists.

Lemma 81 (Decidability of Expansion Judgments).
Given branches \( \Pi \), it is decidable whether:

1. there exists a unique \( \Pi' \) such that \( \Pi \overset{\var}{\sim} \Pi' \);
2. there exist unique \( \Pi_L \) and \( \Pi_R \) such that \( \Pi \overset{\triangledown}{\sim} \Pi_L \parallel \Pi_R \);
3. there exists a unique \( \Pi' \) such that \( \Pi \overset{\text{var}}{\sim} \Pi' \);
4. there exists a unique \( \Pi' \) such that \( \Pi \overset{\downarrow}{\sim} \Pi' \).
5. there exist unique \( \Pi_{\downarrow} \) and \( \Pi_{\downarrow} \) such that \( \Pi \overset{\text{Vec}}{\sim} \Pi_{\downarrow} \parallel \Pi_{\downarrow} \).

Proof. In each part, by induction on \( \Pi \): Every rule either has no premises, or breaks down \( \Pi \) in its nontrivial premise.

Lemma 82 (Expansion Shrinks Size).
We define the size of a pattern \(|p|\) as follows:

\[
\begin{align*}
|x| &= 0 \\
|\_| &= 0 \\
|\langle p, p' \rangle| &= 1 + |p| + |p'| \\
|\emptyset| &= 0 \\
|\text{inj}_1 p| &= 1 + |p| \\
|\text{inj}_2 p| &= 1 + |p| \\
|\emptyset| &= 1 \\
|p :: p'| &= 1 + |p| + |p'|
\end{align*}
\]

We lift size to branches \( \pi = \vec{p} \Rightarrow e \) as follows:

\[|p_1, \ldots, p_n \Rightarrow e| = |p_1| + \ldots + |p_n|\]

We lift size to branch lists \( \Pi = \pi_1 | \ldots | \pi_n \) as follows:

\[|\pi_1 | \ldots | \pi_n| = |\pi_1| + \ldots + |\pi_n|\]

Now, the following properties hold:

1. If \( \Pi \overset{\text{var}}{\sim} \Pi' \) then \(|\Pi| = |\Pi'|\).
2. If \( \Pi \overset{\downarrow}{\sim} \Pi' \) then \(|\Pi| = |\Pi'|\).
3. If \( \Pi \overset{\text{Vec}}{\sim} \Pi' \) then \(|\Pi| \leq |\Pi'|\).
4. If $\Pi \downarrow \Pi_k || \Pi_R$ then $|\Pi| \leq |\Pi_1|$ and $|\Pi| \leq |\Pi_2|$.

5. If $\Pi \xrightarrow{\text{vec}} \Pi_{\Omega} || \Pi_\cdot$ then $|\Pi_\Omega| \leq |\Pi|$ and $|\Pi_\cdot| \leq |\Pi|$.

6. If $\Pi$ guarded and $\Pi \xrightarrow{\text{vec}} \Pi_{\Omega} || \Pi_\cdot$ then $|\Pi_\Omega| < |\Pi|$ and $|\Pi_\cdot| < |\Pi|$.

**Proof.** Properties 1-5 follow by a routine induction on the derivation of the expansion judgement. Since expansion only ever removes pattern constructors, and only ever adds wildcards, it never increases the size of the resulting branch list.

Case 6 for vectors proceeds by induction on the derivation of $\Pi$ guarded, and furthermore depends upon the proof for case 5.

1. **Case**

   $\emptyset, \bar{p} \Rightarrow e \mid \Pi$ guarded

   By inversion on the expansion derivation, we know $\Pi \xrightarrow{\text{vec}} \Pi_{\Omega} || \Pi_\cdot$. By part 5, we know that $|\Pi_\Omega| \leq |\Pi|$ and $|\Pi_\cdot| \leq |\Pi|$.

   Hence $|\bar{p} \Rightarrow e| < |\bar{p} \Rightarrow e|$. By part 5, we know that $|\Pi_\cdot| < |\Pi_\cdot|, \bar{p} \Rightarrow e \mid \Pi$.

   Hence $|\Pi_\cdot| < |\Pi_\cdot|, \bar{p} \Rightarrow e \mid \Pi$.

2. **Case**

   $p :: p', \bar{p} \Rightarrow e \mid \Pi$ guarded

   By inversion on the expansion derivation, we know $\Pi \xrightarrow{\text{vec}} \Pi_{\Omega} || \Pi_\cdot$. By part 5, we know that $|\Pi_\Omega| \leq |\Pi|$ and $|\Pi_\cdot| \leq |\Pi|$.

   Hence $|p, p', \bar{p} \Rightarrow e| < |p :: p', \bar{p} \Rightarrow e|$. By part 5, we know that $|\Pi_\cdot| < |\Pi_\cdot|, \bar{p} \Rightarrow e \mid \Pi$.

   Hence $|\Pi_\cdot| < |\Pi_\cdot|, \bar{p} \Rightarrow e \mid \Pi$.

3. **Case**

   $\cdot, \bar{p} \Rightarrow e \mid \Pi$ guarded

   By inversion on the expansion derivation, we know $\Pi \xrightarrow{\text{vec}} \Pi_{\Omega} || \Pi_\cdot$. By induction, $|\Pi_\Omega| < |\Pi|$ and $|\Pi_\cdot| < |\Pi|$.

   By the definition of size, $\cdot, \bar{p} \Rightarrow e \mid \Pi_\Omega| < |\cdot, \bar{p} \Rightarrow e \mid \Pi$.

   By the definition of size, $\cdot, \bar{p} \Rightarrow e \mid \Pi_\cdot| < |\cdot, \bar{p} \Rightarrow e \mid \Pi$.

4. **Case**

   $x, \bar{p} \Rightarrow e \mid \Pi$ guarded

   Similar to previous case.

**Theorem 2** (Decidability of Coverage).

*Given a context $\Gamma$, branches $\Pi$ and types $\bar{\Lambda}$, it is decidable whether $\Gamma \vdash \Pi$ covers $\bar{\Lambda}$ q is derivable.*
Proof. By induction on, lexicographically, (1) the size \(|\Pi|\) of the branch list \(\Pi\) and then (2) the number of \(\land\) connectives in \(\bar{A}\), and then (3) the size of \(\bar{A}\), considered to be the sum of the sizes \(|A|\) of each type \(A\) in \(\bar{A}\) (treating constraints \(s = t\) as size 1).

(For \(\text{CoversVar} \ \text{Covers}\times \ \text{CoversVec} \ \text{CoversVec} \ \text{CoversVec}!\) and \(\text{Covers+}\) we also use the appropriate part of Lemma 81 (Decidability of Expansion Judgments), as well as Lemma 82 (Expansion Shrinks Size).)

- **Case **\(\text{CoversEmpty}\): No premises.
- **Case **\(\text{CoversVar}\): The number of \(\land\) connectives does not grow, and the size of the branch list stays the same, and \(\bar{A}\) gets smaller.
- **Case **\(\text{Covers}\): The number of \(\land\) connectives and the size of the branch list stays the same, and \(\bar{A}\) gets smaller.
- **Case **\(\text{Covers}/\land\): The size of the branch list stays the same, and the number of \(\land\) connectives in \(\bar{A}\) goes down. This lets us analyze the two possibilities for the coverage-with-conditions judgement:
  - **Case **\(\text{CoversEq}\): The first premise is decidable by Lemma 78 (Decidability of Propositional Judgments) (4). The number of \(\land\) connectives in \(\bar{A}\) gets smaller (note that applying \(\Delta\) as a substitution cannot add \(\land\) connectives).
  - **Case **\(\text{CoversEqBot}\): The premise is decidable by Lemma 78 (Decidability of Propositional Judgments) (4).
- **Case **\(\text{Covers}/\land\): The size of the branch list stays the same, and the number of \(\land\) connectives in \(\bar{A}\) goes down.
- **Case **\(\text{Covers}\times\): The size of the branch list does not grow, the number of \(\land\) connectives stays the same, and \(\bar{A}\) gets smaller, since \(|A_1| + |A_2| < |A_1 \times A_2|\).
- **Case **\(\text{Covers+}\): Here we have \(\bar{A} = (A_1 + A_2, \bar{B})\). In the first premise, we have \((A_1, \bar{B})\), which is smaller than \(\bar{A}\), and in the second premise we have \((A_2, \bar{B})\), which is likewise smaller. (In both premises, the size of the branch list does not grow, and the number of \(\land\) connectives stays the same.)
- **Case **\(\text{CoversVec}\): Since \(\Pi\) guarded is decidable, and \(\Pi \not\succeq \Pi_\| \Pi_\succeq\) is decidable, then we know that the size of the branch lists \(\Pi_\|\) and \(\Pi_\succeq\) is strictly smaller than \(\Pi\).

This lets us analyze the two cases for each premise, noting that the assumption is decidable by Lemma 78 (Decidability of Propositional Judgments) (4).

  - **Case **\(\text{CoversEq}\): The first premise (that \(t = \text{zero}\)) is decidable by Lemma 78 (Decidability of Propositional Judgments) (4). The size of \(\Pi_\|\) is strictly smaller than \(\Pi\)'s size, so we can still appeal to induction (note \(\Delta\) as a substitution cannot add change the size of a branch list).
  - **Case **\(\text{CoversEqBot}\): Decidable by Lemma 78 (Decidability of Propositional Judgments) (4).

The cons case is nearly identical:

  - **Case **\(\text{CoversEq}\): The first premise (that \(t = \text{succ}(n)\)) is decidable by Lemma 78 (Decidability of Propositional Judgments) (4). The size of \(\Pi_\|\) is strictly smaller than \(\Pi\)'s size, so we can still appeal to induction (note \(\Delta\) as a substitution cannot add change the size of a branch list).
  - **Case **\(\text{CoversEqBot}\): Decidable by Lemma 78 (Decidability of Propositional Judgments) (4).

- **Case **\(\text{CoversVec}!\): Since \(\Pi\) guarded is decidable, and \(\Pi \not\succeq \Pi_\| \Pi_\succeq\) is decidable, then we know that the size of the branch lists \(\Pi_\|\) and \(\Pi_\succeq\) is strictly smaller than \(\Pi\).
  - **Case **\(\text{Covers}\): The size of the branch list does not grow, and \(\bar{A}\) gets smaller.
• **Case CoversEq**: The first premise is decidable by Lemma 78 (Decidability of Propositional Judgments) (4). The number of \( \land \) connectives in \( \vec{A} \) gets smaller (note that applying \( \Delta \) as a substitution cannot add \( \land \) connectives).

• **Case CoversEqBot**: Decidable by Lemma 78 (Decidability of Propositional Judgments) (4).

### H’.4 Decidability of Typing

**Theorem 3** (Decidability of Typing).

1. **Synthesis**: Given a context \( \Gamma \), a principality \( p \), and a term \( e \),
   it is decidable whether there exist a type \( A \) and a context \( \Delta \) such that
   \( \Gamma \vdash e \Rightarrow A \ p \vdash \Delta \).

2. **Spines**: Given a context \( \Gamma \), a spine \( s \), a principality \( p \), and a type \( A \) such that \( \Gamma \vdash A \text{ type} \),
   it is decidable whether there exist a type \( B \), a principality \( q \) and a context \( \Delta \) such that
   \( \Gamma \vdash s : A \ p \gg B \ q \vdash \Delta \).

3. **Checking**: Given a context \( \Gamma \), a principality \( p \), a term \( e \), and a type \( B \) such that \( \Gamma \vdash B \text{ type} \),
   it is decidable whether there is a context \( \Delta \) such that
   \( \Gamma \vdash e \Leftarrow B \ p \vdash \Delta \).

4. **Matching**: Given a context \( \Gamma \), branches \( \Pi \), a list of types \( \vec{A} \), a type \( C \), and a principality \( p \), it is decidable whether there exists \( \Delta \) such that \( \Gamma \vdash \Pi :: \vec{A} \Leftarrow C \ p \vdash \Delta \).

   Also, if given a proposition \( P \) as well, it is decidable whether there exists \( \Delta \) such that \( \Gamma / P \vdash \Pi :: \vec{A} ! \Leftarrow C \ p \vdash \Delta \).

**Proof.** For rules deriving judgments of the form

\[
\begin{align*}
\Gamma &\vdash e \Rightarrow - - \vdash \\
\Gamma &\vdash e \Leftarrow B \ p \vdash \\
\Gamma &\vdash s : B \ p \gg - - \vdash \\
\Gamma &\vdash \Pi :: \vec{A} \ q \Leftarrow C \ p \vdash \\
\end{align*}
\]

(where we write “...” for parts of the judgments that are outputs), the following induction measure on such judgments is adequate to prove decidability:

\[
\begin{align*}
\langle e/s/\Pi, \Rightarrow / \gg, \ #\text{large}(B), \ B \ Match, \ \vec{A}, \ \text{match judgment form} \rangle
\end{align*}
\]

where \((...)\) denotes lexicographic order, and where (when comparing two judgments typing terms of the same size) the synthesis judgment (top line) is considered smaller than the checking judgment (second line). That is,

\[
\begin{align*}
\Rightarrow &\prec \Leftarrow / \gg / \text{Match}
\end{align*}
\]

Two match judgments are compared according to, first, the list of branches \( \Pi \) (which is a subterm of the containing case expression, allowing us to invoke the i.h. for the **Case** rule), then the size of the list of types \( \vec{A} \) (considered to be the sum of the sizes \(|A|\) of each type \( A \) in \( \vec{A} \)), and then, finally, whether the judgment is \( \Gamma/P \vdash \ldots \) or \( \Gamma \vdash \ldots \), considering the former judgment (\( \Gamma/P \vdash \ldots \)) to be larger.

Note that this measure only uses the input parts of the judgments, leading to a straightforward decidability argument.

We will show that in each rule deriving a synthesis, checking, spine or match judgment, every premise is smaller than the conclusion.

• **Case EmptySpine**: No premises.
Proof of (Decidability of Typing)

- **Case** $\to$ Spine: In each premise, the expression/spine gets smaller (we have $e$ in the conclusion, $e$ in the first premise, and $s$ in the second premise).

- **Case** $\text{Var}$: No nontrivial premises.

- **Case** Sub: The first premise has the same subject term $e$ as the conclusion, but the judgment is smaller because our measure considers synthesis to be smaller than checking.
  
  The second premise is a subtyping judgment, which by Theorem 1 is decidable.

- **Case** Anno: It is easy to show that the judgment $\Gamma \vdash A ! \text{type}$ is decidable. The second premise types $e$, but the conclusion types $(e : A)$, so the first part of the measure gets smaller.

- **Cases** $\forall I, \forall^\alpha I$: No premises.

- **Case** $\forall$: Both the premise and conclusion type $e$, and both are checking; however, $\#\text{large}(\forall \alpha : \kappa. A_0) < \#\text{large}(\forall \alpha : \kappa. A_0 \land P)$.

We now consider the match rules:

- **Case** $\text{MatchEmpty}$: No premises.

- **Case** $\text{MatchSeq}$: In each premise, the list of branches is properly contained in $\Pi$, making each premise smaller by the first part ("e/s/\Pi") of the measure.

- **Case** $\text{MatchBase}$: The term $e$ in the premise is properly contained in $\Pi$.

- **Cases** $\text{Match}$ $\times$ $\text{Match}$ $+$ $\text{MatchNeg}$ $\text{MatchWild}$: Smaller by part (2) of the measure.

- **Case** $\text{Match}\land$: The premise has a smaller $\vec{A}$, so it is smaller by the $\vec{A}$ part of the measure. (The premise is the other judgment form, so it is larger by the “match judgment form” part, but $\vec{A}$ lexicographically dominates.)

- **Case** $\text{Match}\bot$: For the premise, use Lemma 78 (Decidability of Propositional Judgments) (4).

- **Case** $\text{Match}\uparrow$: Lemma 78 (Decidability of Propositional Judgments) (4) shows that the first premise is decidable. The second premise has the same (single) branch and list of types, but is smaller by the “match judgment form” part of the measure.

November 13, 2018
I’ Determinacy

Lemma 83 (Determinacy of Auxiliary Judgments).

(1) \text{Elimeq}: \text{Given } \Gamma, \sigma, t, \kappa \text{ such that } \text{FEV}(\sigma) \cup \text{FEV}(t) = \emptyset \text{ and } D_1 \vdash \Gamma \parallel \sigma \equiv t : \kappa \equiv \Delta_1^\parallel, \text{ it is the case that } \Delta_1^\parallel = \Delta_2^\parallel.

(2) \text{Instantiation}: \text{Given } \Gamma, \hat{\alpha}, t, \kappa \text{ such that } \hat{\alpha} \in \text{unsolved}(\Gamma) \text{ and } \Gamma \vdash t : \kappa \text{ and } \hat{\alpha} \notin \text{FV}(t) \text{ and } D_1 \vdash \Gamma \parallel \hat{\alpha} \equiv t : \kappa \equiv \Delta_1 \text{ and } D_2 \vdash \Gamma \parallel \hat{\alpha} \equiv t : \kappa \equiv \Delta_2 \text{ it is the case that } \Delta_1 = \Delta_2.

(3) Symmetric instantiation:

\text{Given } \Gamma, \hat{\alpha}, \hat{\beta}, \kappa \text{ such that } \hat{\alpha}, \hat{\beta} \in \text{unsolved}(\Gamma) \text{ and } \hat{\alpha} \neq \hat{\beta} \text{ and } D_1 \vdash \Gamma \parallel \hat{\alpha} \equiv \hat{\beta} : \kappa \equiv \Delta_1 \text{ and } D_2 \vdash \Gamma \parallel \hat{\beta} : \hat{\alpha} : \kappa \equiv \Delta_2 \text{ it is the case that } \Delta_1 = \Delta_2.

(4) Checkeq: \text{Given } \Gamma, \sigma, t, \kappa \text{ such that } D_1 \vdash \Gamma \parallel \sigma \equiv t : \kappa \equiv \Delta_1 \text{ and } D_2 \vdash \Gamma \parallel \sigma \equiv t : \kappa \equiv \Delta_2 \text{ it is the case that } \Delta_1 = \Delta_2.

(5) Elimproc: \text{Given } \Gamma, P \text{ such that } D_1 \vdash \Gamma \parallel P \equiv \Delta_1^\parallel \text{ and } D_2 \vdash \Gamma \parallel P \equiv \Delta_2^\parallel \text{ it is the case that } \Delta_1 = \Delta_2.

(6) Checkprop: \text{Given } \Gamma, P \text{ such that } D_1 \vdash \Gamma \parallel P \text{ true } \equiv \Delta_1 \text{ and } D_2 \vdash \Gamma \parallel P \text{ true } \equiv \Delta_2, \text{ it is the case that } \Delta_1 = \Delta_2.

Proof.

\textbf{Proof of Part (1) \textbf{(Elimeq)}}.

Rule \textbf{ElimeqZero} applies if and only if \( \sigma = t = \text{zero} \).

Rule \textbf{ElimeqSucc} applies if and only if \( \sigma \text{ and } t \text{ are headed by succ} \).

Now suppose \( \sigma = \alpha \).

\begin{itemize}
  \item Rule \textbf{ElimeqUvarRef} applies if and only if \( t = \alpha \). (Rule \textbf{ElimeqClash} cannot apply; rules \textbf{ElimeqUvarL} and \textbf{ElimeqUvarR} have a free variable condition; rules \textbf{ElimeqUvarR\textsubscript{L}} and \textbf{ElimeqUvarR\textsubscript{R}} have a condition that \( \sigma \neq t \).)
  \end{itemize}

In the remainder, assume \( t \neq \text{alpha} \).

\begin{itemize}
  \item If \( \alpha \in \text{FV}(t) \), then rule \textbf{ElimeqUvarL\textsubscript{L}} applies, and no other rule applies (including \textbf{ElimeqUvarR\textsubscript{L}} and \textbf{ElimeqClash}).
  \end{itemize}

In the remainder, assume \( \alpha \notin \text{FV}(t) \).

\begin{itemize}
  \item Consider whether \textbf{ElimeqUvarR\textsubscript{L}} applies. The conclusion matches if we have \( t = \beta \) for some \( \beta \neq \alpha \) (that is, \( \sigma = \alpha \) and \( t = \beta \)). But \textbf{ElimeqUvarR\textsubscript{L}} has a condition that \( \beta \in \text{FV}(\sigma) \), and \( \sigma = \alpha \), so the condition is not satisfied.
  \end{itemize}

In the symmetric case, use the reasoning above, exchanging L’s and R’s in the rule names.

\textbf{Proof of Part (2) \textbf{(Instantiation)}}.

Rule \textbf{InstBin} applies if and only if \( t \) has the form \( t_1 \oplus t_2 \).

Rule \textbf{InstZero} applies if and only if \( t \) has the form zero.

Rule \textbf{InstSucc} applies if and only if \( t \) has the form succ(t_0).

If \( t \) has the form \( \beta \), then consider whether \( \beta \) is declared to the left of \( \hat{\alpha} \) in the given context:

\begin{itemize}
  \item If \( \hat{\beta} \) is declared to the left of \( \hat{\alpha} \), then rule \textbf{InstReach} cannot be used, which leaves only \textbf{InstSolve}.
  \item If \( \hat{\beta} \) is declared to the right of \( \hat{\alpha} \), then \textbf{InstSolve} cannot be used because \( \hat{\beta} \) is not well-formed under \( \Gamma_0 \) (the context to the left of \( \hat{\alpha} \) in \textbf{InstSolve}). That leaves only \textbf{InstReach}.
  \item \( \hat{\alpha} \) cannot be \( \hat{\beta} \), because it is given that \( \hat{\alpha} \notin \text{FV}(t) = \text{FV}(\hat{\beta}) = \{\hat{\beta}\} \).
  \end{itemize}
Proof of Part (3) (Symmetric instantiation).

Suppose that \( \text{InstSolve} \) concluded \( D_1 \). Then \( \Delta_1 \) is the same as \( \Gamma \) with \( \hat{\alpha} \) solved to \( \hat{\beta} \). Moreover, \( \hat{\beta} \) is declared to the left of \( \hat{\alpha} \) in \( \Gamma \). Thus, \( \text{InstSolve} \) cannot conclude \( D_2 \). However, \( \text{InstReach} \) can conclude \( D_2 \), but produces a context \( \Delta_2 \) which is the same as \( \Gamma \) but with \( \hat{\alpha} \) solved to \( \hat{\beta} \). Therefore \( \Delta_1 = \Delta_2 \).

The other possibility is that \( \text{InstReach} \) concluded \( D_1 \). Then \( \Delta_1 \) is the same as \( \Gamma \) with \( \hat{\beta} \) solved to \( \hat{\alpha} \), with \( \hat{\alpha} \) declared to the left of \( \hat{\beta} \) in \( \Gamma \). Thus, \( \text{InstReach} \) cannot conclude \( D_2 \). However, \( \text{InstSolve} \) can conclude \( D_2 \), producing a context \( \Delta_2 \) which is the same as \( \Gamma \) but with \( \hat{\beta} \) solved to \( \hat{\alpha} \). Therefore \( \Delta_1 = \Delta_2 \).

Proof of Part (4) (Checkeq).

Rule \( \text{CheckeqVar} \) applies if and only if \( \sigma = t = \hat{\alpha} \) or \( \sigma = t = \alpha \) (note the free variable conditions in \( \text{CheckeqInstL} \) and \( \text{CheckeqInstR} \)).

Rule \( \text{CheckeqUnit} \) applies if and only if \( \sigma = t = 1 \).

Rule \( \text{CheckeqBin} \) applies if and only if \( \sigma \) and \( t \) are both headed by the same binary connective.

Rule \( \text{CheckeqZero} \) applies if and only if \( \sigma = t = 0 \).

Rule \( \text{CheckeqSucc} \) applies if and only if \( \sigma \) and \( t \) are headed by \( \text{succ} \).

Now suppose \( \sigma = \hat{\alpha} \). If \( t \) is not an existential variable, then \( \text{CheckeqInstr} \) cannot be used, which leaves only \( \text{CheckeqInstL} \). If \( t \) is an existential variable, that is, some \( \hat{\beta} \) (distinct from \( \hat{\alpha} \)), and is unsolved, then both \( \text{CheckeqInstL} \) and \( \text{CheckeqInstR} \) apply, but by part (3), we get the same output context from each.

The \( t = \alpha \) subcase is similar.

Proof of Part (5) (Elimprop). There is only one rule deriving this judgment; the result follows by part (1).

Proof of Part (6) (Checkprop). There is only one rule deriving this judgment; the result follows by part (4).

Lemma 84 (Determinacy of Equivalence).

(1) Propositional equivalence: Given \( \Gamma, P, Q \) such that \( D_1 :: \Gamma \vdash P \equiv Q \vdash \Delta_1 \) and \( D_2 :: \Gamma \vdash P \equiv Q \vdash \Delta_2 \), it is the case that \( \Delta_1 = \Delta_2 \).

(2) Type equivalence: Given \( \Gamma, A, B \) such that \( D_1 :: \Gamma \vdash A \equiv B \vdash \Delta_1 \) and \( D_2 :: \Gamma \vdash A \equiv B \vdash \Delta_2 \), it is the case that \( \Delta_1 = \Delta_2 \).

Proof.

Proof of Part (1) (propositional equivalence). Only one rule derives judgments of this form; the result follows from Lemma 83 (Determinacy of Auxiliary Judgments) (4).

Proof of Part (2) (type equivalence). If neither \( A \) nor \( B \) is an existential variable, they must have the same head connectives, and the same rule must conclude both derivations.

If \( A \) and \( B \) are the same existential variable, then only \( \text{Exvar} \) applies (due to the free variable conditions in \( \text{InstantiateL} \) and \( \text{InstantiateR} \)).

If \( A \) and \( B \) are different unsolved existential variables, the judgment matches the conclusion of both \( \text{InstantiateL} \) and \( \text{InstantiateR} \) but by part (3) of Lemma 83 (Determinacy of Auxiliary Judgments), we get the same output context regardless of which rule we choose.

Theorem 4 (Determinacy of Subtyping).

(1) Subtyping: Given \( \Gamma, e, A, B \) such that \( D_1 :: \Gamma \vdash A <: P B \vdash \Delta_1 \) and \( D_2 :: \Gamma \vdash A <: P B \vdash \Delta_2 \), it is the case that \( \Delta_1 = \Delta_2 \).

Proof. First, we consider whether we are looking at positive or negative subtyping, and then consider the outermost connective of \( A \) and \( B \):
Proof of Theorem 4 (Determinacy of Subtyping)

(1) Checking: Given $\Gamma$, $e$, $A$, $p$ such that $D_1 :: \Gamma \vdash e \equiv A \vdash \Delta_1$ and $D_2 :: \Gamma \vdash e \equiv A \vdash \Delta_2$, it is the case that $\Delta_1 = \Delta_2$.

(2) Synthesis: Given $\Gamma$, $e$ such that $D_1 :: \Gamma \vdash e \Rightarrow B_1 \vdash \Delta_1$ and $D_2 :: \Gamma \vdash e \Rightarrow B_2 \vdash \Delta_2$, it is the case that $B_1 = B_2$ and $p_1 = p_2$ and $\Delta_1 = \Delta_2$.

(3) Spine judgments:

Given $\Gamma$, $e$, $A$, $p$ such that $D_1 :: \Gamma \vdash e : A \vdash C_1 \vdash \Delta_1$ and $D_2 :: \Gamma \vdash e : A \vdash C_2 \vdash \Delta_2$, it is the case that $C_1 = C_2$ and $q_1 = q_2$ and $\Delta_1 = \Delta_2$.

The same applies for derivations of the principality-recovering judgments $\Gamma \vdash e : A \vdash C_k [q_k] \vdash \Delta_k$.

(4) Match judgments:

Given $\Gamma$, $\Pi$, $A$, $p$, $C$ such that $D_1 :: \Gamma \vdash \Pi :: A \vdash C \vdash \Delta_1$ and $D_2 :: \Gamma \vdash \Pi :: A \vdash C \vdash \Delta_2$, it is the case that $\Delta_1 = \Delta_2$.

Given $\Gamma$, $\Pi$, $A$, $p$, $C$ such that $D_1 :: \Gamma \vdash \Pi :: A \vdash C \vdash \Delta_1$ and $D_2 :: \Gamma \vdash \Pi :: A \vdash C \vdash \Delta_2$, it is the case that $\Delta_1 = \Delta_2$.

Proof.

Proof of Part (1) (checking).

The rules with a checking judgment in the conclusion are: $\text{Nil}$, $\text{Cons}$, $\text{Rec}$, $\text{Case}$. The table below shows which rules apply for given $e$ and $A$. The extra “chk-?” column highlights the role of the “chk-?” (“check-intro”) category of syntactic forms: we restrict the introduction rules for $\forall$ and $\exists$ to...
Proof of Theorem 5 (Determinacy of Typing)

Proof of Part (2) (synthesis). Only four rules have a synthesis judgment in the conclusion: \[\text{Var} \quad \text{Anno} \quad \text{Rec} \quad \text{Case}\] and \[\text{Rec}\] applies if and only if \(e\) has the form \(x\). Rule \[\text{Anno}\] applies if and only if \(e\) has the form \((e_0 : A)\).

Otherwise, the judgment can be derived only if \(e\) has the form \(e_1 \ e_2\), by \[\text{Rec}\]

Proof of Part (3) (spine judgments). For the ordinary spine judgment, rule \[\text{EmptySpine}\] applies if and only if the given spine is empty. Otherwise, the choice of rule is determined by the head constructor of the input type: \[\text{Spine}\] applies if and only if \(e\) has the form \((e_0 : A)\).

For the principality-recovering spine judgment: If \(p = \mathit{\text{true}}\), only rule \[\text{SpinePass}\] applies. If \(p = \mathit{\text{false}}\) and \(q = \mathit{\text{true}}\), only rule \[\text{SpinePass}\] applies. If \(p = \mathit{\text{false}}\) and \(q = \mathit{\text{false}}\), then the rule is determined by \[\text{FEV}(e)\]: if \[\text{FEV}(e) = \emptyset\] then only \[\text{SpineRec}\] applies; otherwise, \[\text{FEV}(e) \neq \emptyset\] and only \[\text{SpinePass}\] applies.

Proof of Part (4) (matching). First, the elimination judgment form \(\Gamma \vdash \Pi : \ldots\): It cannot be the case that both \(\Gamma \vdash \sigma \equiv t : \kappa \vdash \bot\) and \(\Gamma \vdash \sigma \equiv t : \kappa \vdash \Theta\), so either \[\text{Match}\] concludes both \(D_1\) and \(D_2\) (and the result follows), or \[\text{MatchEmpty}\] concludes both \(D_1\) and \(D_2\) (in which case, apply the i.h.).

Now the main judgment form, without \(\vdash\) \(\Pi\): either \(\Pi\) is empty, or has length one, or has length greater than one. \[\text{MatchEmpty}\] applies if and only if \(\Pi\) is empty, and \[\text{MatchSeq}\] applies if and only if \(\Pi\) has length greater than one. So in the rest of this part, we assume \(\Pi\) has length one.

Moreover, \[\text{MatchBase}\] applies if and only if \(\bar{A}\) has length zero. So in the rest of this part, we assume the length of \(\bar{A}\) is at least one.
Let $A$ be the first type in $\vec{A}$. Inspection of the rules shows that given particular $A$ and $\rho$, where $\rho$ is the first pattern, only a single rule can apply, or no rule ("∅") can apply, as shown in the following table:

<table>
<thead>
<tr>
<th>$A$</th>
<th>$\exists$</th>
<th>$\wedge$</th>
<th>$+$</th>
<th>$\times$</th>
<th>Vec</th>
<th>other</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{inj}_1 \rho_0$</td>
<td>Match$\exists$</td>
<td>Match$\wedge$</td>
<td>Match$+$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\rho \langle \rho_1, \rho_2 \rangle$</td>
<td>Match$\exists$</td>
<td>Match$\wedge$</td>
<td>$\emptyset$</td>
<td>Match$\times$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$z$</td>
<td>Match$\exists$</td>
<td>MatchNeg</td>
<td>MatchNeg</td>
<td>MatchNeg</td>
<td>MatchNeg</td>
<td>MatchNeg</td>
</tr>
<tr>
<td>$[]$</td>
<td>Match$\exists$</td>
<td>MatchWild</td>
<td>MatchWild</td>
<td>MatchWild</td>
<td>MatchWild</td>
<td>MatchWild</td>
</tr>
<tr>
<td>$\rho_1 :: \rho_2$</td>
<td>Match$\exists$</td>
<td>Match$\wedge$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>MatchCons</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

\[\square\]

### J’ Soundness

#### J’.1 Instantiation

**Lemma 85 (Soundness of Instantiation).**

If $\Gamma \vdash \hat{\alpha} := \tau : k \vdash \Delta$ and $\hat{\alpha} \notin \text{FV}(\Gamma)\tau$ and $\Gamma\tau = \tau$ and $\Delta \rightarrow \Omega$ then $[\Omega]\hat{\alpha} = [\Omega]\tau$.

**Proof.** By induction on the derivation of $\Gamma \vdash \hat{\alpha} := \tau : k \vdash \Delta$.

- **Case** $\Gamma_0 \vdash \tau : k$
  \[
  \frac{}{\Gamma_0, \hat{\alpha} : k, \Gamma_1 \vdash \hat{\alpha} := \tau : k \vdash \Gamma_0, \hat{\alpha} : k = \tau, \Gamma_1} \text{InstSolve}
  \]
  
  \[
  [\Delta]\hat{\alpha} = [\Delta]\tau \quad \text{By definition}
  \]
  \[
  \equiv [\Omega]\hat{\alpha} = [\Omega]\tau \quad \text{By Lemma 29 (Substitution Monotonicity) to each side}
  \]

- **Case** $\hat{\beta} \in \text{unsolved}(\Gamma[\hat{\alpha} : k]|\hat{\beta} : k)$
  \[
  \frac{}{\Gamma[\hat{\alpha} : k]|\hat{\beta} : k] \vdash \hat{\alpha} := \hat{\beta} : k \vdash \Gamma[\hat{\alpha} : k]|\hat{\beta} = \hat{\alpha}} \text{InstReach}
  \]
  
  \[
  [\Delta]\hat{\beta} = [\Delta]\hat{\alpha} \quad \text{By definition}
  \]
  \[
  [\Omega][\Delta]\hat{\beta} = [\Omega][\Delta]\hat{\alpha} \quad \text{Applying } \Omega \text{ to each side}
  \]
  \[
  \equiv [\Omega][\Delta]\hat{\beta} = [\Omega][\Delta]\hat{\alpha} \quad \text{By Lemma 29 (Substitution Monotonicity) to each side}
  \]

- **Case** $\Gamma'$
  \[
  \frac{}{\Gamma_0[\hat{\alpha}_2 : *, \hat{\alpha}_1 : *, \hat{\alpha} : * = \hat{\alpha}_1 \oplus \hat{\alpha}_2] \vdash \hat{\alpha}_1 := \tau_1 \vdash \Theta \vdash \hat{\alpha}_2 := \Theta[\tau_2 : * \vdash \Delta]} \text{InstBin}
  \]
  
  \[
  \Gamma_0[\hat{\alpha} : *] \vdash \hat{\alpha} := \tau_1 \oplus \tau_2 : * \vdash \Delta
  \]
Proof of Lemma 85 \( \text{(Soundness of Instantiation)} \)

\[ \Delta \longrightarrow \Omega \]

Given

\[ \Gamma' \vdash \hat{\alpha}_1 := \tau_1 \vdash \Theta \]

Subderivation

\[ \Theta \longrightarrow \Delta \]

By Lemma 43 \( \text{(Instantiation Extension)} \)

\[ \Theta \longrightarrow \Omega \]

By Lemma 33 \( \text{(Extension Transitivity)} \)

\[ [\Omega] \hat{\alpha}_1 = [\Omega] \tau_1 \]

By i.h.

\[ \Theta \vdash \hat{\alpha}_2 := [\Theta] \tau_2 \vdash \Delta \]

Subderivation

\[ [\Omega] \hat{\alpha}_2 = [\Omega] [\Theta] \tau_2 \]

By i.h.

\[ = [\Omega] \tau_2 \]

By Lemma 29 \( \text{(Substitution Monotonicity)} \)

\[ (\{\tau_1\} + (\{\tau_2\}) = \{\tau_1\} \oplus \{\tau_2\} \]

By above equalities

\[ = [\Omega] (\{\tau_1\} \oplus \{\tau_2\}) \]

By definition of substitution

\[ = \Theta \]

By definition of substitution

\[ \Rightarrow \]

\[ [\Omega] (\tau_1 \oplus \tau_2) = [\Omega] \hat{\alpha} \]

By definition of substitution

\[ \cdot \] 

Case

\[ \Gamma_0[\hat{\alpha} : N] \vdash \hat{\alpha} := \text{zero} : N \vdash \Gamma_0[\hat{\alpha} : N = \text{zero}] \]

Similar to the \( \text{InstZero} \) case.

\[ \cdot \] 

Case

\[ \Gamma_0[\hat{\alpha}_1 : N, \hat{\alpha} : N = \text{succ}(\hat{\alpha}_1)] \vdash \hat{\alpha}_1 := t_1 : N \vdash \Delta \]

\[ \Gamma_0[\hat{\alpha} : N] \vdash \hat{\alpha} := \text{succ}(t_1) : N \vdash \Delta \]

\( \text{InstSucc} \)

Similar to the \( \text{InstBin} \) case, but simpler.

\[ \square \]

Lemma 86 \( \text{(Soundness of Checkeq)} \).

If \( \Gamma \vdash \sigma \triangleq t : \kappa \vdash \Delta \) then \( [\Omega] \sigma = [\Omega] t \).

Proof. By induction on the given derivation.

\[ \cdot \] 

Case \( \text{CheckeqVar} \)

\[ \Gamma \vdash u \equiv u : \kappa \vdash \Gamma \]

\[ \Rightarrow \]

\[ [\Omega] u = [\Omega] u \]

By reflexivity of equality

\[ \cdot \] 

Cases \( \text{CheckeqUnit}, \text{CheckeqZero} \) Similar to the \( \text{CheckeqVar} \) case.

\[ \cdot \] 

Case \( \text{CheckeqSucc} \)

\[ \Gamma \vdash \sigma_0 \equiv t_0 : N \vdash \Delta \]

\[ \Gamma \vdash \text{succ}(\sigma_0) \equiv \text{succ}(t_0) : N \vdash \Delta \]

\[ \Rightarrow \]

\[ \Gamma \vdash \sigma_0 \equiv t_0 : N \vdash \Delta \]

Subderivation

\[ [\Omega] \sigma_0 = [\Omega] t_0 \]

By i.h.

\[ \text{succ}([\Omega] \sigma_0) = \text{succ}([\Omega] t_0) \]

By congruence

\[ \Rightarrow \]

\[ [\Omega] (\text{succ}(\sigma_0)) = [\Omega] (\text{succ}(t_0)) \]

By definition of substitution
Proof of Lemma 86 (Soundness of Checkeq)

**Case**

\[ \Gamma \vdash \sigma_0 \triangleq t_0 : \ast \vdash \Theta \quad \Theta \vdash [\Theta] \sigma_1 \triangleq [\Theta] t_1 : \ast \vdash \Delta \]

\[ \Gamma \vdash \sigma_0 \oplus \sigma_1 \triangleq t_0 \oplus t_1 : \ast \vdash \Delta \]

**CheckeqBin**

- Subderivation
- Given
- By Lemma 46 (Checkeq Extension)
- By Lemma 33 (Extension Transitivity)

\[ \Gamma \vdash \sigma_0 \triangleq t_0 : \mathbb{N} \vdash \Delta \quad [\Theta] \sigma_1 \triangleq [\Theta] t_1 : \ast \vdash \Delta \]

**Subderivation**

\[ \Delta \rightarrow \Omega \quad \Theta \rightarrow \Delta \]

By Lemma 46 (Checkeq Extension) (on 2nd premise)

By Lemma 33 (Extension Transitivity)

\[ [\Omega] \sigma_0 = [\Omega] t_0 \quad [\Omega] \sigma_1 = [\Omega] t_1 \]

By i.h. on first subderivation

By i.h. on second subderivation

**CheckeqInstL**

\[ \Gamma[\Delta] \vdash \hat{\alpha} := t : \kappa \vdash \Delta \]

\[ \Gamma[\Delta] \vdash \hat{\alpha} \triangleq t : \kappa \vdash \Delta \]

\[ \hat{\alpha} \notin FV(t) \]

Premise

**CheckeqInstR**

\[ \Gamma[\Delta] \vdash \hat{\alpha} := \sigma : \kappa \vdash \Delta \]

\[ \Gamma[\Delta] \vdash \sigma \triangleq \hat{\alpha} : \kappa \vdash \Delta \]

\[ \hat{\alpha} \notin FV(t) \]

Premise

By Lemma 86 (Soundness of Checkeq)

Similar to the **CheckeqInstL** case.

Lemma 87 (Soundness of Propositional Equivalence).

*If \( \Gamma \vdash P \equiv Q \vdash \Delta \) where \( \Delta \rightarrow \Omega \) then \( [\Omega]P = [\Omega]Q \).*

**Proof.** By induction on the given derivation.

**Case**

\[ \Gamma \vdash \sigma_1 \triangleq t_1 : \mathbb{N} \vdash \Theta \quad \Theta \vdash [\Theta] \sigma_2 \triangleq [\Theta] t_2 : \mathbb{N} \vdash \Delta \]

\[ \Gamma \vdash (\sigma_1 \circ \sigma_2) \equiv (t_1 \circ t_2) : \vdash \Delta \]

**PropEq**

\[ \Delta \rightarrow \Omega \quad \Theta \rightarrow \Delta \]

Given

By Lemma 46 (Checkeq Extension) (on 2nd premise)

By Lemma 33 (Extension Transitivity)

\[ [\Omega] \sigma_1 = [\Omega] t_1 \quad [\Omega] \sigma_2 = [\Omega] t_2 \]

By Lemma 86 (Soundness of Checkeq)

By Lemma 29 (Substitution Monotonicity)

By Lemma 29 (Substitution Monotonicity)

\[ ([\Theta] \sigma_1 = [\Theta] t_1) \quad ([\Theta] \sigma_2 = [\Theta] t_2) \]

By congruence of equality

\[ ([\Omega] \sigma_1 = [\Omega] t_1) \quad ([\Omega] \sigma_2 = [\Omega] t_2) \]

By congruence of equality

\[ [\Omega] (\sigma_1 = \sigma_2) = [\Omega] (t_1 = t_2) \]

By definition of substitution

\[ [\Omega] (\sigma_1 = \sigma_2) \]

By definition of substitution

\[ [\Omega] (t_1 = t_2) \]

By definition of substitution

\[ [\Omega] (\sigma_1 = \sigma_2) \]

By definition of substitution

\[ [\Omega] (t_1 = t_2) \]

By definition of substitution
Lemma 88 (Soundness of Algorithmic Equivalence).

If $\Gamma \vdash A \equiv B \vdash \Delta$ where $\Delta \rightarrow \Omega$ then $[\Omega]A = [\Omega]B$.

Proof. By induction on the given derivation.

- Case
  \[ \Gamma \vdash \alpha \equiv \alpha \vdash \Gamma \] 
  $\equiv \text{Var}$
  \[ [\Omega]\alpha = [\Omega]\alpha \] By reflexivity of equality

- Cases $\equiv \text{Exvar} \equiv \text{Unit}$ Similar to the $\equiv \text{Var}$ case.

- Case
  \[ \Gamma \vdash A_1 \equiv B_1 \vdash \Theta \quad \Theta \vdash \Theta A_2 \equiv \Theta B_2 \vdash \Delta \] 
  $\equiv \Gamma \vdash A_1 \oplus A_2 \equiv B_1 \oplus B_2 \vdash \Delta$
  $\Delta \rightarrow \Omega$ Given
  $\Theta \vdash \Theta A_2 \equiv \Theta B_2 \vdash \Delta$ Subderivation
  $\Theta \rightarrow \Delta$ By Lemma 49 (Equivalence Extension)
  $\Theta \rightarrow \Omega$ By Lemma 33 (Extension Transitivity)
  $\Gamma \vdash A_1 \equiv B_1 \vdash \Theta$ Subderivation
  $[\Omega]A_1 = [\Omega]B_1$ By i.h.
  $\Delta \rightarrow \Omega$ Given
  $[\Omega]|\Theta A_2 = [\Omega]|\Theta B_2$ By i.h.
  $[\Omega]|A_2 = [\Omega]|B_2$ By Lemma 29 (Substitution Monotonicity)
  $\Rightarrow ([\Omega]|A_1) \oplus ([\Omega]|A_2) = ([\Omega]|B_1) \oplus ([\Omega]|B_2)$ By above equations

- Case $\Gamma, \alpha : \kappa \vdash A_0 \equiv B_0 \vdash \Delta, \alpha : \kappa, \Delta'$
  $\Gamma \vdash \forall \alpha : \kappa. A_0 \equiv \forall \alpha : \kappa. B_0 \vdash \Delta$
  $\Gamma, \alpha : \kappa \vdash A_0 \equiv B_0 \vdash \Delta, \alpha : \kappa, \Delta'$ Subderivation
  $\Delta \rightarrow \Omega$ Given
  $\Gamma, \alpha : \kappa, \cdot \rightarrow \Delta, \alpha : \kappa, \Delta'$ By Lemma 49 (Equivalence Extension)
Proof of Lemma 88 (Soundness of Algorithmic Equivalence)

\[ \Delta' \text{ soft} \]
\[ \Delta, \alpha: \kappa, \Delta' \rightarrow \Omega, \alpha: \kappa, \Omega_Z \]

By Lemma 24 (Soft Extension)

\[ \Gamma, \alpha: \kappa \vdash A_0 \text{ type} \]

By validity on subderivation

\[ \Gamma, \alpha: \kappa \vdash B_0 \text{ type} \]

By well-typing of \( A_0 \)

\[ FV(A_0) \subseteq \text{dom}(\Gamma, \alpha: \kappa) \]

By well-typing of \( B_0 \)

\[ FV(B_0) \subseteq \text{dom}(\Gamma, \alpha: \kappa) \]

By well-typing of \( B_0 \)

\[ \Gamma, \alpha: \kappa \rightarrow \Omega, \alpha: \kappa \]

By \( \rightarrow_{\text{Uvar}} \)

\[ FV(A_0) \subseteq \text{dom}(\Omega, \alpha: \kappa) \]

By Lemma 20 (Declaration Order Preservation)

\[ FV(B_0) \subseteq \text{dom}(\Omega, \alpha: \kappa) \]

By Lemma 20 (Declaration Order Preservation)

\[ [\Omega, \alpha: \kappa, \Omega_Z]A_0 = [\Omega, \alpha: \kappa]A_0 \]

By definition of substitution, since \( FV(A_0) \cap \text{dom}(\Omega_Z) = \emptyset \)

\[ [\Omega, \alpha: \kappa, \Omega_Z]B_0 = [\Omega, \alpha: \kappa]B_0 \]

By definition of substitution, since \( FV(B_0) \cap \text{dom}(\Omega_Z) = \emptyset \)

\[ [\Omega, \alpha: \kappa]A_0 = [\Omega, \alpha: \kappa]B_0 \]

By transitivity of equality

\[ [\Omega]A_0 = [\Omega]B_0 \]

From definition of substitution

\[ \forall \alpha: \kappa. [\Omega]A_0 = [\Omega]B_0 \]

Adding quantifier to each side

\[ [\Omega]([\forall \alpha: \kappa]A_0) = [\Omega]([\forall \alpha: \kappa]B_0) \]

By definition of substitution

---

**Case**

\[ \Gamma \vdash P \equiv Q \land \Theta \]

\[ \Theta \vdash [\Theta]A_0 \equiv [\Theta]B_0 \rightarrow \Delta \]

\[ \Gamma \vdash P \land A_0 \equiv Q \rightarrow B_0 \rightarrow \Delta \]

\[ \Delta \rightarrow \Omega \]

Given

\[ \Theta \vdash [\Theta]A_0 \equiv [\Theta]B_0 \rightarrow \Delta \]

Subderivation

\[ \Theta \rightarrow \Delta \]

By Lemma 49 (Equivalence Extension)

\[ \Theta \rightarrow \Omega \]

By Lemma 33 (Extension Transitivity)

\[ \Gamma \vdash P \equiv Q \land \Theta \]

Subderivation

\[ [\Omega]P = [\Omega]Q \]

By Lemma 87 (Soundness of Propositional Equivalence)

\[ \Theta \vdash [\Theta]A_0 \equiv [\Theta]B_0 \rightarrow \Delta \]

Subderivation

\[ [\Omega][\Theta]A_0 = [\Omega][\Theta]B_0 \]

By i.h.

\[ [\Omega]A_0 = [\Omega]B_0 \]

By Lemma 29 (Substitution Monotonicity)

---

**Case**

\[ \Gamma \vdash P \equiv Q \land \Theta \]

\[ \Theta \vdash [\Theta]A_0 \equiv [\Theta]B_0 \rightarrow \Delta \]

\[ \Gamma \vdash A_0 \land P \equiv B_0 \land Q \rightarrow \Delta \]

Similar to the \( \equiv \) case.

---

**Case**

\[ \Gamma[\hat{\alpha}] \vdash \hat{\alpha} := \tau : \star \rightarrow \Delta \]

\[ \hat{\alpha} \notin FV(\tau) \]

\[ \Gamma[\hat{\alpha}] \vdash \hat{\alpha} \equiv \frac{\tau}{\alpha} \rightarrow \Delta \]

\[ \Gamma[\hat{\alpha}] \vdash \hat{\alpha} := \tau : \star \rightarrow \Delta \]

Subderivation

\[ [\Omega]\hat{\alpha} = [\Omega]\tau \]

By Lemma 85 (Soundness of Instantiation)

---

**Case**

Similar to the \( \equiv \text{Instantiate} \) case.

---

**J.2 Soundness of Checkprop**

Lemma 89 (Soundness of Checkprop).

If \( \Gamma \vdash P \text{ true} \rightarrow \Delta \) and \( \Delta \rightarrow \Omega \) then \( \Psi \vdash [\Omega]P \text{ true} \).
Proof of Lemma 89 (Soundness of Checkprop)

\[ \text{lem:checkprop-soundness} \]

Proof. By induction on the derivation of \( \Gamma \vdash P \text{ true} \vdash \Delta \).

- **Case** \( \Gamma \vdash \sigma \doteq t : N \vdash \Delta \)

  \[ \Gamma \vdash \sigma \doteq t \text{ true} \vdash \Delta \]

  \[ \text{CheckpropEq} \]

  \( \Gamma \vdash \sigma \doteq t : N \vdash \Delta \) \quad \text{Subderivation}

  \[ [\Omega]\sigma = [\Omega]t \]

  \[ \text{By Lemma 86 (Soundness of Checkeq)} \]

  \[ \Psi \vdash [\Omega](\sigma = t) \text{ true} \]

  \[ \text{By def. of subst.} \]

\[ \square \]

J'.3 Soundness of Eliminations (Equality and Proposition)

Lemma 90 (Soundness of Equality Elimination).

If \([\Gamma]\sigma = \sigma \) and \([\Gamma]t = t \) and \( \Gamma \vdash \sigma : \kappa \) and \( \Gamma \vdash t : \kappa \) and \( \text{FEV}(\sigma) \cup \text{FEV}(t) = \emptyset \), then:

1. If \( \Gamma / \sigma \doteq t : \kappa \vdash \Delta \)
   
   then \( \Delta = (\Gamma, \Theta) \) where \( \Theta = (\alpha_1 = t_1, \ldots, \alpha_n = t_n) \) and
   
   for all \( \Omega \) such that \( \Gamma \rightarrow \rightarrow \Omega \)
   
   and all \( t' \) such that \( \Omega \vdash t' : \kappa' \),
   
   it is the case that \( [\Omega](\Theta)\vdash t' : \kappa'\) where \( \theta = \text{mgu}(\sigma, t) \).

2. If \( \Gamma / \sigma \doteq \kappa \vdash \perp \) then \( \text{mgu}(\sigma, t) = \perp \) (that is, no most general unifier exists).

Proof. First, we need to recall a few properties of term unification.

(i) If \( \alpha \) is a term, then \( \text{mgu}(\alpha, \alpha) = id \).

(ii) If \( f \) is a unary constructor, then \( \text{mgu}(f(\sigma), f(t)) = \text{mgu}(\sigma, t) \), supposing that \( \text{mgu}(\sigma, t) \) exists.

(iii) If \( f \) is a binary constructor, and \( \sigma = \text{mgu}(f(\sigma_1, \sigma_2), f(t_1, t_2)) \) and \( \sigma_1 = \text{mgu}(\sigma_1, t_1) \)

and \( \sigma_2 = \text{mgu}(\sigma_1, [\sigma_1\sigma_2]) \), then \( \sigma = \sigma_2 \circ \sigma_1 = \sigma_1 \circ \sigma_2 \).

(iv) If \( \alpha \notin \text{FV}(t) \), then \( \text{mgu}(\alpha, t) = (\alpha = t) \).

(v) If \( f \) is an \( n \)-ary constructor, and \( \sigma_i \) and \( t_i \) (for \( i \leq n \)) have no unifier, then \( f(\sigma_1, \ldots, \sigma_n) \) and \( f(t_1, \ldots, t_n) \) have no unifier.

We proceed by induction on the derivation of \( \Gamma / \sigma \doteq t : \kappa \vdash \Delta \), proving both parts with a single induction.

- **Case** \( \Gamma / \alpha \doteq \alpha : \kappa \vdash \Gamma \)

  \[ \text{ElimeqUvarRefl} \]

  Here we have \( \Delta = \Gamma \), so we are in part (1).

  Let \( \theta = id \) (which is \( \text{mgu}(\sigma, \sigma) \)).

  We can easily show \( [id][\Omega](\alpha = \alpha) = [\Omega, \alpha] = [\Omega, \cdot] \).

- **Case** \( \Gamma / \text{zero} \doteq \text{zero} : N \vdash \Gamma \)

  \[ \text{ElimeqZero} \]

  Similar to the \[ \text{ElimeqUvarRefl} \] case.
Proof of Lemma 90 (Soundness of Equality Elimination) \lem:elimeq-soundness

• Case \( \Gamma / t_1 \equiv t_2 : \mathbb{N} \vdash \Delta^⊥ \)

\[ \Gamma / \text{succ}(t_1) \equiv \text{succ}(t_2) : \mathbb{N} \vdash \Delta^⊥ \] \ElimeqSucc

We distinguish two subcases:

– Case \( \Delta^⊥ = \Delta \):

Since we have the same output context in the conclusion and premise, the “for all \( t′ \)…” part follows immediately from the i.h. (1).

The i.h. also gives us \( \theta = \text{mgu}(t_1, t_2) \).

Let \( \theta = \theta_0 \). By property (ii), \( \text{mgu}(t_1, t_2) = \text{mgu}((\text{succ}(t_1), \text{succ}(t_2)) = \theta \).

– Case \( \Delta^⊥ = \bot \):

\[ \Gamma / t_1 \equiv t_2 : \mathbb{N} \vdash \bot \] Subderivation

\( \text{mgu}(t_1, t_2) = \bot \) By i.h. (2)

\( \text{mgu}(\text{succ}(t_1), \text{succ}(t_2)) = \bot \) By contrapositive of property (ii)

• Case \( \alpha \notin \text{FV}(t) \) \( (\alpha = -) \notin \Gamma \)

\[ \Gamma / \alpha \equiv t : \kappa \vdash \Gamma, \alpha = t \] \ElimeqUvarL

Here \( \Delta \neq \bot \), so we are in part (1).

\[ [\Omega, \alpha = t]t' = [\Omega][t/\alpha][\Omega]t' \] By a property of substitution

\[ = [\Omega][t/\alpha]t' \] By a property of substitution

\[ = [\Omega][\theta][\Omega]t' \] By \( \text{mgu}(\alpha, t) = (\alpha/t) \)

\( \text{mgu}(\alpha, t) = (\alpha/t) \) By a property of substitution (\( \theta \) creates no evars)

• Case \( \alpha \notin \text{FV}(t) \) \( (\alpha = -) \notin \Gamma \)

\[ \Gamma / t \equiv \alpha : \kappa \vdash \Gamma, \alpha = t \] \ElimeqUvarR

Similar to the \ElimeqUvarL case.

• Case \[ \Gamma / \tau_1 \equiv \tau_1' : * \vdash \Theta \]

\[ \Theta / [\Theta] \tau_1 \equiv [\Theta] \tau_2' : * \vdash \Delta^⊥ \] \ElimeqUnit

Similar to the \ElimeqUvarRefl case.

• Case \[ \Gamma / \tau_1 \equiv \tau_1' : * \vdash \Theta \]

\[ \Theta / [\Theta] \tau_1 \equiv [\Theta] \tau_2' : * \vdash \Delta^⊥ \] \ElimeqBin

Either \( \Delta^⊥ \) is some \( \Delta \), or it is \( \bot \).

– Case \( \Delta^⊥ = \Delta \):


Proof of Lemma 90 (Soundness of Equality Elimination)

\[ \Gamma \vdash \tau_1 \odot \tau'_1 : \star \vdash \Theta \]

Subderivation

By i.h. (1)

(IH-1st) \[ [\Omega, \Delta_1]u_1 = \theta_1[\Omega]u_1 \]

\[ \theta_1 = \text{mgu}(\tau_1, \tau'_1) \]

" for all \( \Omega \vdash u_1 : \kappa' \)

\[ \Theta \vdash [\Omega] \tau_1 \odot [\Theta] \tau'_1 : \star \vdash \Delta \]

Subderivation

By i.h. (1)

(IH-2nd) \[ [\Omega, \Delta_1, \Delta_2]u_2 = [\theta_2][\Omega, \Delta_1]u_2 \]

\[ \theta_2 = \text{mgu}(\tau_2, \tau'_2) \]

" for all \( \Omega \vdash u_2 : \kappa' \)

Suppose \( \Omega \vdash u : \kappa' \).

\[ [\Omega, \Delta_1, \Delta_2]u = [\theta_2][\Omega, \Delta_1]u \]

By (IH-2nd), with \( u_2 = u \)

\[ = [\theta_2][\theta_1][\Omega]u \]

By (IH-1st), with \( u_1 = u \)

\[ = [\Omega][\theta_2 \circ \theta_1]u \]

By a property of substitution

\[ \theta_2 \circ \theta_1 = \text{mgu}((\tau_1 \odot \tau_2), (\tau'_1 \odot \tau'_2)) \]

By property (iii) of substitution

– Case \( \Delta = \bot \):

Use the i.h. (2) on the second premise to show \( \text{mgu}(\tau_2, \tau'_2) = \bot \), then use property (v) of unification to show \( \text{mgu}((\tau_1 \odot \tau_2), (\tau'_1 \odot \tau'_2)) = \bot \).

- Case

\[ \Gamma \vdash \tau_1 \odot \tau'_1 : \star \vdash \bot \]

\[ \Gamma \vdash \tau_1 \odot \tau_2 \odot \tau'_2 \odot \tau'_1 : \star \vdash \bot \] \hspace{1cm} \text{ElimeqBinBot}

Similar to the \( \bot \) subcase for \text{ElimeqSucc}, but using property (v) instead of property (ii).

- Case

\[ \sigma \not\equiv t \]

\[ \Gamma \vdash \sigma \equiv t : \kappa \vdash \bot \] \hspace{1cm} \text{ElimeqClash}

Since \( \sigma \not\equiv t \), we know \( \sigma \) and \( t \) have different head constructors, and thus no unifier. \( \square \)
### Theorem 6 (Soundness of Algorithmic Subtyping)

If $[\Gamma]A = A$ and $[\Gamma]B = B$ and $\Gamma \vdash A$ type and $\Gamma \vdash B$ type and $\Delta \rightarrow \Omega$ and $\Gamma \vdash A \triangleleft B$ then $[\Omega]A \triangleleft [\Omega]B$.

**Proof.** By induction on the given derivation.

1. **Case** $B$ not headed by $\forall$

   $\Gamma, \triangleright \alpha, \Theta \vdash [\hat{\alpha}/\alpha]A_0 \triangleleft B \triangleleft \Delta, \triangleright \alpha, \Theta \triangleleft_{\triangleright \alpha}$

   Let $\Omega' = (\Omega, \triangleright \alpha, \Theta)$.

   $\Gamma, \triangleright \alpha, \Theta \vdash k$ $\triangleright \hat{\alpha} : k$

   $\Delta \rightarrow \Omega$

   $\Omega \vdash B$ type

   $[\Omega'](\Delta, \triangleright \alpha, \Theta) \vdash [\hat{\alpha}/\alpha]A_0 \leq B$

   By i.h.

   $\Omega \vdash B$ type

   $[\Omega'](\Delta, \triangleright \alpha, \Theta) \vdash [\hat{\alpha}/\alpha]A_0 \leq \Omega$

   By above equality

   $\Gamma, \triangleright \alpha, \Theta \vdash k$

   $\Delta, \triangleright \alpha, \Theta \vdash k$

   $[\Omega'](\Delta, \triangleright \alpha, \Theta) \vdash [\hat{\alpha}/\alpha]A_0 \leq [\Omega]B$

   By distributivity of substitution

2. **Case** $\ll$: Similar to the $\ll_{\triangleright \alpha}$ case.

3. **Case** $\ll_{\triangleright \alpha}$

   $\Gamma, \beta : k \vdash A \triangleleft B \triangleright \Delta, \beta : k, \Theta$

   $\Gamma \vdash A \triangleleft \forall \beta : k, B \triangleright \Delta$ $\ll_{\triangleright \alpha}$

---

*Proof of Theorem 6 (Soundness of Algorithmic Subtyping)*

*thm:subtyping-soundness*
Proof of Theorem 6 (Soundness of Algorithmic Subtyping) thm:subtyping-soundness

Γ, β : κ ⊢ A <:_ B₀ ⊢ Δ, β : κ, Θ

By Lemma 25 (Filling Completes)

Subderivation

Γ ⊢ A type
Γ, β : κ ⊢ A type
Γ ⊢ ∀ β : κ. B₀ type

By Lemma 35 (Suffix Weakening)

Given

Γ, β : κ ⊢ B₀ type

By inversion (ForallWF)

[Ω'](Δ, β : κ, Θ) ⊢ [Ω']A ≤ [Ω']B₀

Γ, β : κ ⊢ Δ, β : κ, Θ

Θ is soft By Lemma 22 (Extension Inversion) (i)


By def. of substitution

[Ω]Δ ⊢ [Ω]A <:_ [Ω](∀ β : κ. B₀)

By ≤∀R

• Case <:_L Similiar to the <:_R case.

• Case

Γ ⊢ A ≡ B ⊢ Δ

Γ ⊢ A <:_P B ⊢ Δ <:_Equiv

Subderivation

Δ ⊢ Ω


By Lemma 38 (Soundness of Algorithmic Equivalence)

Γ ⊢ Δ

By Lemma 49 (Equivalence Extension)

Γ ⊢ A type

By Lemma 16 (Substitution for Type Well-Formedness)


By ≤RefP

Γ ⊢ A <:_ B ⊢ Δ

neg(A)

Γ ⊢ A <:_ P B ⊢ Δ <:_L

By inversion

neg(A)

nonpos(B)

By inversion

nonpos(B)

nonpos(A)

since neg(A)


By induction

Γ ⊢ A ≡ B ⊢ Δ

nonpos(A)

neg(B)

Γ ⊢ A <:_ P B ⊢ Δ <:_P

By def. of substitution

Similar to the <:_L case.
J’.4 Soundness of Typing

Theorem 7 (Soundness of Match Coverage).

1. If $\Gamma \vdash \Pi$ covers $\vec{A}$ and $\Gamma \vdash \vec{A}$ types and $[\Gamma]\vec{A} = \vec{A}$ and $\Gamma \rightarrow \Omega$ then $[\Omega]\Gamma \vdash \Pi$ covers $\vec{A}$.

2. If $\Gamma / P \vdash \Pi$ covers $\vec{A}$! and $\Gamma \rightarrow \Omega$ and $\Gamma \vdash \vec{A}$! types and $[\Gamma]\vec{A} = \vec{A}$ and $[\Gamma]\Pi = \Pi$ then $[\Omega]\Gamma / P \vdash \Pi$ covers $\vec{A}$!.

Proof. By mutual induction on the given algorithmic coverage derivation.

1. • Case

   \[
   \begin{array}{c}
   \frac{\Gamma \vdash \vec{A} :: B \rightarrow \Delta \quad \text{pos}(A) \quad \text{nonneg}(B)}{
   \Gamma \vdash \vec{A} :: B \rightarrow \Delta}
   \end{array}
   \]

   Similar to the $\llcorner \llcorner L$ case.

   • Case

   \[
   \begin{array}{c}
   \frac{\Gamma \vdash \vec{A} :: B \rightarrow \Delta \quad \text{nonneg}(A) \quad \text{pos}(B)}{
   \Gamma \vdash \vec{A} :: B \rightarrow \Delta}
   \end{array}
   \]

   Similar to the $\llcorner \llcorner L$ case.

   $\blacksquare$

Proof of Theorem 6 (Soundness of Algorithmic Subtyping)
Proof of Theorem 7 (Soundness of Match Coverage) thm:coverage-soundness

- Case \[ \frac{\Gamma \vdash \Gamma t_1 \equiv \Gamma t_2 : \kappa \vdash \bot}{\Gamma \vdash t_1 \vdash \Pi \text{ covers } \vec{A} !} \]

\[ \text{Subderivation} \]

- Case \[ \frac{\mu \text{g}(\Gamma t_1, \Gamma t_2) = \bot}{\text{By Lemma 90 (Soundness of Equality Elimination) (2)}} \]

\[ \text{By given equality} \]

\[ \Rightarrow \frac{\Omega \Gamma / t_1 = t_2 \vdash \Pi \text{ covers } \vec{A}}{\text{By DeclCoversEqBot}} \]

\[ \square \]

Lemma 91 (Well-formedness of Algorithmic Typing).

Given \( \Gamma \) ctx:

(i) If \( \Gamma \vdash e \Rightarrow A \ p \vdash \Delta \) then \( \Delta \vdash A \ p \) type.

(ii) If \( \Gamma \vdash s : A \ p \gg B \ q \vdash \Delta \) and \( \Gamma \vdash A \ p \) type then \( \Delta \vdash B \ q \) type.

Proof.

1. Suppose \( \Gamma \vdash e \Rightarrow A \ p \vdash \Delta \):

- Case \[ \frac{(x : A \ p) \in \Gamma}{\Gamma \vdash x \Rightarrow [\Gamma]A \ p \vdash \Gamma} \]

\[ \text{Var} \]

\( \Gamma = (\Gamma_0, x : A \ p, \Gamma_1) \) \( (x : A \ p) \in \Gamma \)

\( \Gamma \vdash A \ p \) type \( \text{Follows from } \Gamma \) ctx

- Case \[ \frac{\Gamma \vdash A ! \text{ type}}{\Gamma \vdash (e : A) \Rightarrow [\Delta]A ! \vdash \Gamma} \]

\[ \text{Anno} \]

\( \Gamma \vdash A ! \text{ type} \) \( \text{By inversion} \)

\( \Delta \rightarrow \Delta \) \( \text{By Lemma 51 (Typing Extension)} \)

\( \Delta \vdash A ! \text{ type} \) \( \text{By Lemma 41 (Extension Weakening for Principal Typing)} \)

\[ \Rightarrow \frac{\Delta \vdash [\Delta]A ! \text{ type}}{\text{By Lemma 39 (Principal Agreement) (i)}} \]

2. Suppose \( \Gamma \vdash s : A \ p \gg B \ q \vdash \Delta \) and \( \Gamma \vdash A \ p \) type:
Proof of Lemma 91 (Well-formedness of Algorithmic Typing).

• Case
  \[ \Gamma \vdash A \rightarrow B \vdash \text{EmptySpine} \]
  \[ \Gamma \vdash A \vdash \text{Given} \]

• Case
  \[ \Gamma \vdash e \triangleleft A \vdash \Theta \quad \Theta \vdash s : [\Theta]B \vdash C \vdash \Delta \]
  \[ \Gamma \vdash e s : A \rightarrow B \vdash C \vdash \Delta \quad \rightarrow \text{Spine} \]
  \[ \Gamma \vdash A \vdash B \vdash \text{type} \quad \text{Given} \]
  \[ \Gamma \vdash B \vdash \text{type} \quad \text{By Lemma 42 (Inversion of Principal Typing)} \]
  \[ \Theta \vdash B \vdash \text{type} \quad \text{By Lemma 41 (Extension Weakening for Principal Typing)} \]
  \[ \Theta \vdash [\Theta]B \vdash \text{type} \quad \text{By Lemma 40 (Right-Hand Subst. for Principal Typing)} \]
  \[ \Delta \vdash C \vdash q \vdash \text{type} \quad \text{By induction} \]

• Case
  \[ \Gamma, \alpha : \kappa \vdash e s : [\alpha/\alpha]A \vdash C \vdash \Delta \quad \rightarrow \text{Spine} \]
  \[ \Gamma \vdash \forall \alpha : \kappa. A \vdash \text{type} \quad \text{Given} \]
  \[ \Gamma \vdash A \vdash \text{type} \quad \text{By Lemma 42 (Inversion of Principal Typing)} \]
  \[ \Gamma \vdash \forall \alpha : \kappa. A \vdash \text{type} \quad \text{By Lemma 41 (Extension Weakening for Principal Typing)} \]
  \[ \Gamma, \alpha : \kappa, \alpha : \kappa \vdash A \vdash \text{type} \quad \text{By weakening} \]
  \[ \Gamma, \alpha : \kappa \vdash [\alpha/\alpha]A \vdash \text{type} \quad \text{By substitution} \]
  \[ \Delta \vdash C \vdash q \vdash \text{type} \quad \text{By induction} \]

• Case
  \[ \Gamma \vdash P \vdash \text{true} \vdash \Theta \quad \Theta \vdash e s : [\Theta]A \vdash C \vdash \Delta \quad \rightarrow \text{Spine} \]
  \[ \Gamma \vdash P \vdash A \vdash \text{type} \quad \text{Given} \]
  \[ \Gamma \vdash P \vdash A \vdash \text{prop} \quad \text{By Lemma 42 (Inversion of Principal Typing)} \]
  \[ \Gamma \vdash A \vdash \text{type} \quad \text{By Lemma 41 (Extension Weakening for Principal Typing)} \]
  \[ \Theta \vdash A \vdash \text{type} \quad \text{By Lemma 40 (Right-Hand Subst. for Principal Typing)} \]
  \[ \Delta \vdash C \vdash q \vdash \text{type} \quad \text{By induction} \]

• Case
  \[ \Theta \vdash \alpha_2 : \ast, \alpha_1 : \ast, \alpha : \ast \vdash \alpha_1 \rightarrow \alpha_2 \vdash e s : (\alpha_1 \rightarrow \alpha_2) \vdash C \vdash \Delta \quad \rightarrow \text{Spine} \]
  \[ \Theta \vdash \alpha_1 \rightarrow \alpha_2 \vdash \text{type} \quad \text{By rules} \]
  \[ \Delta \vdash C \vdash q \vdash \text{type} \quad \text{By induction} \]

\[ \square \]

Theorem 8 (Eagerness of Types).

(i) If \( D \) derives \( \Gamma \vdash e \triangleleft A \vdash \Delta \) and \( \Gamma \vdash A \vdash \text{type} \) and \( A = [\Gamma]A \) then \( D \) is eager.
(ii) If \( D \) derives \( \Gamma \vdash e \Rightarrow A \ p \vdash \Delta \) then \( D \) is eager.

(iii) If \( D \) derives \( \Gamma \vdash s : A \ p \Rightarrow B \ q \vdash \Delta \) and \( \Gamma \vdash A \ p \text{ type and } A = [\Gamma]A \) then \( D \) is eager.

(iv) If \( D \) derives \( \Gamma \vdash s : A \ p \Rightarrow B \ q \vdash \Delta \) and \( \Gamma \vdash A \ p \text{ type and } A = [\Gamma]A \) then \( D \) is eager.

(v) If \( D \) derives \( \Gamma \vdash \Pi :: \vec{\alpha} q \Leftarrow C \ p \vdash \Delta \) and \( \Gamma \vdash \vec{\alpha} q \text{ types and } [\Gamma]\vec{\alpha} = \vec{\alpha} \) and \( \Gamma \vdash C \ p \text{ type} \) then \( D \) is eager.

(vi) If \( D \) derives \( \Gamma / P \vdash \Pi :: \vec{\alpha} ! \Leftarrow C \ p \vdash \Delta \) and \( \Gamma \vdash P \text{ prop and } \text{FEV}(P) = \emptyset \) and \( \Gamma \vdash \vec{\alpha} ! \text{ types and } \Gamma \vdash C \ p \text{ type} \) then \( D \) is eager.

Proof. By induction on the given derivation.

Part (i), checking

- **Case** \( \text{Rec} \) By i.h. (i).
- **Case** \( \text{Sub} \) By i.h. (ii) and (i).
- **Case** \( \forall \) By i.h. (i).
- **Case** \( \wedge \) Substitution is idempotent, so in the last premise \( [\Theta][\Theta]A_0 = [\Theta]A_0 \) and we can use the i.h. (i).
- **Case** \( \exists \) Similar to the \( \forall \) case.
- **Case** \( \text{Nil} \) This rule has no subderivations of the relevant form, so the case is trivial.
- **Case** \( \rightarrow \) By i.h. (i).
- **Case** \( \times \) In the premise, \( [\Gamma_0][\Theta_1;*, \vec{\alpha}_2;*, \vec{\alpha}_*: = \vec{\alpha}_2, x : \vec{\alpha}_1] = \vec{\alpha}_2 \) so we can use the i.h. (i).
- **Case** \( +_{k} \) By i.h. (i).
- **Case** \( +_{\vec{\alpha}_k} \) Similar to the \( \rightarrow \vec{\alpha} \) case.
- **Case** \( \times \) By i.h. (i) on the first subderivation, then i.h. (i) on the second subderivation (using the fact that \( [\Theta][\Theta]A_2 = [\Theta]A_2 \)).
- **Case** \( \times \vec{\alpha}_k \) Similar to the \( \rightarrow \vec{\alpha} \) case.
- **Case** \( \text{Nil} \) This rule has no subderivations of the relevant form, so the case is trivial.
- **Case** \( \text{Cons} \) By i.h. (i) on the subderivations typing \( e_1 \) and \( e_2 \), using \( [\Gamma'][\Gamma']A_0 = [\Gamma']A_0 \) and \( [\Theta][\Theta](\text{Vec} \ \vec{\alpha} A_0) = [\Theta](\text{Vec} \ \vec{\alpha} A_0) \).

**Case**

\[
\begin{align*}
\Gamma \vdash e \Rightarrow B \ q & \Leftarrow [\Theta]A \ p \vdash \Delta \\
\Delta \vdash \text{case}(e, \Pi) & \Leftarrow A \ p \vdash \Delta
\end{align*}
\]

**Case**

\[
\begin{align*}
\Gamma \vdash e & \Rightarrow B \ ! \vdash \Theta \\
\Delta \vdash \text{case}(e, \Pi) & \Leftarrow A \ p \vdash \Delta
\end{align*}
\]

**Case**

\[
\begin{align*}
\Gamma \vdash e & \Rightarrow B \ q \vdash \Theta \\
\Delta \vdash \text{case}(e, \Pi) & \Leftarrow A \ p \vdash \Delta
\end{align*}
\]

By Definition 8, the given derivation is eager.
Proof of Theorem 8 (Eagerness of Types)  thm:eagerness

Part (ii), synthesis

- **Case** \[\text{Var}\]  Substitution is idempotent: \([\Gamma][\Gamma]A_0 = [\Gamma]A_0\).
  
  By inversion, \(\Delta = \Gamma\) and \(A = [\Gamma]A_0\) where \((x : A_0)p \in \Gamma\).

  Using the above equations, we have
  
  \[
  [\Gamma][\Gamma]A_0 = [\Gamma]A_0 \\
  [\Gamma]A = A \\
  [\Delta]A = A
  \]

  This rule has no subderivations, so there is nothing else to show.

- **Case** \[\text{Anno}\]  By inversion, \(\Delta = \Gamma\) and \(A = [\Gamma]A_0\).

  Substitution is idempotent, so \([\Gamma][\Gamma]A_0 = [\Gamma]A_0\) and we can use the i.h. (i) to show that the checking subderivation is eager.

  The type in the conclusion is \([\Delta]A_0\), which by idempotence is equal to \([\Delta][\Delta]A_0\). Since \(A = [\Delta]A_0\), we have \(A = [\Delta]A\).

- **Case**  
  \[
  \Gamma \vdash e \Rightarrow B \ p \ (-\Theta) \\
  \Theta \vdash s : B \ p \gg A \ [q] \ -\Delta
  \]

  \[
  \Gamma \vdash e \ s \Rightarrow A \ q \ -\Delta
  \]

  Substitution is idempotent, so \([\Theta][\Theta]A_0 = [\Theta]A_0\), and we can apply the i.h. showing \(C = [\Delta]C\) and that all subderivations are eager. Since the output type and output context of the conclusion are \(C\) and \(\Delta\), the same as the premise, we have \(C = [\Delta]C\).

Parts (iii) and (iv), spines

- **Case**  
  \[
  \Gamma, \hat{\alpha} : \kappa \vdash e \ s_0 : [\hat{\alpha}/\alpha]A_0 \ f \gg C \ q \ -\Delta
  \]

  \[
  \Gamma \vdash e \ s_0 : \forall \alpha : \kappa. A_0 \ p \gg C \ q \ -\Delta
  \]

  It is given that \([\Gamma](\forall \alpha : \kappa. A_0) = (\forall \alpha : \kappa. A_0)\).

  Therefore, \([\Gamma]A_0 = A_0\).

  Since \(\hat{\alpha}\) is not solved in \(\Gamma, \hat{\alpha} : \kappa\), we also have

  \[
  [\Gamma, \hat{\alpha} : \kappa][\hat{\alpha}/\alpha]A_0 = [\hat{\alpha}/\alpha]A_0
  \]

  By i.h., \(C = [\Delta]C\) and all subderivations are eager. Since the output type and output context of the conclusion are \(C\) and \(\Delta\), the same as the premise, we have \(C = [\Delta]C\).

- **Case**  
  \[
  \Gamma \vdash \ p \text{ true} \ -\Theta \\
  \Theta \vdash e \ s_0 : [\Theta]A_0 \ p \gg C \ q \ -\Delta
  \]

  \[
  \Gamma \vdash e \ s_0 : \ p \gg A_0 \ p \gg C \ q \ -\Delta
  \]

  Substitution is idempotent, so \([\Theta][\Theta]A_0 = [\Theta]A_0\), and we can apply the i.h. showing \(C = [\Delta]C\) and that all subderivations are eager. Since the output type and output context of the conclusion are \(C\) and \(\Delta\), the same as the premise, we have \(C = [\Delta]C\).
• Case **SpineRecover** By i.h. (iii).

• Case **SpinePass** By i.h. (iii).

• Case

\[
\frac{\Gamma \vdash e \in A_1 \vdash \Theta \quad \Theta \vdash s : [\Theta]A_2 \vdash C \quad \Delta}{\Gamma \vdash e \circ s : A_1 \rightarrow A_2 \vdash C \quad \Delta} \quad \rightarrow Spine
\]

We have \([\Gamma]A_1 = A_1\). Therefore, \([\Gamma]A_1 = A_1\). By i.h. on the first subderivation, its subderivations are eager.

Substitution is idempotent, so \([\Theta][\Theta]A_2 = [\Theta]A_2\). By i.h. on the second subderivation, \([\Delta]C = C\) (and its subderivations are eager).

Since the output type and output context of the conclusion are \(C\) and \(\Delta\), the same as the premise, we have \(C = [\Delta]C\); we also showed that all subderivations are eager.

• Case

\[
\frac{\Gamma_0[\lambda_2 : *, \lambda_1 : *, \lambda : * = \lambda_1 \rightarrow \lambda_2] \vdash e \circ s_0 : (\lambda_1 \rightarrow \lambda_2) \vdash C \quad \Delta}{\Gamma_0[\lambda : *] \vdash e \circ s_0 : \lambda \vdash C \quad \Delta} \quad \rightarrow Spine
\]

By definition of substitution,

\[
[\Gamma_0[\lambda_2 : *, \lambda_1 : *, \lambda : * = \lambda_1 \rightarrow \lambda_2]](\lambda_1 \rightarrow \lambda_2) = (\lambda_1 \rightarrow \lambda_2)
\]

Therefore, we can apply the i.h.

Since the output type and output context of the conclusion are \(C\) and \(\Delta\), the same as the premise, we have \(C = [\Delta]C\); we also showed that all subderivations are eager.

**Parts (v) and (vi), pattern matching**

Part (v), rules [MatchEmpty], etc.: By i.h. (v) and, in [MatchBase], i.h. (i). [MatchSeq] By i.h. (v), using idempotency of substitution for \(\bar{A}\).

Part (vi), rule [Match]. trivial. Part (vi), rule [MatchUnify] by the assumption \(\Gamma \vdash \bar{A}!\) types, the vector \(\bar{A}\) has no existential variables at all, so in the second premise, \(\bar{A} = [\Gamma]\bar{A}\) and we can apply the i.h. (v). \(\square\)

**Theorem 9** (Soundness of Algorithmic Typing).

Given \(\Delta \rightarrow \Omega\):

(i) If \(\Gamma \vdash e \in A \quad \rightarrow \Delta\) and \(\Gamma \vdash A \quad p\) type and \(A = [\Gamma]A\) then \([\Omega]\Delta \vdash [\Omega]e \in [\Omega]A\).

(ii) If \(\Gamma \vdash e \in A \quad \rightarrow \Delta\) then \([\Omega]\Delta \vdash [\Omega]e \vdash [\Omega]A\).

(iii) If \(\Gamma \vdash s : A \quad \rightarrow B \quad \rightarrow \Delta\) and \(\Gamma \vdash A \quad p\) type and \(A = [\Gamma]A\) then \([\Omega]\Delta \vdash [\Omega]s : [\Omega]A\quad p \vdash [\Omega]B\).

(iv) If \(\Gamma \vdash s : A \quad \rightarrow B \quad \rightarrow \Delta\) and \(\Gamma \vdash A \quad p\) type and \(A = [\Gamma]A\) then \([\Omega]\Delta \vdash [\Omega]s : [\Omega]A\quad p \vdash [\Omega]B\quad [\Omega]\).

(v) If \(\Gamma \vdash \Pi : \bar{A} \quad q \in C \quad \rightarrow \Delta\) and \(\Gamma \vdash \bar{A}!\) types and \([\Gamma]\bar{A} = \bar{A}\) and \(\Gamma \vdash C \quad p\) type then \(p \vdash [\Omega]\Delta : [\Omega]\Pi \! : [\Omega]\bar{A} \quad q \in [\Omega]C\).

(vi) If \(\Gamma / P \vdash \Pi \vdash \bar{A}! \in C \quad \rightarrow \Delta\) and \(\Gamma \vdash P \quad p\) type and \(FEV(P) = \emptyset\) and \([\Gamma]P = P\) and \(\Gamma \vdash \bar{A}!\) types and \(\Gamma \vdash C \quad p\) type then \([\Omega]\Delta / [\Omega]P \vdash [\Omega]\Pi \vdash [\Omega]\bar{A} \! : [\Omega]C \quad p\).
Proof. By induction, using the measure in Definition[7]

Where the i.h. is used, we elide the reasoning establishing the condition \( \Gamma A = A \) for parts (i), (iii), (iv), (v) and (vi): this condition follows from Theorem[8] which ensures that the appropriate condition holds for all subderivations.

- **Case**
  \[ (x: A p) \in \Gamma \]
  \[ \Gamma \vdash x \Rightarrow \Gamma A p \vdash \Gamma \]
  \[ \text{Var} \]
  \[ \{x: A p\} \in \Gamma \]
  \[ \{x: A p\} \in \Delta \]
  \[ \Delta \rightarrow \Omega \]
  \[ \{x: [\Omega]A p\} \in [\Omega]\Gamma \]
  \[ [\Omega]\Gamma \vdash [\Omega]x \Rightarrow [\Omega]A p \]
  \[ \text{By Lemma}[9] \text{ (Uvar Preservation)} \] (ii)
  \[ \Delta \rightarrow \Omega \]
  \[ \Gamma \rightarrow \Omega \]
  \[ \{\Omega\}A = [\Omega][\Gamma]A \]
  \[ \text{By Lemma}[29] \text{ (Substitution Monotonicity)} \] (iii)
  \[ [\Omega]\Gamma \vdash [\Omega]x \Rightarrow [\Omega][\Gamma]A p \]
  \[ \text{By above equality} \]

- **Case**
  \[ \Gamma \vdash e \Rightarrow A q \vdash \Theta \]
  \[ \Theta \vdash A \leq \text{join}(\text{pol}(B), \text{pol}(A)) B \vdash \Delta \]
  \[ \Gamma \vdash e \Leftarrow B p \vdash \Delta \]
  \[ \text{Sub} \]
  \[ \Gamma \vdash e \Rightarrow A q \vdash \Theta \]
  \[ \Theta \vdash A \leq \text{join}(\text{pol}(B), \text{pol}(A)) B \vdash \Delta \]
  \[ \text{Subderivation} \]
  \[ \Theta \rightarrow \Delta \]
  \[ \Delta \rightarrow \Omega \]
  \[ \Theta \rightarrow \Omega \]
  \[ [\Omega]\Theta \vdash [\Omega]e \Rightarrow [\Omega]A q \]
  \[ \text{By i.h.} \]
  \[ [\Omega]\Theta = [\Omega]\Delta \]
  \[ \text{By Lemma}[33] \text{ (Extension Transitivity)} \]
  \[ [\Omega]\Delta \vdash [\Omega]e \Rightarrow [\Omega]A q \]
  \[ \text{By above equality} \]
  \[ \Theta \vdash A \leq \text{join}(\text{pol}(B), \text{pol}(A)) B \vdash \Delta \]
  \[ \text{Subderivation} \]
  \[ [\Omega]\Delta \vdash [\Omega]A \leq \text{join}(\text{pol}(B), \text{pol}(A)) \]
  \[ \text{By Theorem}[6] \]
  \[ [\Omega]\Delta \vdash [\Omega]e \Leftarrow [\Omega]B p \]
  \[ \text{By DeclSub} \]

- **Case**
  \[ \Gamma \vdash A_0! \text{ type} \]
  \[ \Gamma \vdash e_0 \Leftarrow [\Gamma]A_0! \vdash \Delta \]
  \[ \Gamma \vdash (e_0 : A_0) \Rightarrow [\Delta]A_0! \vdash \Delta \]
  \[ \text{Anno} \]
  \[ \Gamma \vdash e_0 \Leftarrow [\Gamma]A_0! \vdash \Delta \]
  \[ \text{Subderivation} \]
  \[ [\Omega]\Delta \vdash [\Omega]e_0 \Leftarrow [\Omega][\Gamma]A_0! \]
  \[ \text{By i.h.} \]
  \[ \Gamma \vdash A_0! \text{ type} \]
  \[ \text{Subderivation} \]
  \[ \Gamma \vdash A_0 \text{ type} \]
  \[ \text{By inversion} \]

\[ \text{FEV}(A_0) = \emptyset \]
Proof of Theorem 9 (Soundness of Algorithmic Typing)

\[ \Gamma \rightarrow \Delta \]

By Lemma 51 (Typing Extension)

\[ \Delta \rightarrow \Omega \]

Given

\[ \Gamma \rightarrow \Omega \]

By Lemma 33 (Extension Transitivity)

\[ \Omega \vdash A_0 \text{ type} \]

By Lemma 36 (Extension Weakening (Sorts))

\[ [\Omega] \omega \Omega \rightarrow [\Omega] \Delta \]

By Lemma 16 (Substitution for Type Well-Formedness)

\[ [\Omega] \Delta \vdash [\Omega] A_0 \text{ type} \]

By Lemma 54 (Completing Stability)

\[ [\Omega][\Gamma] A_0 = [\Omega] A_0 \]

By Lemma 29 (Substitution Monotonicity) (iii)

\[ [\Omega] \Delta \vdash [\Omega] e_0 \iff [\Omega] A_0 \]

By above equality

\[ \Gamma \vdash () \iff 1 \vdash \Delta \]

\[ [\Omega] \Delta \vdash () \iff 1 \vdash [\Omega] \Delta \]

By Dec11

\[ [\Omega] \Delta \vdash () \iff [\Omega] 1 \vdash [\Omega] \Delta \]

By definition of substitution

\[ \Gamma_0 \mathcal{\&} : \ast \vdash () \iff \mathcal{\&} \vdash [\Gamma_0 \mathcal{\&} : \ast = 1] \]

By definition of context application

\[ \Gamma_0 \mathcal{\&} : \ast = 1 \rightarrow \Omega \]

Given

\[ [\Omega] \mathcal{\&} = [\Omega] [\Delta] \mathcal{\&} \]

By Lemma 29 (Substitution Monotonicity) (i)

\[ = [\Omega] 1 \]

By definition of context application

\[ = 1 \]

By definition of context application

\[ [\Omega] \Delta \vdash () \iff 1 \vdash [\Omega] \mathcal{\&} \]

By Dec11

\[ [\Omega] \Delta \vdash [\Omega] () \iff [\Omega] [\Delta] \mathcal{\&} \]

By above equality

\[ \nu \text{chk-I} \]

\[ \Gamma, \alpha : \kappa \vdash v \iff A_0 \vdash \Delta, \alpha : \kappa, \Theta \]

\[ \Gamma \vdash v \iff \forall \alpha : \kappa. A_0 \vdash \Delta \]

By Vl

\[ \Delta \rightarrow \Omega \]

Given

\[ \Delta, \alpha \rightarrow \Omega, \alpha \]

By Uvar

\[ \Gamma, \alpha \rightarrow \Delta, \alpha, \Theta \]

By Lemma 51 (Typing Extension)

\[ \Theta \text{ soft} \]

By Lemma 22 (Extension Inversion) (i) (with \( \Gamma_R = \cdot \), which is soft)

\[ \Delta, \alpha, \Theta \rightarrow \Omega, \alpha, \Theta \]

By Lemma 25 (Filling Completes)
Proof of Theorem 9 (Soundness of Algorithmic Typing)

\[ \Gamma, \alpha \vdash \nu \iff A_0 \vdash \Delta' \] Subderivation

\[ [\Omega']\Delta' \vdash (\Omega)\nu \iff [\Omega']A_0 \vdash \nu \] By i.h.

\[ [\Omega']A_0 = [\Omega]A_0 \] By Lemma [17] (Substitution Stability)

\[ [\Omega']\Delta' \vdash (\Omega)\nu \iff [\Omega]A_0 \vdash \nu \] By above equality

\[ \Delta, \alpha, \Theta \rightarrow \Omega, \alpha, \Theta \] Above

\[ \Theta \rightarrow \Delta \] Above

\[ \Theta = [(\Omega)\Delta, \alpha) \] By Lemma [53] (Softness Goes Away)

\[ [\Omega]\Delta \vdash [\Omega]\nu \iff \forall \alpha. [\Omega]A_0 \vdash \nu \] By above equality

\[ [\Omega]\Delta \vdash [\Omega]\nu \iff [\Omega](\forall \alpha. A_0) \vdash \nu \] By definition of substitution

- **Case** \( \Gamma, \hat{\alpha} : k \vdash e \ s_0 : [\hat{\alpha}\!/\alpha]A_0 \not\equiv C q \vdash \Delta \)

\[ \Gamma \vdash e \ s_0 : \forall \alpha : k. [\Omega]A_0 \not\equiv C q \vdash \Delta \] \text{Subderivation}

\[ [\Omega]\Delta \vdash [\Omega](e \ s_0) : \forall \alpha : [\Omega]\hat{\alpha}\!/\alpha A_0 \not\equiv [\Omega]C q \] By i.h.

\[ [\Omega]\Delta \vdash [\Omega](e \ s_0) : [(\Omega)\hat{\alpha}\!/\alpha [\Omega]A_0 \not\equiv [\Omega]C q \] By a property of substitution

\[ [\Omega]\Delta \vdash [\Omega](e \ s_0) : \forall \alpha : k. [\Omega]A_0 \not\equiv [\Omega]C q \] By def. of substitution

- **Case** \( e \ \text{chk-I} \)

\[ \Gamma \vdash \text{true} \vdash \Theta \quad \Theta \vdash e \equiv [\Theta]A_0 \vdash \Delta \] \text{Subderivation}

\[ \Gamma \vdash e \equiv A_0 \land P \vdash \Delta \] \text{Subderivation}

\[ \Theta \rightarrow \Delta \] Given

\[ \Theta \rightarrow \Delta \] By Lemma [51] (Typing Extension)

\[ \Theta \rightarrow \Omega \] By Lemma [33] (Extension Transitivity)

\[ [\Omega]\Theta \vdash [\Omega]P \ \text{true} \] By Lemma [89] (Soundness of Checkprop)

\[ [\Omega]\Delta \vdash [\Omega]P \ \text{true} \] By Lemma [56] (Confluence of Completeness)

\[ \Theta \vdash e \equiv [\Theta]A_0 \vdash \Delta \] Subderivation

\[ [\Omega]\Delta \vdash [\Omega]e \equiv ([\Omega][\Theta]A_0) \vdash \Delta \] By i.h.

\[ [\Omega]\Delta \vdash [\Omega]e \equiv ([\Omega][\Theta]A_0) \land [\Omega]P \vdash \Delta \] By above equality

\[ [\Omega]\Theta A_0 = [\Omega]A_0 \] By Lemma [29] (Substitution Monotonicity) (iii)

\[ [\Omega]\Delta \vdash [\Omega]e \equiv ([\Omega]A_0) \land [\Omega]P \vdash \Delta \] By def. of substitution
Proof of Theorem 9 (Soundness of Algorithmic Typing)

• Case \( \Gamma \vdash t = \text{zero} \true \vdash \Delta \)
  \[ \Gamma \vdash \text{[]} \leftrightarrow (\text{Vec } \alpha A) \ p \vdash \Delta \]

\( \Gamma \vdash t = \text{zero} \true \vdash \Delta \)
Subderivation
\( \Delta \rightarrow \Omega \)
Given
\( [\Omega] \Delta \vdash [\Omega](t = \text{zero}) \true \)
By Lemma 89 (Soundness of Checkprop)
\( [\Omega] \Delta \vdash [\Omega]t = \text{zero} \true \)
By def. of substitution
\[ [\Omega] \Delta \vdash [\Omega] \text{[]} \leftrightarrow (\text{Vec } [\Omega]t [\Omega]A) \ p \]
By DeclNil

• Case \( \Gamma', e_1 \vdash e_2 \leftrightarrow (\text{Vec } \alpha A) \ p \vdash \Delta \)

\[ \Gamma', e_1 : e_2 \leftrightarrow (\text{Vec } \alpha A) \ p \vdash \Delta \]
Subderivation
\( \Delta \rightarrow \Omega \)
Given
\( \Gamma' \rightarrow \Theta \)
By Lemma 51 (Typing Extension)
\( \Theta \rightarrow \Delta, \alpha, \Delta' \)
By Lemma 51 (Typing Extension)
\( \Delta, \alpha, \Delta' \rightarrow \Omega' \)
By Lemma 25 (Filling Completeness)
\( \Gamma' \rightarrow \Omega' \)
By Lemma 33 (Extension Transitivity)
\( [\Omega'] \Gamma' \vdash [\Omega'](t = \text{succ}(\alpha)) \true \)
By Lemma 89 (Soundness of Checkprop)
\( [\Omega'](\Lambda, \alpha, \Delta') \rightarrow [\Omega'](t = \text{succ}(\alpha)) \true \)
By Lemma 56 (Confluence of Completeness)
\( [\Omega'](\Lambda, \alpha, \Delta') \rightarrow [\Omega](t = \text{succ}(\alpha)) \true \)
By Lemma 17 (Substitution Stability)
\( [\Omega](\Lambda \vdash [\Omega](t = \text{succ}(\alpha)) \true \)
By Lemma 52 (Context Partitioning) + thinning
\[ 1 \]
By def. of substitution

\( \Gamma' \vdash e_1 \leftrightarrow [\Gamma'] A_0 \ p \vdash \Theta \)
Subderivation
\( [\Omega'] \Theta \vdash [\Omega]e_1 \leftrightarrow ([\Omega'] A_0) p \)
By i.h.
\( [\Omega'] A_0 = [\Omega'] A_0 \)
By Lemma 29 (Substitution Monotonicity) (iii)
\( [\Omega'] \Theta \vdash [\Omega]e_1 \leftrightarrow [\Omega'] A_0 p \)
By above equality
\[ 2 \]
Similar to above

\( \Theta \vdash e_2 \leftrightarrow [\Theta](\text{Vec } \alpha A) \ f \vdash \Delta, \alpha, \Delta' \)
Subderivation
\( [\Omega'](\Lambda, \alpha, \Delta') \rightarrow [\Omega'] \Theta \leftrightarrow ([\Omega'](\text{Vec } \alpha A) \ f \)
By i.h.
\( [\Omega] \Delta \vdash [\Omega]e_2 \leftrightarrow [\Omega](\text{Vec } \alpha A) \ f \)
Similar to above
\[ 3 \]
By def. of substitution
\( [\Omega] \Delta \vdash [\Omega]e_1 : e_2 \leftrightarrow [\Omega](\text{Vec } t A_0) p \)
By DeclCons (premises: 1, 2, 3)
\[ 4 \]
By def. of substitution
Proof of Theorem 9 (Soundness of Algorithmic Typing) thm:typing-soundness

• Case \( e \) chk-I  
  \[ \Gamma, \hat{x} : \kappa \vdash e \iff [\hat{x}/\alpha]A_0 \vdash \Delta \]
  
  Subderivation

  \[ [\Omega]\Lambda \vdash [\Omega]e \iff [\Omega][\hat{x}/\alpha]A_0 \]

  By a property of substitution

\[ \Gamma, \hat{x} : \kappa \vdash e \iff \exists \alpha : \kappa. A_0 \Downarrow \Delta \]

  \[ \Gamma, \hat{x} : \kappa \vdash \Delta \]

  By Lemma 51 (Typing Extension)

  \[ \Delta \vdash \hat{x} : \kappa \]

  By Lemma 36 (Extension Weakening (Sorts))

  \[ \Delta \vdash \Omega \]

  Given

  \[ [\Omega]\Lambda \vdash [\Omega]e \iff [\Omega]x \kappa. A_0 \]

  By Lemma 58 (Bundled Substitution for Sorting)

  Subderivation

  \[ \Omega \vdash [\Omega]A_0 \]

  By Lemma 30 (Typing Extension)

  \[ [\Omega]A_0 \Downarrow \Delta \]

  By above equalities

  \[ \Delta \vdash \exists \kappa. A_0 \]

  By repeated \( \Downarrow \) Eqn

  \[ \exists \kappa. A_0 \Downarrow \Omega \]

  By Lemma 33 (Extension Transitivity)

  \[ \exists \kappa. A_0 \Downarrow \Delta \]

  By Lemma 33 (Extension Transitivity)

  \[ \exists \kappa. A_0 \Downarrow \Omega \]

  By a property of substitution

\[ \exists \exists \kappa. A_0 \Downarrow \Delta \]

  \[ \exists \exists \kappa. A_0 \Downarrow \Omega \]

  By a property of substitution

• Case \( v \) chk-I  
  \[ \Gamma, \varpi / P \vdash \Theta^+ \]
  
  \[ \Theta^+ \vdash v \iff [\Theta^+]A_0 \vdash [\Delta, \varpi, \Delta'] \]

  \[ \Gamma \vdash v \iff \theta \Downarrow \Delta \]

  \[ \Gamma \vdash \Lambda \Downarrow \text{type} \]

  Given

  \[ \text{FEV}(\Gamma)[A] = \emptyset \]

  By inversion on rule PrincipalWF

  \[ \text{FEV}(\Gamma)[P] = \emptyset \]

  A = (P \Downarrow A_0)

  \[ \Gamma, \varpi / P \vdash \Theta^+ \]

  Subderivation

  \[ \Gamma, \varpi / \sigma \Downarrow t : \kappa \Downarrow \Theta^+ \]

  By inversion

  \[ \text{FEV}(\Gamma)[\sigma] \cup \text{FEV}(\Gamma)[t] = \emptyset \]

  By \text{FEV}(\Gamma)[P] = \emptyset above

  \[ \Theta^+ = [\Gamma, \varpi, \Theta] \]

  By Lemma 50 (Soundness of Equality Elimination)

  \[ [\Omega, \Theta]t' = [\Theta]f[\Gamma, \varpi]t' \]

  \[ \theta = \text{mgu}(\sigma, t) \]

  " (for all \( \Omega ' \) extending \( \Gamma, \varpi \) and \( t' \) s.t. \( \Omega ' \Downarrow t' : \kappa' \))

  \[ \Delta \rightarrow \Omega \]

  Given

  \[ \Theta^+ \rightarrow \Delta, \varpi, \Delta' \]

  By Lemma 51 (Typing Extension)

  \[ \Delta, \varpi, \Theta \rightarrow \Delta, \varpi, \Delta' \]

  By above equalities

  Let \( \Theta^+ = [\Omega, \varpi, \Delta'] \).

  \[ \Delta, \varpi, \Theta \rightarrow \Omega, \varpi, \Delta' \]

  By repeated \( \rightarrow \) Eqn

  \[ \Theta^+ \rightarrow \Omega^+ \]

  By Lemma 33 (Extension Transitivity)

  \[ [\Omega', \Theta]B = [\Theta][\Gamma, \varpi]B \]

  By Lemma 95 (Substitution Upgrade) (i)

  (for all \( \Omega ' \) extending \( \Gamma, \varpi \) and \( B \) s.t. \( \Omega ' \Downarrow B : \kappa' \))
Proof of Theorem 9 (Soundness of Algorithmic Typing)

\[ \Theta^+ \vdash v \iff [\Theta^+]A_0 ! : \vdash, \vartriangleright_P, \Delta' \]

Subderivation

\[ [\Omega^+] (\Delta, \vartriangleright_P, \Delta') \vdash [\Omega]v \iff [\Omega^+] (\Theta^+)A_0 ! \]

By i.h.

\[ \Gamma, \vartriangleright_P, \Theta \rightarrow \Omega, \vartriangleright_P, \Delta' \]

By Lemma 33 [Extension Transitivity]

\[ \Gamma \rightarrow \Omega \]

By Lemma 22 [Extension Inversion]

\[ [\Omega^+] (\Theta^+)A_0 = [\Omega^+]A_0 \]

By Lemma 29 [Substitution Monotonicity]

\[ = [\theta] (\Omega, \vartriangleright_P)A_0 \]

Above, with \((\Omega, \vartriangleright_P)\) as \(\Omega'\) and \(A_0\) as \(B\)

\[ = [\theta] (\Omega)A_0 \]

By def. of substitution

\[ [\Omega, \vartriangleright_P, \Theta] (\Delta, \vartriangleright_P, \Delta') = [\theta] (\Omega) \Delta \]

By Lemma 95 [Substitution Upgrade (iii)]

\[ [\theta] (\Omega) \Delta \vdash [\Omega] [\theta]v \iff [\theta] (\Omega)A_0 ! \]

By above equalities

\[ [\Omega^+] (\Delta, \vartriangleright_P, \Delta') / (\sigma = t) \vdash [\Omega]v \iff [\Omega]A_0 ! \]

By DeclCheckUnify

\[ [\Omega^+] (\Delta, \vartriangleright_P, \Delta') = [\Omega] \Delta \]

From def. of context application

\[ [\Omega] \Delta / (\sigma = t) \vdash [\Omega]v \iff [\Omega]A_0 ! \]

By above equality

\[ [\Omega] \Delta \vdash [\Omega]v \iff (\sigma = t) \supset [\Omega]A_0 ! \]

By DeclRef

\[ [\Omega] \Delta \vdash [\Omega]v \iff ([\Omega] (\sigma = t)) \supset [\Omega]A_0 ! \]

By above FEV condition

\[ \boxed{\text{Case } v \text{ chk-l}} \]

\[ \Gamma \vdash v \iff P : \bot \Delta \rightarrow \Gamma \]

Subderivation

\[ \Gamma, \vartriangleright_P / P : \bot \Delta \rightarrow \Gamma \]

Subderivation

\[ \Gamma, \vartriangleright_P / \sigma \equiv t : \kappa : \bot \Delta \rightarrow \Gamma \]

By inversion

\[ P = (\sigma = t) \]

""

FEV([\Gamma] \sigma) \cup FEV([\Gamma] t) = \emptyset

As in \boxed{\text{case}} (above)

mgu(\sigma, t) = \bot

By Lemma 90 [Soundness of Equality Elimination]

\[ [\Omega] \Delta / (\sigma = t) \vdash [\Omega]v \iff [\Omega]A_0 ! \]

By DeclCheckRef

\[ [\Omega] \Delta \vdash [\Omega]v \iff (\sigma = t) \supset [\Omega]A_0 ! \]

By DeclRef

\[ [\Omega] \Delta \vdash [\Omega]v \iff ([\Omega] (\sigma = t)) \supset [\Omega]A_0 ! \]

By def. of subst.

\[ \boxed{\text{Let } \Omega' = \Omega.} \]

\[ \boxed{\text{Case } \Gamma \vdash P \text{ true } \vdash \Theta \vdash e \ s_0 : [\Theta]A_0 \ p \gg C \ q \vdash \Delta} \]

\[ \Gamma \vdash e \ s_0 : P \supset A_0 \ p \gg C \ q \vdash \Delta \]

Subderivation

\[ \Theta \vdash e \ s_0 : [\Theta]A_0 \ p \gg C \ q \vdash \Delta \]

Subderivation

\[ \Theta \rightarrow \Delta \]

By Lemma 51 [Typing Extension]

\[ \Delta \rightarrow \Omega \]

Given

\[ \Theta \rightarrow \Omega \]

By Lemma 23 [Extension Transitivity]
Proof of Theorem 9 (Soundness of Algorithmic Typing)

\[ \Omega \Delta \vdash [\Omega](e\ s_0) : [\Omega][\Theta]A_0 \ p \gg [\Omega]C q \]

By i.h.

\[ [\Omega][\Theta]A_0 = [\Omega]A_0 \]

By Lemma 29 (Substitution Monotonicity) (iii)

\[ [\Omega] \Delta \vdash [\Omega](e\ s_0) : [\Omega]A_0 \ p \gg [\Omega]C q \]

By above equality

\[ \Gamma \vdash P\ true \vdash [\Theta] \]

Subderivation

\[ [\Omega][\Theta] \vdash [\Omega]P\ true \]

By Lemma 97 (Completeness of Checkprop)

\[ [\Omega][\Theta] = [\Omega]\Delta \]

By Lemma 56 (Confluence of Completeness)

\[ [\Omega] \Delta \vdash [\Omega]P\ true \]

By above equality

\[ [\Omega] \Delta \vdash [\Omega](e\ s_0) : ([\Omega]P) \supset [\Omega]A_0 \ p \gg [\Omega]C q \]

By Def. of subst.

\[ [\Omega] \Delta \vdash [\Omega](e\ s_0) : ([\Omega](P \supset A_0)) \ p \gg [\Omega]C q \]
• Case $\Gamma, x: A_1 p \vdash e_0 \iff A_2 p \vdash \Delta, x: A_1 p, \Theta$

\[
\begin{align*}
\Delta &\rightarrow \Omega \\
\Delta, x: A_1 p &\rightarrow \Omega, x: [\Omega]A_1 p \\
\Theta &\text{ soft}
\end{align*}
\]

Given

By $\rightarrow \text{Var}$

By Lemma 51 (Typing Extension)

By Lemma 22 (Extension Inversion) (v)

(with $\Gamma_R = \cdot$, which is soft)

Subderivation

[\Omega']\Delta' \vdash [\Omega]e_0 \iff [\Omega']A_2 p

By i.h.

By Lemma 17 (Substitution Stability)

By above equality

Above

[\Omega']\Delta' = ([\Omega]\Delta, x: [\Omega]A_1 p)

By Lemma 53 (Softness Goes Away)

By above equality

\[
\begin{align*}
[\Omega]\Delta &\vdash \lambda x. [\Omega]e_0 \iff ([\Omega]A_1) \rightarrow ([\Omega]A_2) p \\
\iff [\Omega]\Delta &\vdash [\Omega](\lambda x. e_0) \iff [\Omega](A_1 \rightarrow A_2) p
\end{align*}
\]

By definition of substitution

• Case $v \text{ chk-l}$ $\Gamma, x: A p \vdash v \iff A p \vdash \Delta, x: A p, \Theta$

\[
\Gamma \vdash \text{rec} x. v \iff A p \vdash \Delta
\]

Rec

Similar to the $\rightarrow$ case, applying $\text{DeclRec}$ instead of $\text{Decl} \rightarrow$
• Case \[ \Gamma[\bar{\alpha}_1:*;\bar{\alpha}_2:*;\bar{\alpha}:* = \bar{\alpha}_1 \rightarrow \bar{\alpha}_2], x: \bar{\alpha}_1 \vdash_0 e_0 \iff \bar{\alpha}_2 \vdash_1 \Delta, x: \bar{\alpha}_1 \vdash_1 \Theta \] 

\[ \quad \Gamma[\bar{\alpha} : *) \vdash_0 \lambda x. e_0 \iff \bar{\alpha} \vdash_1 \Delta \quad \text{[by Lemma 51 (Typing Extension)]} \]

\[ \Gamma[\bar{\alpha}_1:*;\bar{\alpha}_2:*;\bar{\alpha}:* = \bar{\alpha}_1 \rightarrow \bar{\alpha}_2], x: \bar{\alpha}_1 \vdash_1 \Delta, x: \bar{\alpha}_1 \vdash_1 \Theta \quad \text{[soft]} \]

\[ \quad \Gamma[\bar{\alpha}_1:*;\bar{\alpha}_2:*;\bar{\alpha}:* = \bar{\alpha}_1 \rightarrow \bar{\alpha}_2] \rightarrow \Delta \quad \text{[by i.h.]} \]

\[ \Delta \rightarrow \Omega \quad \text{[by definition of substitution]} \]

\[ \Delta, x: \bar{\alpha}_1 \vdash_1 \Omega, x: \varnothing(\bar{\alpha}_1) \rightarrow \bar{\alpha}_1 \vdash_1 \Theta \quad \text{[by definition of context substitution]} \]

\[ \Delta, x: \bar{\alpha}_1, \Theta \rightarrow \Omega, x: \varnothing(\bar{\alpha}_1), \Theta \quad \text{[by above equalities]} \]

\[ \quad \Omega \Delta \vdash \lambda x. [\Omega \Delta \vdash \varnothing(\bar{\alpha}_1) \rightarrow \Omega \Delta \vdash \bar{\alpha}_2] \quad \text{[by Definition of Substitution]} \]

\[ \quad \Omega \Delta \vdash \lambda x. [\Omega \Delta \vdash \varnothing(\bar{\alpha}_1) \rightarrow \Omega \Delta \vdash \bar{\alpha}_2] \quad \text{[by Definition of Substitution]} \]

\[ \quad \Omega \Delta \vdash \lambda x. [\Omega \Delta \vdash \varnothing(\bar{\alpha}_1) \rightarrow \Omega \Delta \vdash \bar{\alpha}_2] \quad \text{[by above equality]} \]

\[ \quad \Omega \Delta \vdash \lambda x. (\varnothing(\bar{\alpha}_1) \rightarrow \bullet) \quad \text{[by Lemma 33 (Extension Transitivity)]} \]

\[ \quad \Omega \Delta \vdash \lambda x. (\varnothing(\bar{\alpha}_1) \rightarrow \bullet) \quad \text{[by above equality]} \]

\[ \quad \varnothing \vdash [\Omega \Delta \vdash \varnothing(\bar{\alpha}_1) \rightarrow \Omega \Delta \vdash \bar{\alpha}_2] \quad \text{[by above equality]} \]

\[ \quad \varnothing \vdash [\Omega \Delta \vdash \varnothing(\bar{\alpha}_1) \rightarrow \Omega \Delta \vdash \bar{\alpha}_2] \quad \text{[by above equality]} \]

\[ \quad \varnothing \vdash [\Omega \Delta \vdash \varnothing(\bar{\alpha}_1) \rightarrow \Omega \Delta \vdash \bar{\alpha}_2] \quad \text{[by above equality]} \]
Proof of Theorem 9 (Soundness of Algorithmic Typing) \[\text{thm:typing-soundness}\]

\[\begin{align*}
\Omega|\Theta &\vdash [\Omega]|s_0 : [\Omega]|A \gg [\Omega]|C \ [p] & \text{By i.h.} \\
\Rightarrow & \quad [\Omega]|\Delta \vdash [\Omega]|(e_0 \ s_0) \Rightarrow [\Omega]|C \ p & \text{By rule Decl→E}
\end{align*}\]

\[\begin{itemize}
\item \textbf{Case} \quad \Gamma \vdash s : A \gg C \ j \vdash \Delta \\
\quad \text{FEV}(C) = \emptyset \quad \text{SpineRecover} \\
\quad \Gamma \vdash s : A \gg C \ [1] \vdash \Delta \\
\quad \text{Subderivation} \\
\quad [\Omega] \Gamma \vdash [\Omega]|s : [\Omega]|A \gg [\Omega]|C \ q & \text{By i.h.}
\end{itemize}\]

We show the quantified premise of \[\text{DeclSpineRecover}\], namely,

\[\text{for all } C'.\]

\[\text{if } [\Omega]|\Theta \vdash s : [\Omega]|A \gg C' \ j \text{ then } C' = [\Omega]|C\]

Suppose we have \(C'\) such that \([\Omega]|\Gamma \vdash s : [\Omega]|A \gg C'\). To apply \[\text{DeclSpineRecover}\], we need to show \(C' = [\Omega]|C\).

\[\begin{align*}
[\Omega]|\Gamma &\vdash [\Omega]|s : [\Omega]|A \gg C' \ j' & \text{Assumption} \\
\Omega_{\text{canon}} &\rightarrow \Omega & \text{By Lemma \[\text{Canonical Completion}\]} \\
dom(\Omega_{\text{canon}}) &\rightarrow \dom(\Gamma) & \text{"} \\
\Gamma &\rightarrow \Omega_{\text{canon}} & \text{"} \\
[\Omega]|\Gamma &\Rightarrow [\Omega_{\text{canon}}]|\Gamma & \text{By Lemma \[\text{Multiple Confluence}\]} \\
[\Omega]|A &\Rightarrow [\Omega_{\text{canon}}]|A & \text{By Lemma \[\text{Completing Completeness}\] (ii)} \\
[\Omega_{\text{canon}}]|\Gamma &\vdash [\Omega_{\text{canon}}]|s : [\Omega_{\text{canon}}]|A \gg C' \ j' & \text{By above equalities} \\
\Gamma &\vdash s : [\Gamma]|A \gg C'' \ q \vdash \Delta'' & \text{By Theorem \[\text{DeclSpineRecover}\] (iii)} \\
\Omega_{\text{canon}} &\rightarrow \Omega'' & \text{"} \\
\Delta'' &\rightarrow \Omega'' & \text{"} \\
C' &\Rightarrow [\Omega'']|C'' & \text{"} \\
\Gamma &\vdash s : [\Gamma]|A \gg C'' \ q \vdash \Delta'' & \text{Above} \\
[\Gamma]|A &\Rightarrow A & \text{Given} \\
\Gamma &\vdash s : A \gg C'' \ q \vdash \Delta'' & \text{By above equality} \\
\Gamma &\vdash s : A \gg C \ j \vdash \Delta & \text{Subderivation} \\
C'' &\Rightarrow C \text{ and } q = j \text{ and } \Delta'' = \Delta & \text{By Theorem \[\text{DeclSpineRecover}\]}
\end{align*}\]

We have thus shown the above “for all \(C'\ldots\)” statement.

\[\Rightarrow [\Omega]|\Gamma \vdash [\Omega]|s : [\Omega]|A \gg [\Omega]|C \ [!] \quad \text{By \[\text{DeclSpineRecover}\]}\]
Proof of Theorem 9 (Soundness of Algorithmic Typing)

Case
\[ \Gamma \vdash s : A p \Rightarrow C q \vdash \Delta \quad ((p = f) \text{ or } (q = !) \text{ or } (\text{FEV}(C) \neq \emptyset)) \]
\[ \Gamma \vdash s : A p \Rightarrow C [q] \vdash \Delta \]
\[ \Gamma \vdash s : A p \Rightarrow C q \vdash \Delta \]
Subderivation
\[ [\Omega] \vdash [\Omega] s : [\Omega] A p \Rightarrow [\Omega] C q \quad \text{By i.h.} \]
\[ [\Omega] \vdash [\Omega] s : [\Omega] A p \Rightarrow [\Omega] C [q] \quad \text{By DeclSpinePass} \]

Case
\[ \Gamma \vdash : A p \Rightarrow A p \dashv \Gamma \]
EmptySpine
\[ [\Omega] \vdash : [\Omega] A p \Rightarrow [\Omega] A p \quad \text{By DeclEmptySpine} \]

Case
\[ \Gamma \vdash e_0 \Leftarrow A_1 p \vdash \Theta \quad \Theta \vdash s_0 : [\Theta] A_2 p \Rightarrow C q \vdash \Delta \]
\[ \Gamma \vdash e_0 s_0 : A_1 \rightarrow A_2 p \Rightarrow C q \vdash \Delta \]
\[ \Delta \rightarrow \Omega \quad \Theta \rightarrow \Delta \quad \text{Given} \]
\[ \Theta \rightarrow \Omega \quad \text{By Lemma [51] (Typing Extension)} \]
\[ \Theta \rightarrow \Omega \quad \text{By Lemma [33] (Extension Transitivity)} \]
\[ \Gamma \vdash e_0 \Leftarrow A_1 p \vdash \Theta \quad \text{Subderivation} \]
\[ [\Omega] \Theta \vdash [\Omega] e_0 \Leftarrow [\Omega] A_1 p \quad \text{By i.h.} \]
\[ [\Omega] \Theta = [\Omega] \Delta \quad \text{By Lemma [56] (Confluence of Completeness)} \]
\[ [\Omega] \Delta \vdash [\Omega] e_0 \Leftarrow [\Omega] A_1 p \quad \text{By above equality} \]
\[ [\Omega] \Delta \vdash [\Omega] s_0 : [\Omega] A_2 \quad \text{Subderivation} \]
\[ [\Omega] \Delta \vdash [\Omega] s_0 : [\Omega] A_2 \rightarrow [\Omega] A_1 p \Rightarrow C q \quad \text{By i.h.} \]
\[ [\Omega] \Delta \vdash [\Omega] s_0 : [\Omega] A_2 \rightarrow [\Omega] A_1 p \Rightarrow C q \quad \text{By above equality} \]
\[ [\Omega] \Delta \vdash [\Omega] (e_0 s_0) : \Theta A_1 \rightarrow A_2 p \Rightarrow C q \quad \text{By Decl↓Spine} \]
\[ [\Omega] \Delta \vdash [\Omega] (e_0 s_0) : \Theta A_1 \rightarrow A_2 p \Rightarrow C q \quad \text{By def. of subst.} \]

Case
\[ \Gamma \vdash e_0 \Leftarrow A_k p \vdash \Delta \]
\[ \Gamma \vdash \text{inj}_k e_0 \Leftarrow A_1 + A_2 p \vdash \Delta \quad \text{+I_k} \]
\[ \Gamma \vdash e_0 \Leftarrow A_k p \vdash \Delta \quad \text{Subderivation} \]
\[ [\Omega] \Delta \vdash [\Omega] e_0 \Leftarrow [\Omega] A_k p \quad \text{By i.h.} \]
\[ [\Omega] \Delta \vdash \text{inj}_k [\Omega] e_0 \Leftarrow ([\Omega] A_1) + ([\Omega] A_2) p \quad \text{By Decl+I_k} \]
\[ [\Omega] \Delta \vdash [\Omega] (\text{inj}_k e_0) \Leftarrow [\Omega] (A_1 + A_2) p \quad \text{By def. of substitution} \]

Case
\[ \Gamma[\alpha_1 : \ast, \alpha_2 : \\ast, \alpha : \\ast = \alpha_1 + \alpha_2] \vdash e_0 \Leftarrow \alpha_k \vdash \Delta \]
\[ \Gamma[\alpha : \ast] \vdash \text{inj}_k e_0 \Leftarrow \alpha \vdash \Delta \quad \text{+I_k\alpha} \]
Proof of Theorem 9 (Soundness of Algorithmic Typing)

\[ \Gamma, \ldots, \alpha : \ast \vdash e_0 \iff \alpha_k \gamma \vdash \Delta \]

Subderivation

\[ |\Omega| \alpha \vdash |\Omega| e_0 \iff |\Omega| \alpha_k \gamma \vdash \Delta \]

By i.h.

\[ |\Omega| \alpha \vdash \text{inj}_k [|\Omega| e_0] \Rightarrow ([|\Omega| \alpha_1] + ([|\Omega| \alpha_2] \gamma \vdash \Delta \]

By \text{Decl} + ||_k

\[ ([|\Omega| \alpha_1] + ([|\Omega| \alpha_2] = [|\Omega| \alpha \gamma \vdash \Delta \]

Similar to the \(-|\alpha|\) case (above)

\[ |\Omega| \alpha \vdash [|\Omega| e_0] \Rightarrow [|\Omega| \alpha \gamma \vdash \Delta \]

By above equality / def. of subst.

\begin{itemize}
  \item Case \( \Gamma \vdash e_1 \iff A_1 \ p \vdash \Theta \quad \Theta \vdash e_2 \iff [\Theta] A_2 \ p \vdash \Delta \)
    \begin{itemize}
      \item Subderivation
        \[ \Theta \vdash e_2 \iff [\Theta] A_2 \ p \vdash \Delta \]
        By Lemma 51 (Typing Extension)
      \item \( \Theta \rightarrow \Delta \)
        By Lemma 51 (Typing Extension)
      \item \( \Theta \rightarrow \Omega \)
        By Lemma 33 (Extension Transitivity)
      \item \( \Gamma \vdash e_1 \iff A_1 \ p \vdash \Theta \)
        By i.h.
      \item \[ |\Omega| \Theta \vdash |\Omega| e_1 \iff |\Omega| A_1 \ p \]
        By Lemma 33 (Extension Transitivity)
      \item \[ |\Omega| \alpha \vdash [|\Omega| e_1] \iff [|\Omega| A_1] \ p \]
        Given
    \end{itemize}
    \[ \Gamma \vdash (e_1, e_2) \iff A_1 \times A_2 \ p \vdash \Delta \]
    By def. of substitution
\end{itemize}

\begin{itemize}
  \item Case \( \Gamma, \ldots, \alpha : \ast \vdash e_1 \iff \alpha_1 \gamma \vdash \Delta \)
    \[ \Theta \vdash e_2 \iff [\Theta] \alpha_2 \gamma \vdash \Delta \]
    By i.h.
    \[ \Theta \rightarrow \Delta \]
    By Lemma 51 (Typing Extension)
    \[ \Theta \rightarrow \Omega \]
    By Lemma 33 (Extension Transitivity)
    \[ \Gamma \vdash e_1 \iff \alpha_1 \gamma \vdash \Delta \]
    Subderivation
    \[ |\Omega| \Theta \vdash |\Omega| e_1 \iff |\Omega| \alpha_1 \gamma \vdash \Delta \]
    By i.h.
    \[ |\Omega| \Theta \vdash |\Omega| \alpha \gamma \vdash \Delta \]
    By Lemma 56 (Confluence of Completeness)
    \[ |\Omega| \alpha \vdash |\Omega| e_1 \iff |\Omega| \alpha_1 \gamma \vdash \Delta \]
    By above equality
    \[ \Theta \vdash e_2 \iff [\Theta] \alpha_2 \gamma \vdash \Delta \]
    Subderivation
    \[ |\Omega| \alpha \vdash [|\Omega| e_2] \iff [|\Omega| \alpha_2 \gamma \vdash \Delta \]
    By i.h.
    \[ |\Omega| \Theta \alpha_2 \gamma \vdash [\Theta] \alpha_2 \gamma \vdash \Delta \]
    By Lemma 29 (Substitution Monotonicity)
    \[ |\Omega| \alpha \vdash [|\Omega| e_2] \iff [|\Omega| \alpha_2 \gamma \vdash \Delta \]
    By above equality
    \[ |\Omega| \alpha \vdash ([|\Omega| e_1], [\Omega] e_2) \iff ([|\Omega| \alpha_1] \times [\Omega] \alpha_2 \gamma \vdash \Delta \]
    By \text{Decl} + ||_k
    \[ ([|\Omega| \alpha_1] \times [\Omega] \alpha_2 \gamma \vdash \Delta \]
    Similar to the \(-|\alpha|\) case (above)
    \[ ([|\Omega| \alpha_1] \times [\Omega] \alpha_2 \gamma \vdash \Delta \]
    By above equality
Proof of Theorem 9 (Soundness of Algorithmic Typing)  \[\text{thm:typing-soundness}\]

- **Case** $\Gamma[\Delta_2 : \star, \Delta_1 : \star, \Delta : \Delta_1 \rightarrow \Delta_2] \vdash e_0 \ s_0 : (\Delta_1 \rightarrow \Delta_2) \ j \supset C \ j \vdash \Delta$

  - **Case** Case $\ldots, \Delta : \Delta_1 \rightarrow \Delta_2] \vdash e_0 \ s_0 : (\Delta_1 \rightarrow \Delta_2) \ j \supset C \ j \vdash \Delta$

    - **Case** By Lemma 7 (Soundness of Match Coverage), $B$
    - **Case** By Lemma 5 (Determinacy of Typing), we know $\Theta$
    - **Case** Assume $\Theta$
    - **Case** Assume $\Delta \vdash \Pi :: [\Theta]B \ q \iff [\Theta]C \ p \vdash \Delta$  

      - **Case** $\Gamma \vdash e_0 \Rightarrow B \ ! - \Theta$

        - **Case** Subderivation
        - **Case** By Lemma 51 (Typing Extension)
        - **Case** $\Theta \rightarrow \Delta$
        - **Case** By Lemma 33 (Extension Transitivity)
        - **Case** $\Theta \rightarrow \Omega$
        - **Case** By Lemma 56 (Confluence of Completeness)
        - **Case** $\Theta \vdash \Pi :: [\Theta]B \iff [\Theta]C \ p \vdash \Delta$

      - **Case** $\Gamma \vdash e_0 \Rightarrow B \ ! - \Theta$

        - **Case** Subderivation
        - **Case** By Lemma 63 (Well-Formed Outputs of Typing) (Synthesis)
        - **Case** $\Theta \vdash B \ ! - \Theta$

      - **Case** $\Gamma \vdash C \ p \vdash \Delta$

        - **Case** Given
        - **Case** By Lemma 51 (Typing Extension)
        - **Case** $\Gamma \vdash \Theta$

      - **Case** $\Theta \vdash C \ p \vdash \Delta$

        - **Case** By Lemma 41 (Extension Weakening for Principal Typing)
        - **Case** $\Theta \vdash C \ p \vdash \Delta$

      - **Case** $\Theta \vdash [\Theta]C \ p \vdash [\Theta]C \ p$

    - **Case** $\Theta \vdash [\Theta]C \ p \vdash [\Theta]C \ p$

      - **Case** By Lemma 40 (Right-Hand Subst. for Principal Typing)

      - **Case** $\Theta \vdash [\Theta]C \ p \vdash [\Theta]C \ p$

      - **Case** By Lemma 29 (Substitution Monotonicity)

      - **Case** By Lemma 29 (Substitution Monotonicity)

      - **Case** By above equalities

Assume $\Omega$ such that $\Delta \rightarrow \Omega$.
Assume $D$ such that $[\Omega]\Delta \vdash e \Rightarrow D \ q$.
Hence $[\Omega]\Gamma \vdash e \Rightarrow D \ q$.
By Theorem 12, there exist $B$ and $\Theta'$ such that $\Gamma \vdash e_0 \Rightarrow B \ ! - \Theta'$ and $\Omega \rightarrow \Delta' \rightarrow \Omega'$ and $D = [\Omega']B'$ and $B' = [\Theta']B$.
By Lemma 5 (Determinacy of Typing), we know $\Theta' = \Theta$ and $B' = B$, which means $D = [\Omega]B$.
By Lemma 7 (Soundness of Match Coverage), $[\Omega]\Delta \vdash [\Omega]\Pi :: [\Omega]B \ q$.
Hence $[\Omega]\Delta \vdash [\Omega]\Pi :: [\Omega]B \ q$.
By rule DeclCase, $[\Omega]\Delta \vdash [\Omega]\Pi :: [\Theta]C \ p$

Part (v):

- **Case** MatchEmpty Apply rule DeclMatchEmpty

- **Case** Case $\ldots, \Delta : \Delta_1 \vdash e_0 \ s_0 : (\Delta_1 \rightarrow \Delta_2) \ j \supset C \ j \vdash \Delta$

  - **Case** By i.h. and DeclMatchBase

  - **Case** By i.h. and DeclMatchBase
Proof of Theorem 9 (Soundness of Algorithmic Typing)

• Case

\[
\Gamma \vdash \pi :: \tilde{A} q \leftarrow C p \vdash \Theta \quad \Theta \vdash \Pi' :: \tilde{A} q \leftarrow C p \vdash \Delta
\]

\[
\Gamma \vdash \pi \downarrow \Pi' :: \tilde{A} q \leftarrow C p \vdash \Delta
\]

Apply the i.h. to each premise, using lemmas for well-formedness under \(\Theta\); then apply \text{DeclMatchNeg}.

• Cases

Apply the i.h. and the corresponding declarative match rule.

• Cases \text{MatchNil}, \text{MatchPlus}

We have \(\Gamma \vdash \tilde{A}!\) types, so the first type in \(\tilde{A}\) has no free existential variables. Apply the i.h. and the corresponding declarative match rule.

• Case

\[A\text{ not headed by }\land\text{ or }\exists\]

\[
\Gamma, z : A! \vdash \tilde{\rho} \Rightarrow e' :: \tilde{A} q \leftarrow C p \vdash \Delta, z : A!, \Delta'
\]

\[
\Gamma \vdash z, \tilde{\rho} \Rightarrow e :: A, \tilde{A} q \leftarrow C p \vdash \Delta
\]

Construct \(\Omega'\) and show \(\Delta, z : A!, \Delta' \rightarrow \Omega'\) as in the \(\rightarrow\) case. Use the i.h., then apply rule \text{DeclMatchNeg}.

Part (vi):

• Case

\[
\Gamma / \sigma \vdash \tau : \kappa \vdash \bot
\]

\[
\Gamma / \sigma \vdash \tilde{\rho} e \vdash \tilde{A}! \leftarrow C p \downarrow \Gamma
\]

\[
\Gamma / \sigma \vdash \tau : \kappa \vdash \bot
\]

Subderivation

\[
[\Gamma](\sigma = \tau) = (\sigma = \tau)
\]

Given

\[
(\sigma = \tau) = [\Gamma](\sigma = \tau)
\]

Given

\[
\Omega(\sigma = \tau)
\]

By Lemma 29 (Substitution Monotonicity) (i)

\[
\text{mgu}(\sigma, \tau) = \bot
\]

By Lemma 90 (Soundness of Equality Elimination)

\[
\text{mgu}(\Omega(\sigma), [\Omega] \tau) = \bot
\]

By above equality

\[
\xRightarrow{\text{by i.h.}} [\Omega](\sigma = \tau) \vdash [\Omega](\tilde{\rho} e) :: [\Omega] \tilde{A} \leftarrow [\Omega] C p
\]

By \text{DeclMatchNeg}.

• Case

\[
\Gamma, \Theta \vdash \sigma :: \tau : \kappa \vdash \Gamma'
\]

\[
\Gamma' \vdash \tilde{\rho} e \vdash \tilde{A} q \leftarrow C p \vdash \Delta', \Gamma, \Theta
\]

\[
\Gamma / \sigma \vdash \tau : \kappa \vdash \Gamma'
\]

Subderivation

\[
(\sigma = \tau) = [\Gamma](\sigma = \tau)
\]

Given

\[
(\sigma = \tau) = [\Gamma](\sigma = \tau)
\]

By Lemma 29 (Substitution Monotonicity) (i)

\[
\Gamma' = ([\Gamma, \Theta], \Theta)
\]

By Lemma 90 (Soundness of Equality Elimination)

\[
\Theta = ((\alpha_1 = t_1), \ldots, (\alpha_n = t_n))
\]

"""""""

\[
\theta = \text{mgu}(\Omega(\sigma), [\Omega] \tau)
\]

"""""""

\[
[\Omega, \Theta, t'] = [\theta](\Omega, [\Theta] t')
\]

"""""""

\(\forall \Omega, \Theta \vdash t' : \kappa'

\[
\Gamma, \Theta \vdash \tilde{\rho} e \vdash \tilde{A} q \leftarrow C p \vdash \Delta', \Gamma, \Theta
\]

Subderivation

\[
[\Omega, \Theta](\Delta', \Theta) \vdash [\Omega, \Theta](\tilde{\rho} e) :: [\Omega, \Theta] \tilde{A} \leftarrow [\Omega, \Theta] C p
\]

By i.h.
Proof of Theorem 9 (Soundness of Algorithmic Typing)

**K’ Completeness**

**K’.1 Completeness of Auxiliary Judgments**

Lemma 92 (Completeness of Instantiation).

*Given* \( \Gamma \rightarrow \Omega \) and \( \text{dom}(\Gamma) = \text{dom}(\Omega) \) and \( \Gamma \vdash \tau : \kappa \) and \( \tau = [\Gamma] \tau \) and \( \hat{\alpha} \in \text{unsolved}(\Gamma) \) and \( \hat{\alpha} \notin \text{FV}(\tau) \):

If \( [\Omega] \hat{\alpha} = [\Omega] \tau \) then there are \( \Delta, \Omega' \) such that \( \Omega \rightarrow \Omega' \) and \( \Delta \rightarrow \Omega' \) and \( \text{dom}(\Delta) = \text{dom}(\Omega') \) and \( \Gamma \vdash \hat{\alpha} := \tau : \kappa \rightarrow \Delta \).

*Proof.* By induction on \( \tau \).

We have \( [\Omega] \Gamma \vdash [\Omega] \hat{\alpha} \leq^{P} [\Omega] \Lambda \). We now case-analyze the shape of \( \tau \).

- **Case** \( \tau = \hat{\beta} \):
  
  \( \hat{\alpha} \notin \text{FV}(\hat{\beta}) \) Given
  
  \( \hat{\alpha} \neq \hat{\beta} \) From definition of \( \text{FV}(\cdot) \)
  
  \( \hat{\beta} \in \text{unsolved}(\Gamma) \) From \( [\Gamma] \hat{\beta} = \hat{\beta} \)
  
  Let \( \Omega' = \Omega \).

  \[\varepsilon \Gamma \rightarrow \Omega'\] By Lemma 32 (Extension Reflexivity)

Now consider whether \( \hat{\alpha} \) is declared to the left of \( \hat{\beta} \), or vice versa.

- **Case** \( \Gamma = \Gamma_0[\hat{\alpha} : \kappa][\hat{\beta} : \kappa] \):
  
  Let \( \Delta = \Gamma_0[\hat{\alpha} : \kappa][\hat{\beta} : \kappa = \hat{\alpha}] \).

  \( \Gamma \vdash \hat{\alpha} := \hat{\beta} : \kappa \rightarrow \Delta \) By \( \text{InstReach} \)

  \( [\Omega] \hat{\alpha} = [\Omega] \hat{\beta} \) Given

  \( \Gamma \rightarrow \Omega \) Given

  \[\varepsilon \Delta \rightarrow \Omega \] By Lemma 22 (Parallel Extension Solution)

  \[\varepsilon \text{dom}(\Delta) = \text{dom}(\Omega') \] dom(\(\Delta\)) = dom(\(\Gamma\)) and dom(\(\Omega')\)) = dom(\(\Omega\))

- **Case** \( \Gamma = \Gamma_0[\hat{\beta} : \kappa][\hat{\alpha} : \kappa] \):
  
  Similar, but using \( \text{InstSolve} \) instead of \( \text{InstReach} \)

- **Case** \( \tau = \alpha \):
  
  We have \( [\Omega] \hat{\alpha} = \alpha \), so (since \( \Omega \) is well-formed), \( \alpha \) is declared to the left of \( \hat{\alpha} \) in \( \Omega \).

  We have \( \Gamma \rightarrow \Omega \).

  By Lemma 21 (Reverse Declaration Order Preservation), we know that \( \alpha \) is declared to the left of \( \hat{\alpha} \) in \( \Gamma \); that is, \( \Gamma = \Gamma_1[\alpha : \kappa][\hat{\alpha} : \kappa] \).
Proof of Lemma 92 (Completeness of Instantiation)

Let $\Delta = \Gamma_1[\alpha : \kappa][\hat{\alpha} : \kappa = \alpha]$ and $\Omega' = \Omega$.

By $\text{InstSolve}$, $\Gamma_1[\alpha : \kappa][\hat{\alpha} : \kappa = \alpha] \vdash \hat{\alpha} : \kappa = \Delta$.

By Lemma 27 (Parallel Extension Solution), $\Gamma_1[\alpha : \kappa][\hat{\alpha} : \kappa = \alpha] \rightarrow \Omega$.

We have $\text{dom}(\Delta) = \text{dom}(\Gamma)$ and $\text{dom}(\Omega') = \text{dom}(\Omega)$; therefore, $\text{dom}(\Delta) = \text{dom}(\Omega')$.

- Case $\tau = 1$:
  Similar to the $\tau = \alpha$ case, but without having to reason about where $\alpha$ is declared.

- Case $\tau = 0$:
  Similar to the $\tau = 1$ case.

- Case $\tau = \tau_1 \oplus \tau_2$:
  
  $\Omega[\hat{\alpha} = [\Omega](\tau_1 \oplus \tau_2)]$ 
  
  Given

  $\tau_1 \oplus \tau_2 = [\Gamma](\tau_1 \oplus \tau_2)$ 
  
  By definition of substitution

  $\tau_1 = [\Gamma]\tau_1$ 
  
  By definition of substitution and congruence

  $\tau_2 = [\Gamma]\tau_2$ 
  
  Similarly

  $\hat{\alpha} \notin \text{FV}(\tau_1 \oplus \tau_2)$ 
  
  Given

  $\hat{\alpha} \notin \text{FV}(\tau_1)$ 
  
  From definition of $\text{FV}(\cdot)$

  $\hat{\alpha} \notin \text{FV}(\tau_2)$ 
  
  Similarly

  $\Gamma = \Gamma_0[\hat{\alpha} : \star]$ 
  
  By $\hat{\alpha} \in \text{unsolved}(\Gamma)$

  $\Gamma \rightarrow \Omega$ 
  
  Given

  $\Gamma_0[\hat{\alpha} : \star] \rightarrow \Gamma_0[\hat{\alpha}_2 : \star, \hat{\alpha}_1 : \star, \hat{\alpha} : \star]$ 
  
  By Lemma 23 (Deep Evar Introduction) (i) twice

  $\ldots, \hat{\alpha}_2, \hat{\alpha}_1 \vdash \hat{\alpha}_1 \oplus \hat{\alpha}_2 : \star$ 
  
  Straightforward

  $\Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha}] 
  \rightarrow \Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \oplus \hat{\alpha}_2]$ 
  
  By Lemma 23 (Deep Evar Introduction) (ii)

  $\Gamma_0[\hat{\alpha}] 
  \rightarrow \Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \oplus \hat{\alpha}_2]$ 
  
  By Lemma 33 (Extension Transitivity)

(In the last few lines above, and the rest of this case, we omit the "\star" annotations in contexts.)

Since $\hat{\alpha} \in \text{unsolved}(\Gamma)$ and $\Gamma \rightarrow \Omega$, we know that $\Omega$ has the form $\Omega_0[\hat{\alpha} = \tau_0]$.

To show that we can extend this context, we apply Lemma 23 (Deep Evar Introduction) (iii) twice to introduce $\hat{\alpha}_2 = \tau_2$ and $\hat{\alpha}_1 = \tau_1$, and then Lemma 28 (Parallel Variable Update) to overwrite $\tau_0$:

$$\Omega_0[\hat{\alpha} = \tau_0] \rightarrow \Omega_0[\hat{\alpha}_2 = \tau_2, \hat{\alpha}_1 = \tau_1, \hat{\alpha} = \hat{\alpha}_1 \oplus \hat{\alpha}_2]$$

We have $\Gamma \rightarrow \Omega$, that is,

$$\Gamma_0[\hat{\alpha}] \rightarrow \Omega_0[\hat{\alpha} = \tau_0]$$

By Lemma 26 (Parallel Admissibility) (i) twice, inserting unsolved variables $\hat{\alpha}_2$ and $\hat{\alpha}_1$ on both contexts in the above extension preserves extension:

$$\Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha}] \rightarrow \Omega_0[\hat{\alpha}_2 = \tau_2, \hat{\alpha}_1 = \tau_1, \hat{\alpha} = \tau_0]$$

By Lemma 26 (Parallel Admissibility) (ii) twice

$$\Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \oplus \hat{\alpha}_2] \rightarrow \Omega_0[\hat{\alpha}_2 = \tau_2, \hat{\alpha}_1 = \tau_1, \hat{\alpha} = \hat{\alpha}_1 \oplus \hat{\alpha}_2]$$

By Lemma 28 (Parallel Variable Update)

Since $\hat{\alpha} \notin \text{FV}(\tau)$, it follows that $[\Gamma_1] \tau = [\Gamma] \tau = \tau$.

Therefore $\hat{\alpha}_1 \notin \text{FV}(\tau_1)$ and $\hat{\alpha}_1, \hat{\alpha}_2 \notin \text{FV}(\tau_2)$.

By Lemma 53 (Completing Completeness) (i) and (iii), $[\Omega_1] \Gamma_1 = [\Omega] \Gamma$ and $[\Omega_1] \hat{\alpha}_1 = \tau_1$.

By i.h., there are $\Delta_2$ and $\Omega_2$ such that $\Gamma_1 \vdash \hat{\alpha}_1 : \kappa \rightarrow \Delta_2$ and $\Delta_2 \rightarrow \Omega_2$ and $\Omega_1 \rightarrow \Omega_2$. 
Proof of Lemma 92 (Completeness of Instantiation)\lem:instantiation-completeness

Next, note that \([\Delta_2]|\Delta_2|\tau_2 = |\Delta_2|\tau_2\).
By Lemma 64 (Left Unsolvedness Preservation), we know that \(\Delta_2 \in \text{unsolved}(\Delta_2)\).
By Lemma 65 (Left Free Variable Preservation), we know that \(\Delta_2 \notin \text{FV}(|\Delta_2|\tau_2)\).
By Lemma 33 (Extension Transitivity), \(\Omega \rightarrow \Omega_2\).
We know \([\Omega_2]|\Delta_2 = [\Omega]|\Gamma\) because:
\[
[\Omega_2]|\Delta_2 = [\Omega_2]|\Omega_2 \quad \text{By Lemma 54 (Completing Stability)}
= [\Omega]|\Omega \quad \text{By Lemma 55 (Completing Completeness) (iii)}
= [\Omega]|\Gamma \quad \text{By Lemma 54 (Completing Stability)}
\]
By Lemma 55 (Completing Completeness) (i), we know that \([\Omega_2]|\Delta_2 = |\Omega_1|\Delta_2 = |\tau_2\).
By Lemma 55 (Completing Completeness) (i), we know that \([\Omega_2]|\tau_2 = |\Omega]|\tau_2\).
Hence we know that \([\Delta_2]|\Delta_2 \leq [\Omega_2]|\tau_2\).
By i.h., we have \(\Delta \rightarrow \Omega\) and \(\Delta_2 \rightarrow \Delta_2 \leq [\Omega_2]|\tau_2\).
By rule InstBin, \(\Gamma \vdash \hat{\alpha} : \tau : \kappa \rightarrow \Delta\).
By Lemma 33 (Extension Transitivity), \(\Omega \rightarrow \Omega\).

- Case \(\tau = \text{succ}(\tau_0)\):
  Similar to the case for \(\tau_1 \oplus \tau_2\) case, but simpler. \(\square\)

Lemma 93 (Completeness of \text{Checkeq}).

Given \(\Gamma \rightarrow \Omega\) and \(\text{dom}(\Gamma) = \text{dom}(\Omega)\)
and \(\Gamma \vdash \sigma : \kappa\) and \(\Gamma \vdash \tau : \kappa\)
and \(\Omega|\sigma = |\Omega|\tau\)
then \(\Gamma \vdash [\Gamma]|\sigma = [\Gamma]|\tau : \kappa \rightarrow \Delta\)
where \(\Delta \rightarrow \Omega\) and \(\text{dom}(\Delta) = \text{dom}(\Omega')\) and \(\Omega \rightarrow \Omega'\).

Proof. By mutual induction on the sizes of \([\Gamma]|\sigma\) and \([\Gamma]|\tau\).
We distinguish cases of \([\Gamma]|\sigma\) and \([\Gamma]|\tau\).

- Case \([\Gamma]|\sigma = [\Gamma]|\tau = 1\):
  \(\Gamma \vdash 1 \equiv 1 : * \vdash \Gamma\)
  By \text{CheckeqUnit}
  Let \(\Omega' = \Omega\).
  \(\Gamma \rightarrow \Omega\)
  \(\Delta \rightarrow \Omega\)
  \(\text{dom}(\Gamma) = \text{dom}(\Omega)\)
  \(\Omega \rightarrow \Omega'\)
  By Lemma 32 (Extension Reflexivity)

- Case \([\Gamma]|\sigma = [\Gamma]|\tau = 0\):
  Similar to the case for 1, applying \text{CheckeqZero} instead of \text{CheckeqUnit}

- Case \([\Gamma]|\sigma = [\Gamma]|\tau = \alpha\):
  Similar to the case for 1, applying \text{CheckeqVar} instead of \text{CheckeqUnit}

- Case \([\Gamma]|\sigma = \hat{\alpha}\) and \([\Gamma]|\tau = \hat{\beta}\):
  - If \(\hat{\alpha} = \hat{\beta}\): Similar to the case for 1, applying \text{CheckeqVar} instead of \text{CheckeqUnit}
  - If \(\hat{\alpha} \neq \hat{\beta}\):
Proof of Lemma 93 (Completeness of Checkeq)

\[ \Gamma \rightarrow \Omega \]
\[ \hat{\alpha} \notin FV(\hat{\beta}) \]

Given
\[ [\Omega] \sigma = [\Omega] t \]
\[ [\Omega][\Gamma] \sigma = [\Omega][\Gamma] t \]
\[ [\Omega][\alpha] = [\Omega][\Gamma] t \]
\[ \text{dom}(\Gamma) = \text{dom}(\Omega) \]

Given
\[ \Gamma \vdash \hat{\alpha} := [\Gamma] t : \kappa \vdash \Delta \]

By Lemma 29 (Substitution Monotonicity) (i) twice
\[ \Omega \rightarrow \Omega' \]
\[ \Delta \rightarrow \Omega' \]
\[ \text{dom}(\Delta) = \text{dom}(\Omega') \]

By Lemma 92 (Completeness of Instantiation) By CheckeqInstL

• Case \( [\Gamma] \sigma = \hat{\alpha} \) and \( [\Gamma] t = 1 \) or zero or \( \alpha \):
   Similar to the previous case, except:

\[ \hat{\alpha} \notin FV(\hat{\beta}) \]

By definition of FV(–)

and similarly for 1 and \( \alpha \).

• Case \( [\Gamma] t = \hat{\alpha} \) and \( [\Gamma] \sigma = 1 \) or zero or \( \alpha \): Symmetric to the previous case.

• Case \( [\Gamma] \sigma = \hat{\alpha} \) and \( [\Gamma] t = \text{succ}([\Gamma] t_0) \):

If \( \hat{\alpha} \notin FV([\Gamma] t_0) \), then \( \hat{\alpha} \notin FV([\Gamma] t) \). Proceed as in the previous several cases.

The other case, \( \hat{\alpha} \in FV([\Gamma] t_0) \), is impossible:

We have \( \hat{\alpha} \notin [\Gamma] t_0 \).
Therefore \( \hat{\alpha} \text{ succ}([\Gamma] t_0) \), that is, \( \hat{\alpha} \text{ succ}([\Gamma] t) \).
By a property of substitutions, \( [\Omega][\alpha] \text{ succ}([\Omega][\Gamma] t) \).
Since \( \Gamma 
\rightarrow \Omega \), by Lemma 29 (Substitution Monotonicity) (i), \( [\Omega][\Gamma] t = [\Omega] t \), so \( [\Omega][\alpha] \text{ succ}([\Omega] t \).
But it is given that \([\Omega][\alpha] = [\Omega] \), a contradiction.

• Case \( [\Gamma] t = \hat{\alpha} \) and \( [\Gamma] \sigma = \text{succ}([\Gamma] \sigma_0) \): Symmetric to the previous case.

• Case \( [\Gamma] \sigma = [\Gamma] \sigma_1 \oplus [\Gamma] \sigma_2 \) and \( [\Gamma] t = [\Gamma] t_1 \oplus [\Gamma] t_2 \):

\[ \Gamma \rightarrow \Omega \]
\[ \Theta \rightarrow [\Theta][\Gamma] \sigma_1 : \star \vdash \Theta \]
\[ \Omega \rightarrow [\Omega][\Gamma] \sigma_2 : \star \vdash \Delta \]

Given
By i.h.
""
""
By i.h.
""
""
By Lemma 33 (Extension Transitivity)

\[ \Gamma \vdash [\Gamma] \sigma_1 \oplus [\Gamma] \sigma_2 : [\Gamma] t_1 \oplus [\Gamma] t_2 \vdash \star \vdash \Delta \]

By CheckeqBin
Proof of \textbf{Lemma 93} (Completeness of Checkeq). 
\textbf{Lemma 94} (Completeness of Elimeq).
If \([\Gamma] \sigma = \tau\) and \([\Gamma] t = t\) and \(\Gamma \vdash \sigma : \kappa\) and \(\Gamma \vdash t : \kappa\) and \(\text{FEV}(\sigma) \cup \text{FEV}(t) = \emptyset\) then:

\begin{enumerate}[(1)]
  \item If \(\text{mgu}(\sigma, t) = \emptyset\) then \(\Gamma / \sigma \equiv t : \kappa \vdash \Delta\) where \(\Delta\) has the form \(\alpha_1 = t_1, \ldots, \alpha_n = t_n\) and for all \(u\) such that \(\Gamma \vdash u : \kappa\), it is the case that \([\Gamma, \Delta] u = \emptyset([\Gamma] u)\).
  \item If \(\text{mgu}(\sigma, t) = \bot\) (that is, no most general unifier exists) then \(\Gamma / \sigma \equiv t : \kappa \vdash \bot\).
\end{enumerate}

\textbf{Proof}. By induction on the structure of \([\Gamma] \sigma\) and \([\Gamma] t\).

\begin{itemize}
  \item Case \([\Omega] \sigma = t = \text{zero}\):
    \begin{itemize}
      \item \(\text{mgu}(\text{zero}, \text{zero}) = \cdot\) By properties of unification
      \(\Gamma / \text{zero} \equiv \text{zero} : \mathbb{N} \vdash \Gamma\) By rule \texttt{ElimeqZero}
      Suppose \(\Gamma \vdash u : \kappa'\).
      \(\Gamma, \Delta] u = [\Gamma] u\) where \(\Delta = \cdot\)
      \(= \emptyset([\Gamma] u)\) where \(\emptyset\) is the identity
    \end{itemize}
  \item Case \(\sigma = \text{succ}(\sigma')\) and \(t = \text{succ}(t')\):
    \begin{itemize}
      \item Case \(\text{mgu}(\text{succ}(\sigma'), \text{succ}(t')) = \emptyset\):
        \(\text{mgu}(\sigma', t') = \text{mgu}(\text{succ}(\sigma'), \text{succ}(t')) = \emptyset\) By properties of unification
        \(\text{succ}(\sigma') = [\Gamma] \text{succ}(\sigma')\) Given
        \(= \text{succ}(\text{succ}(\sigma'))\) By definition of substitution
        \(\sigma' = [\Gamma] \sigma'\) By injectivity of successor
        \(\text{succ}(t') = [\Gamma] \text{succ}(t')\) Given
        \(= \text{succ}(\text{succ}(t'))\) By definition of substitution
        \(t' = [\Gamma] t'\) By injectivity of successor
        \(\Gamma / \sigma' \equiv t' : \mathbb{N} \vdash \Gamma, \Delta\) By i.h.
        \(\emptyset([\Gamma] u)\) for all \(u\) such that \(\ldots\)"
        \(\Gamma / \text{succ}(\sigma') \equiv \text{succ}(t') : \mathbb{N} \vdash \Gamma, \Delta\) By rule \texttt{ElimeqSucc}
      \item Case \(\text{mgu}(\text{succ}(\sigma'), \text{succ}(t')) = \bot\):
        \(\text{mgu}(\sigma', t') = \text{mgu}(\text{succ}(\sigma'), \text{succ}(t')) = \bot\) By properties of unification
        \(\text{succ}(\sigma') = [\Gamma] \text{succ}(\sigma')\) Given
        \(= \text{succ}(\text{succ}(\sigma'))\) By definition of substitution
        \(\sigma' = [\Gamma] \sigma'\) By injectivity of successor
        \(\text{succ}(t') = [\Gamma] \text{succ}(t')\) Given
        \(= \text{succ}(\text{succ}(t'))\) By definition of substitution
        \(t' = [\Gamma] t'\) By injectivity of successor
        \(\Gamma / \sigma' \equiv t' : \mathbb{N} \vdash \bot\) By i.h.
        \(\Gamma / \text{succ}(\sigma') \equiv \text{succ}(t') : \mathbb{N} \vdash \bot\) By rule \texttt{ElimeqSucc}
    \end{itemize}
\end{itemize}
Case $\sigma = \sigma_1 \oplus \sigma_2$ and $t = t_1 \oplus t_2$:

First we establish some properties of the subterms:

$$\begin{align*}
\sigma_1 \oplus \sigma_2 &= [\Gamma](\sigma_1 \oplus \sigma_2) & \text{Given} \\
&= [\Gamma]\sigma_1 \oplus [\Gamma]\sigma_2 & \text{By definition of substitution} \\
\Rightarrow [\Gamma]\sigma_1 &= \sigma_1 & \text{By injectivity of } \oplus \\
[\Gamma]\sigma_2 &= \sigma_2 & \text{By injectivity of } \oplus \\
t_1 \oplus t_2 &= [\Gamma](t_1 \oplus t_2) & \text{Given} \\
&= [\Gamma]t_1 \oplus [\Gamma]t_2 & \text{By definition of substitution} \\
\Rightarrow [\Gamma]t_1 &= t_1 & \text{By injectivity of } \oplus \\
[\Gamma]t_2 &= t_2 & \text{By injectivity of } \oplus \\
\end{align*}$$

- **Subcase** $\text{mgu}(\sigma, t) = \bot$:
  
  * **Subcase** $\text{mgu}(\sigma_1, t_1) = \bot$:
    
    $\Gamma \vdash \sigma_1 \triangleq t_1 : \kappa \dashv \bot$ \quad By i.h.
    $\Gamma \vdash \sigma_1 \oplus \sigma_2 \triangleq t_1 \oplus t_2 : \kappa \dashv \bot$ \quad By rule ElimeqBinBot

  * **Subcase** $\text{mgu}(\sigma_1, t_1) = \emptyset$ and $\text{mgu}(\theta_1(\sigma_2), \theta_1(t_2)) = \bot$:
    
    $\Gamma \vdash \sigma_1 \triangleq t_1 : \kappa \dashv \Gamma, \Delta_1$ \quad By i.h.
    $[\Gamma, \Delta_1]u = \theta_1([\Gamma]u)$ for all $u$ such that . . . 

    $[\Gamma, \Delta_1]\sigma_2 = \theta_1([\Gamma]\sigma_2)$ \quad Above line with $\sigma_2$ as $u$
    $= \theta_1(\sigma_2)$ \quad $[\Gamma]\sigma_2 = \sigma_2$
    $[\Gamma, \Delta_1]t_2 = \theta_1([\Gamma]t_2)$ \quad Above line with $t_2$ as $u$
    $= \theta_1(t_2)$ \quad Since $[\Gamma]\sigma_2 = \sigma_2$
    $\text{mgu}([\Gamma, \Delta_1]\sigma_2, [\Gamma, \Delta_1]t_2) = \theta_2$ \quad By transitivity of equality

    $[\Gamma, \Delta_1][\Gamma, \Delta_1]\sigma_2 = [\Gamma, \Delta_1]\sigma_2$ \quad By Lemma [29] (Substitution Monotonicity)
    $[\Gamma, \Delta_1][\Gamma, \Delta_1]t_2 = [\Gamma, \Delta_1]t_2$ \quad By Lemma [29] (Substitution Monotonicity)

    $\Gamma, \Delta_1 / [\Gamma, \Delta_1]\sigma_2 \triangleq [\Gamma, \Delta_1]t_2 : \kappa \dashv \bot$ \quad By i.h.
    $\Rightarrow \Gamma / \sigma_1 \oplus \sigma_2 \triangleq t_1 \oplus t_2 : \kappa \dashv \bot$ \quad By rule ElimeqBin

  - **Subcase** $\text{mgu}(\sigma, t) = \emptyset$:
    
    $\text{mgu}(\sigma_1 \oplus \sigma_2, t_1 \oplus t_2) = \emptyset = \emptyset \circ \emptyset$ \quad By properties of unifiers
    $\text{mgu}(\sigma_1, t_1) = \emptyset$ \quad $\Rightarrow$
    $\text{mgu}(\theta_1(\sigma_2), \theta_1(t_2)) = \emptyset$ \quad $\Rightarrow$
    $\Gamma / \sigma_1 \triangleq t_1 : \kappa \dashv \Gamma, \Delta_1$ \quad By i.h.

    * $[\Gamma, \Delta_1]u = \theta_1([\Gamma]u)$ for all $u$ such that . . . 

    $[\Gamma, \Delta_1]\sigma_2 = \theta_1([\Gamma]\sigma_2)$ \quad Above line with $\sigma_2$ as $u$
    $= \theta_1(\sigma_2)$ \quad $[\Gamma]\sigma_2 = \sigma_2$
    $[\Gamma, \Delta_1]t_2 = \theta_1([\Gamma]t_2)$ \quad Above line with $t_2$ as $u$
    $= \theta_1(t_2)$ \quad $[\Gamma]\sigma_2 = \sigma_2$
    $\text{mgu}([\Gamma, \Delta_1]\sigma_2, [\Gamma, \Delta_1]t_2) = \theta_2$ \quad By transitivity of equality
Proof of Lemma 94 (Completeness of Elimeq).

Let \( \Gamma, \Delta \) be a context.

1. By Lemma 29 (Substitution Monotonicity):
   \[ \Gamma, \Delta \sigma_1 \vdash \Gamma, \Delta \sigma_2 \]

2. By Lemma 29 (Substitution Monotonicity):
   \[ \Gamma, \Delta \vdash \Gamma, \Delta t \]

By i.h.:

\[ \Gamma, \Delta \vdash \Gamma, \Delta \sigma \]

By rule ElimeqBin:

Suppose \( \Gamma \vdash u : \kappa' \).

- Subcase \( \alpha \in \text{FV}(t) \):
  \[ \text{mg}u(\alpha, t) = \perp \]
  By properties of unification
  \[ \Gamma / \alpha \vdash t : \kappa \]
  By ElimeqUvarL.

- Subcase \( \alpha \notin \text{FV}(t) \):
  \[ \text{mg}u(\alpha, t) = [t/\alpha] \]
  By properties of unification
  \[ \Gamma \alpha = \alpha \]
  By ElimeqUvarL.

Suppose \( \Gamma \vdash u : \kappa' \).

\[ \Gamma, \alpha = \Gamma[u/\alpha] \]

By definition of substitution
\[ = \Gamma[t/\alpha][u] \]
By properties of substitution
\[ = [t/\alpha][u] \]
\[ = \Gamma t = t \]

- Case \( \sigma = \alpha \):
  Similar to previous case.

Lemma 95 (Substitution Upgrade).

If \( \Delta \) has the form \( \alpha_1 = t_1, \ldots, \alpha_n = t_n \)
and, for all \( u \) such that \( \Gamma \vdash u : \kappa \), it is the case that \( \Gamma, \Delta \vdash \theta(\Gamma u) \),
then:

1. If \( \Gamma \vdash A \) type then \( \Gamma, \Delta A = \theta(\Gamma A) \).
2. If \( \Gamma \rightarrow \Omega \) then \( \theta(\Gamma) \Omega = \theta(\Gamma \Omega) \).
3. If \( \Gamma \rightarrow \Omega \) then \( \theta(\Gamma, \Delta) = \theta(\Gamma \Delta) \).
4. If \( \Gamma \rightarrow \Omega \) then \( \theta(\Gamma, \Delta) e = \theta(\Gamma e) \).

Proof. Part (i): By induction on the given derivation, using the given “for all” at the leaves.

Part (ii): By induction on the given derivation, using part (i) in the Var case.

Part (iii): By induction on \( \Delta \). In the base case (\( \Delta = \) ), use part (ii). Otherwise, use the i.h. and the definition of context substitution.

Part (iv): By induction on \( e \), using part (i) in the \( e = (e_0 : A) \) case.
**Lemma 96** (Completeness of Propequiv).
Given $\Gamma \rightarrow \Omega$
and $\Gamma \vdash P \text{ prop}$ and $\Gamma \vdash Q \text{ prop}$
and $[\Omega]P = [\Omega]Q$
then $\Gamma \vdash [\Gamma]P \equiv [\Gamma]Q \rightarrow \Delta$
where $\Delta \rightarrow \Omega'$ and $\Omega \rightarrow \Omega'$.

*Proof.* By induction on the given derivations. There is only one possible case:

- **Case**

  $\Gamma \vdash \sigma_1 : N \quad \Gamma \vdash \sigma_2 : N$

  $\Gamma \vdash \sigma_1 = \sigma_2 \text{ prop}$

  $\Gamma \vdash \sigma_1 = \sigma_2 \text{ prop}$

  $\Delta \rightarrow \Omega_0$

  $\Delta \rightarrow \Omega'$

  $\Theta \rightarrow \Omega_0$

  $\Theta \rightarrow \Omega_0$

  $\Gamma \vdash \sigma_2 : N$

  $\Theta \vdash \sigma_2 : N$

  $\Theta \vdash \tau_2 : N$

  $\Theta \vdash [\Theta]\tau_1 \equiv [\Theta]\tau_2 : N \rightarrow \Delta$

  By Lemma 93 (Completeness of Checkeq)

  Given

  Definition of substitution

  Subderivation

  Subderivation

  By Lemma 93 (Completeness of Checkeq)

  Similarily

  By Extension Transitivity

  By above equalities

  By above equalities

  $\Gamma \vdash ([\Gamma]\sigma_1 = [\Theta]\sigma_2) \equiv ([\Gamma]\tau_1 = [\Theta]\tau_2) \rightarrow \Gamma$

  $\Gamma \vdash ([\Gamma]\sigma_1 = [\Gamma]\sigma_2) \equiv ([\Gamma]\tau_1 = [\Gamma]\tau_2) \rightarrow \Gamma$

  By PropEq

  By above equalities

---

**Lemma 97** (Completeness of Checkprop).
If $\Gamma \rightarrow \Omega$ and $\text{dom}(\Gamma) = \text{dom}(\Omega)$
and $\Gamma \vdash P \text{ prop}$
and $[\Gamma]P = P$
and $[\Omega]P \vdash \Omega$ true
and $\Gamma \vdash \Omega \vdash [\Omega]P$ true
then $\Gamma \vdash P \text{ true } \rightarrow \Delta$
where $\Delta \rightarrow \Omega'$ and $\Omega \rightarrow \Omega'$ and $\text{dom}(\Delta) = \text{dom}(\Omega')$.

*Proof.* Only one rule, DeclCheckpropEq, can derive $[\Omega]P \vdash \Omega$ true, so by inversion, $P$ has the form $\{t_1 = t_2\}$ where $[\Omega]t_1 = [\Omega]t_2$.
By inversion on $\Gamma \vdash \{t_1 = t_2\} \text{ prop}$, we have $\Gamma \vdash t_1 : N$ and $\Gamma \vdash t_2 : N$.
Then by Lemma 93 (Completeness of Checkeq), $\Gamma \vdash [\Gamma]t_1 \equiv [\Gamma]t_2 : N \rightarrow \Delta$ where $\Delta \rightarrow \Omega'$ and $\Omega \rightarrow \Omega'$.
By CheckpropEq, $\Gamma \vdash \{t_1 = t_2\}$ true $\rightarrow \Delta$. 

---

November 13, 2018
K′.2 Completeness of Equivalence and Subtyping

Lemma 98 (Completeness of Equiv).

If $\Gamma \rightarrow \Omega$ and $\Gamma \vdash A$ type and $\Gamma \vdash B$ type
and $[\Omega]A = [\Omega]B$
then there exist $\Delta$ and $\Omega'$ such that $\Delta \rightarrow \Omega'$ and $\Omega \rightarrow \Omega'$ and $\Gamma \vdash [\Gamma]A \equiv [\Gamma]B \vdash \Delta$.

Proof. By induction on the derivations of $\Gamma \vdash A$ type and $\Gamma \vdash B$ type.

We distinguish cases of the rule concluding the first derivation. In the first four cases (ImpliesWF, WithWF, ForallWF, ExistsWF), it follows from $[\Omega]A = [\Omega]B$ and the syntactic invariant that $\Omega$ substitutes terms $t$ (rather than types $A$) that the second derivation is concluded by the same rule. Moreover, if none of these three rules concluded the first derivation, the rule concluding the second derivation must not be ImpliesWF, WithWF, ForallWF, or ExistsWF either.

Because $\Omega$ is predicative, the head connective of $[\Gamma]A$ must be the same as the head connective of $[\Omega]A$.

We distinguish cases that are imposs. (impossible), fully written out, and similar to fully-written-out cases. For the lower-right case, where both $[\Gamma]A$ and $[\Gamma]B$ have a binary connective $\oplus$, it must be the same connective.

The Vec type is omitted from the table, but can be treated similarly to $\supset$ and $\land$.

$$
\begin{array}{llllllll}
\land & \lor \ & \forall \beta. \ B' \ & \exists \beta. \ B' \ & 1 \ & \alpha \ & \beta \ & B_1 \oplus B_2
\hline
\lor & \text{Implies} & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{imposs.} \\
\land & \text{imposs.} & \text{With} & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{imposs.} \\
\forall \alpha. \ A' & \text{imposs.} & \text{imposs.} & \text{Forall} & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{imposs.} \\
\exists \alpha. \ A' & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{Exists} & \text{imposs.} & \text{imposs.} & \text{imposs.} \\
1 & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{2.Units} & \text{imposs.} & \text{2.BEx.Unit} \\
\alpha & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{2.Uvars} & \text{imposs.} \\
\& & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{2.AEx.Unit} & \text{2.AEx.Uvar} & \text{imposs.} \\
A_1 \oplus A_2 & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{imposs.} & \text{2.AEx.Bin} \\
\end{array}
$$

- Case $\Gamma \vdash P$ prop $\Gamma \vdash A_0$ type $\Gamma \vdash P \supset A_0$ type ImpliesWF

This case of the rule concluding the first derivation coincides with the Implies entry in the table.

We have $[\Omega]A = [\Omega]B$, that is, $[\Omega](P \supset A_0) = [\Omega]B$.

Because $\Omega$ is predicative, $B$ must have the form $Q \supset B_0$, where $[\Omega]P = [\Omega]Q$ and $[\Omega]A_0 = [\Omega]B_0$. 

---

Proof of Lemma 98 (Completeness of Equiv). lem:equiv-completeness
Proof of Lemma 98 (Completeness of Equiv)

\[
\begin{align*}
\Gamma &\vdash P \text{ prop} & \text{Subderivation} \\
\Gamma &\vdash A_0 \text{ type} & \text{Subderivation} \\
\Gamma &\vdash Q \supset B_0 \text{ type} & \text{Given} \\
\Gamma &\vdash Q \text{ prop} & \text{By inversion on rule impliesWF} \\
\Gamma &\vdash B_0 \text{ type} & \text{"} \\
\Gamma &\vdash [\Gamma]P \equiv [\Gamma]Q \vdash \Theta & \text{By Lemma 96 (Completeness of Propequiv)} \\
\Theta &\rightarrow \Omega_0 & \text{"} \\
\Omega &\rightarrow \Omega_0 & \text{"}
\end{align*}
\]

\[
\begin{align*}
\Gamma &\rightarrow \emptyset & \text{By Lemma 48 (Prop Equivalence Extension)} \\
\Gamma &\vdash A_0 \text{ type} & \text{Above} \\
\Gamma &\vdash B_0 \text{ type} & \text{Above} \\
[\emptyset]A_0 &\equiv [\emptyset]B_0 & \text{Above} \\
[\emptyset]A_0 &\equiv [\emptyset]B_0 & \text{By Lemma 55 (Completing Completeness) (ii) twice} \\
\Gamma &\vdash [\Gamma]A_0 \equiv [\Gamma]B_0 \vdash \Delta & \text{By i.h.} \\
\equiv &\Delta & \text{"} \\
\Omega_0 &\rightarrow \Omega' & \text{"}
\end{align*}
\]

\[
\begin{align*}
\equiv &\Omega & \rightarrow \Omega' & \text{By Lemma 33 (Extension Transitivity)} \\
\Gamma &\vdash ([\Gamma]P \supset [\Gamma]A_0) \equiv ([\Gamma]Q \supset [\Gamma]B_0) \vdash \Delta & \text{By \equiv \rightarrow} \\
\equiv &\Gamma &\vdash [\Gamma](P \supset A_0) \equiv [\Gamma](Q \supset B_0) \vdash \Delta & \text{By definition of substitution}
\end{align*}
\]

**Case WithWF**: Similar to the impliesWF case, coinciding with the With entry in the table.

**Case** \(\Gamma, \alpha : \kappa \vdash A_0 \text{ type} \)

\[
\begin{align*}
\Gamma &\vdash \forall \alpha : \kappa. A_0 \text{ type} & \text{ForallWF} \\
\Gamma &\vdash \Delta & \text{Given} \\
\Gamma, \alpha : \kappa &\rightarrow \emptyset, \alpha : \kappa & \text{By \rightarrow \rightarrow \text{Uvar}} \\
\Gamma, \alpha : \kappa &\vdash B_0 & \text{Subderivation} \\
[\emptyset]A_0 &\equiv [\emptyset]B_0 & \text{O predicate} \\
[\emptyset]A_0 &\equiv [\emptyset]B_0 & \text{From definition of substitution} \\
\Gamma, \alpha : \kappa &\vdash [\Gamma]A_0 \equiv [\Gamma]B_0 \vdash \Delta_0 & \text{By i.h.} \\
\Delta_0 &\rightarrow \Omega_0 & \text{"} \\
\Omega, \alpha : \kappa &\rightarrow \emptyset & \text{"}
\end{align*}
\]

\[
\begin{align*}
\equiv &\emptyset & \text{\Omega \rightarrow \Omega' and } \Omega_0 = (\Omega', \alpha : \kappa, \ldots) & \text{By Lemma 22 (Extension Inversion) (i)} \\
\Delta_0 &\equiv (\Delta, \alpha : \kappa, \Delta') & \text{By Lemma 22 (Extension Inversion) (i)} \\
\equiv &\Delta & \rightarrow \Omega' & \text{"} \\
\Gamma &\vdash \forall \alpha : \kappa. \left[\Gamma\right]A_0 \equiv \forall \alpha : \kappa. \left[\Gamma\right]B_0 \vdash \Delta & \text{By \equiv \forall} \\
\equiv &\Gamma &\vdash [\Gamma](\forall \alpha : \kappa. A_0) \equiv [\Gamma](\forall \alpha : \kappa. B_0) \vdash \Delta & \text{By definition of substitution}
\end{align*}
\]

**Case ExistsWF**: Similar to the ForallWF case. (This is the Exists entry in the table.)

**Case** BinWF**: If BinWF also concluded the second derivation, then the proof is similar to the impliesWF case, but on the first premise, using the i.h. instead of Lemma 96 (Completeness of Propequiv). This is the 2.Bins entry in the lower right corner of the table.
If $\text{BinWF}$ did not conclude the second derivation, we are in the $2.AEx.\text{Bin}$ or $2.BEx.\text{Bin}$ entries; see below.

In the remainder, we cover the $4 \times 4$ region in the lower right corner, starting from $2.\text{Units}$. We already handled the $2.\text{Bins}$ entry in the extreme lower right corner. At this point, we split on the forms of $[\Gamma]A$ and $[\Gamma]B$ instead; in the remaining cases, one or both types is atomic (e.g. $2.\text{Uvars}$, $2.AEx.\text{Bin}$) and we will not need to use the induction hypothesis.

- **Case 2.\text{Units}:** $[\Gamma]A = [\Gamma]B = 1$
  
  \[\Gamma \vdash 1 \equiv 1 \rightarrow \Gamma\]
  
  By $\equiv\text{Unit}$
  
  \[\Gamma \rightarrow \Omega\]
  
  Given
  
  Let $\Omega' = \Omega$.
  
  \[\Delta \rightarrow \Omega\]
  
  $\Delta = \Gamma$
  
  \[\Omega \rightarrow \Omega'\]
  
  By Lemma 32 (Extension Reflexivity) and $\Omega' = \Omega$

- **Case 2.\text{Uvars}:** $[\Gamma]A = [\Gamma]B = \alpha$

  \[\Gamma \rightarrow \Omega\]
  
  Given
  
  Let $\Omega' = \Omega$.
  
  \[\Gamma \vdash \alpha \equiv \alpha \rightarrow \Gamma\]
  
  By $\equiv\text{Var}$
  
  \[\Delta \rightarrow \Omega\]
  
  $\Delta = \Gamma$
  
  \[\Omega \rightarrow \Omega'\]
  
  By Lemma 32 (Extension Reflexivity) and $\Omega' = \Omega$

- **Case 2.AExUnit**: $[\Gamma]A = \hat{\alpha}$ and $[\Gamma]B = 1$

  \[\Gamma \rightarrow \Omega\]
  
  Given
  
  $1 = [\Omega]1$
  
  By definition of substitution
  
  $\hat{\alpha} \notin FV(1)$
  
  By definition of $FV(\cdot)$
  
  $[\Omega][\hat{\alpha}] = [\Omega]1$
  
  Given
  
  \[\Gamma \vdash \hat{\alpha} : 1 : \star \rightarrow \Delta\]
  
  By Lemma 92 (Completeness of Instantiation) (1)
  
  \[\Omega \rightarrow \Omega'\]
  
  "
  
  \[\Delta \rightarrow \Omega'\]
  
  "
  
  $1 = [\Gamma]1$
  
  By definition of substitution
  
  $\hat{\alpha} \notin FV(1)$
  
  By definition of $FV(\cdot)$
  
  \[\Gamma \vdash \hat{\alpha} \equiv 1 \rightarrow \Delta\]
  
  By $\equiv\text{Instantiate}$

- **Case 2.BExUnit**: $[\Gamma]A = 1$ and $[\Gamma]B = \hat{\alpha}$

  Symmetric to the 2.AExUnit case.

- **Case 2.AExUvar**: $[\Gamma]A = \hat{\alpha}$ and $[\Gamma]B = \alpha$

  Similar to the 2.AEx.Unit case, using $\hat{\beta} = [\Omega]\beta = [\Gamma]\beta$ and $\hat{\alpha} \notin FV(\beta)$.

- **Case 2.BExUvar**: $[\Gamma]A = 1$ and $[\Gamma]B = \hat{\alpha}$

  Symmetric to the 2.AExUvar case.

- **Case 2.AEx.SameEx**: $[\Gamma]A = \hat{\alpha} = \hat{\beta} = [\Gamma]B$
Proof of [Lemma 98](Completeness of Equiv) (lem:equiv-completeness)

\[ \Gamma \vdash \alpha \equiv \alpha \vdash \Gamma \]  
By \[\equiv\text{Exvar} (\alpha = \beta)\]
\[|\Gamma|\alpha = \alpha \]  
\[\alpha \text{ unsolved in } \Gamma\]
\[\Gamma \vdash |\Gamma|\alpha \equiv |\Gamma|\beta \vdash \Gamma \]  
By above equality + \[\alpha = \beta\]
\[\Gamma \rightarrow \Omega \]  
Given
\[\Delta \rightarrow \Omega \]  
\[\Delta = \Gamma\]

Let \[\Omega' = \Omega.\]

\[\alpha \quad \beta \]
\[\begin{align*}
\alpha 
\quad \beta
\end{align*}\]

- **Case 2.AEx.OtherEx:** \[|\Gamma|A = \alpha \text{ and } |\Gamma|B = \beta \text{ and } \alpha \neq \beta\]

  Either \(\alpha \in \text{FV}(|\Gamma|\beta)\), or \(\alpha \notin \text{FV}(|\Gamma|\beta)\).

  - \(\alpha \in \text{FV}(|\Gamma|\beta)\):
    
    We have \(\alpha \leq |\Gamma|\beta\).
    
    Therefore \(\alpha = |\Gamma|\beta\), or \(\alpha < |\Gamma|\beta\).
    
    But we are in Case 2.AEx.OtherEx, so the former is impossible.
    
    Therefore, \(\alpha < |\Gamma|\beta\).
    
    By a property of substitutions, \(|\Omega|\alpha < |\Omega|\beta|.|\beta\).
    
    Since \(\Gamma \rightarrow \Omega\), by Lemma 29 (Substitution Monotonicity) (iii), \(|\Omega|\beta = |\Omega|\beta\), so \(|\Omega|\alpha < |\Omega|\beta|.|\beta\).
    
    But it is given that \(|\Omega|\alpha = |\Omega|\beta\), a contradiction.

  - \(\alpha \notin \text{FV}(|\Gamma|\beta)\):
    
    \[\Gamma \vdash : |\Gamma|\beta \vdash \Delta\]  
    By Lemma 92 (Completeness of Instantiation)
    
    \[\Gamma \vdash \alpha \equiv |\Gamma|\beta \vdash \Delta\]  
    By \[\equiv\text{Instantiatel}\]
    
    \[\Delta \rightarrow \Omega'\]  
    
    \[\Omega \rightarrow \Omega'\]  

- **Case 2.AEx.Bin:** \[|\Gamma|A = \alpha \text{ and } |\Gamma|B = B_1 \oplus B_2\]

  Since \(|\Gamma|B\) is an arrow, it cannot be exactly \(\alpha\). By the same reasoning as in the previous case (2.AEx.OtherEx), \(\alpha \notin \text{FV}(|\Gamma|\beta)\).

  \[\Gamma \vdash \alpha := |\Gamma|B \rightarrow \Delta\]  
  By Lemma 92 (Completeness of Instantiation)
  \[\equiv\]
  \[\Delta \rightarrow \Omega'\]
  
  \[\Omega \rightarrow \Omega'\]

\[\begin{align*}
\alpha 
\quad B_1 \oplus B_2
\end{align*}\]

- **Case 2.BEx.Bin:** \[|\Gamma|A = A_1 \oplus A_2 \text{ and } |\Gamma|B = \beta\]

  Symmetric to the 2.AEx.Bin case, applying \[\equiv\text{InstantiateL}\] instead of \[\equiv\text{Instantiatel}\].

\[\square\]

**Theorem 10** (Completeness of Subtyping).

If \(\Gamma \rightarrow \Omega\) and \(\text{dom}(\Gamma) = \text{dom}(\Omega)\) and \(\Gamma \vdash A\) type and \(\Gamma \vdash B\) type and \(|\Omega|\Gamma = |\Omega|A \leq_P |\Omega|B\) then there exist \(\Delta\) and \(\Omega'\) such that \(\Delta \rightarrow \Omega'\)
and \(\text{dom}(\Delta) = \text{dom}(\Omega')\)
and \(\Omega \rightarrow \Omega'\)
and \(\Gamma \vdash |\Gamma|A <^P |\Gamma|B \rightarrow \Delta\).

**Proof.** By induction on the number of \(\forall \exists\) quantifiers in \(|\Omega|A\) and \(|\Omega|B\).

It is straightforward to show \(\text{dom}(\Delta) = \text{dom}(\Omega')\); for examples of the necessary reasoning, see the proof of Theorem 12.

We have \(|\Omega|\Gamma \vdash |\Omega|A \leq_{\text{join}(\text{pol}(B), \text{pol}(A))} |\Omega|B|\).

Proof of [Theorem 10] (Completeness of Subtyping) (thm:subtyping-completeness)
Proof of Theorem 10 (Completeness of Subtyping) thm:subtyping-completeness

- Case \( [\Omega] \Gamma \vdash \delta \) nonpos([\Omega] A)

\[
\begin{array}{c}
\frac{[\Omega] \Gamma \vdash [\Omega] A \leq [\Omega] B}{[\Omega] \Gamma \vdash [\Omega] A \leq [\Omega] B} \text{ Refl}
\end{array}
\]

First, we observe that, since applying \( \Omega \) as a substitution leaves quantifiers alone, the quantifiers that head \( A \) must also head \( B \). For convenience, we alpha-vary \( B \) to quantify over the same variables as \( A \).

- If \( A \) is headed by \( \forall \), then \( [\Omega] A = (\forall \alpha : \kappa. [\Omega] A_0) = (\forall \alpha : \kappa. [\Omega] B_0) = [\Omega] B \).
  
  Let \( \Gamma_0 = (\Gamma, \alpha : \kappa, \alpha, \hat{\alpha} : \kappa) \).
  
  Let \( \Omega_0 = (\Omega, \alpha : \kappa, \alpha, \hat{\alpha} : \kappa = \alpha) \).

  * If pol(\( A_0 \)) \( \in \{-, 0\} \), then:
    
    (We elide the straightforward use of lemmas about context extension.)

    \[
    \begin{array}{c}
    [\Omega_0] \Gamma_0 \vdash [\Omega] A_0 \leq [\Omega] B_0 \hspace{1cm} \text{By Refl-} \\
    [\Omega_0] \Gamma_0 \vdash [\Omega] A_0 \leq [\Omega] B_0 \hspace{1cm} \text{By def. of subst.} \\
    \Delta \rightarrow \Omega' \\
    \Omega \rightarrow \Omega' \\
    \end{array}
    \]

    * If pol(\( A_0 \)) \( = + \), then proceed as above, but apply \( \text{Refl}+ \) instead of \( \text{Refl}− \) and apply \( \text{< : } \) instead of \( \text{< : } \) after applying the i.h. (Rule \( \text{< : } \) also works.)

  - If \( A \) is not headed by \( \forall \):
    
    We have \( \text{nonneg}([\Omega] A) \). Therefore \( \text{nonneg}(A) \), and thus \( A \) is not headed by \( \exists \). Since the same quantifiers must also head \( B \), the conditions in rule \( \text{< : } \) are satisfied.

\[
\begin{array}{c}
\frac{[\Gamma] A \equiv [\Gamma] B \vdash \Delta}{[\Gamma] A \equiv [\Gamma] B \vdash \Delta} \text{ Given} \\
\frac{[\Gamma] A \equiv [\Gamma] B \vdash \Delta}{[\Gamma] A \equiv [\Gamma] B \vdash \Delta} \text{ By Lemma 98 (Completeness of Equiv)} \\
\frac{[\Gamma] A \equiv [\Gamma] B \vdash \Delta}{[\Gamma] A \equiv [\Gamma] B \vdash \Delta} \text{ By \( \text{< : } \) } \\
\end{array}
\]

- Case \( \text{< : } \) Symmetric to the \( \text{< : } \) case, using \( \text{< : } \) (or \( \text{< : } \)), and \( \text{< : } \) instead of \( \text{< : } \) and \( \text{< : } \).

- Case \( [\Omega] \Gamma \vdash \tau : \kappa \)

\[
\frac{[\Omega] \Gamma \vdash [\tau / \alpha][\Omega] A_0 \leq \Omega [\Omega] B}{[\Omega] \Gamma \vdash \forall \alpha : \kappa. [\Omega] A_0 \leq [\Omega] B} \text{ < : VL}
\]

We begin by considering whether or not \( [\Omega] B \) is headed by a universal quantifier.

- \( [\Omega] B = (\forall \beta : \kappa'. B') \):
Proof of Theorem 10 (Completeness of Subtyping) 

The remaining steps are similar to the \( \leq \forall R \) case.

- \( [\Omega] B \) not headed by \( \forall \):
  \[
  [\Omega] \Gamma \vdash \tau : \kappa \quad \text{Subderivation}
  \]
  \[
  \Gamma \rightarrow \Omega \quad \text{Given}
  \]
  \[
  \Gamma, \triangleright, \delta : \kappa \rightarrow \Omega, \triangleright, \delta : \kappa = \tau
  \]
  \[
  \text{By \text{Marker}}
  \]
  \[
  \Omega_0 = [\Omega_0] (\Gamma, \triangleright, \delta : \kappa)
  \]
  \[
  \text{By definition of context application (lines 16, 13)}
  \]

\[
[\Omega] \Gamma \vdash [\tau/\alpha][\Omega] A_0 \leq [\Omega] B
\]

\[
[\Omega_0] (\Gamma, \triangleright, \delta : \kappa) \vdash [\tau/\alpha][\Omega_0] A_0 \leq [\Omega] B
\]

\[
[\Omega_0] (\Gamma, \triangleright, \delta : \kappa) \vdash [\Omega_0] \delta/\alpha[\Omega_0] A_0 \leq [\Omega] B
\]

\[
[\Omega_0] (\Gamma, \triangleright, \delta : \kappa) \vdash [\Omega_0] \delta/\alpha[\Omega_0] A_0 \leq [\Omega_0] B
\]

\[
[\Omega_0] (\Gamma, \triangleright, \delta : \kappa) \vdash [\Omega] A_0 \leq [\Omega] B
\]

\[
[\Omega_0] (\Gamma, \triangleright, \delta : \kappa) \vdash [\Omega_0] A_0 \leq [\Omega] B
\]

By definition of substitution

\[
\Gamma, \triangleright, \delta : \kappa \vdash [\Gamma]\delta/\alpha[\Gamma] A_0 \leq \vdash [\Gamma] B \vdash \Delta
\]

By i.h. (A lost a quantifier)

\[
\Delta_0 \rightarrow \Omega''
\]

\[
\Omega_0 \rightarrow \Omega''
\]

- \( V \):
  \[
  \Gamma, \triangleright, \delta : \kappa \rightarrow \Delta_0
  \]
  \[
  \text{By Lemma 50 (Subtyping Extension)}
  \]
  \[
  \Delta_0 = (\Delta, \triangleright, \delta, \Theta)
  \]
  \[
  \text{By Lemma 22 (Subtyping Inversion) (ii)}
  \]
  \[
  \Gamma \rightarrow \Delta
  \]
  \[
  \text{"}
  \]
  \[
  \Omega'' = (\Omega', \triangleright, \delta, \Omega_Z)
  \]
  \[
  \text{By Lemma 22 (Subtyping Inversion) (ii)}
  \]
  \[
  \Delta \rightarrow \Omega'
  \]
  \[
  \text{"}
  \]
  \[
  \Omega_0 \rightarrow \Omega''
  \]
  \[
  \text{Above}
  \]
  \[
  \Omega, \triangleright, \delta : \kappa = \tau \rightarrow \Omega', \triangleright, \delta, \Omega_Z
  \]
  \[
  \text{By above equalities}
  \]
  \[
  \text{By Lemma 22 (Subtyping Inversion) (ii)}
  \]

- \( \forall \):
  \[
  \Gamma, \triangleright, \delta : \kappa \vdash [\Gamma]\forall/\alpha[\Gamma] A_0 \leq \vdash [\Gamma] B \vdash \Delta
  \]
  \[
  \text{By above equality} \Delta_0 = (\Delta, \triangleright, \delta, \Theta)
  \]
  \[
  \Gamma \vdash \forall/\alpha[\Gamma] A_0 \leq \vdash [\Gamma] B \vdash \Delta
  \]
  \[
  \text{By def. of subst.} ([\Gamma]\delta = \delta \text{ and } [\Gamma]\alpha = \alpha)
  \]
  \[
  \Gamma[B \not\text{ headed by } \forall
  \]
  \[
  \text{From the case assumption}
  \]
  \[
  \Gamma \vdash \forall\alpha : \kappa, [\Gamma] A_0 \leq \vdash [\Gamma] B \vdash \Delta
  \]
  \[
  \text{By \text{< \forall L}}
  \]
  \[
  \Gamma \vdash [\Gamma]\forall\alpha : \kappa, A_0 \leq \vdash [\Gamma] B \vdash \Delta
  \]
  \[
  \text{By definition of substitution}
  \]

- Case

\[
[\Omega] \Gamma, \beta : \kappa \vdash [\Omega] A \leq [\Omega] B_0
\]

\[
[\Omega] \Gamma \vdash [\Omega] A \leq \forall \beta : \kappa, [\Omega] B_0
\]

\[
\leq \forall R
\]
Proof of **Theorem 10** (Completeness of Subtyping)

\[ B = \forall \beta : \kappa. B_0 \]
\[ \Omega \vdash [\Omega]A \leq \Omega B \]  
Given
\[ \Omega \vdash [\Omega]A \leq \forall \beta. [\Omega]B_0 \]  
By above equality
\[ [\Omega, \beta : \kappa](\Omega, \beta : \kappa) \leq [\Omega, \beta : \kappa]B_0 \]  
Subderivation
\[ \Gamma, \beta : \kappa \vdash [\Gamma](\beta, \kappa)A \leq [\Gamma](\beta, \kappa)B_0 \]  
By definitions of substitution
\[ \Gamma \vdash A' \]  
By i.h. (B lost a quantifier)
\[ \Delta' \rightarrow \Omega_\delta' \]
\[ \Omega, \beta : \kappa \rightarrow \Omega' \]  
By definition of substitution

\[ \Gamma, \beta : \kappa \rightarrow \Delta' \]  
By Lemma **43** (Instantiation Extension)
\[ [\Gamma, \beta : \kappa, \Theta] \rightarrow [\Gamma, \beta : \kappa, \Omega_R] \]  
By Lemma **22** (Extension Inversion) (i)
\[ \Gamma \vdash [\Gamma]A \leq [\Gamma]B_0 \]  
By above equality
\[ \Omega \rightarrow \Omega' \]
By above equality

\[ \Gamma \vdash [\Gamma]A \leq [\Gamma]B_0 \rightarrow \Delta, \beta : \kappa, \Theta \]  
By Lemma **33** (Extension Transitivity)
\[ \Gamma \vdash [\Gamma]A \leq [\Gamma](\forall \beta : \kappa, B_0) \rightarrow \Delta \]  
By i.c. \( \forall \beta \)

* Case
\[ \Omega, \gamma : \kappa \vdash [\Omega]A_0 \leq [\Omega]B \]
\[ [\Omega]A \leq [\exists \alpha : \kappa, A_0] \leq [\Omega]B \]  
\[ \Lambda = [\exists \alpha : \kappa, A_0] \]  
\[ [\Omega]A \leq [\exists \alpha : \kappa, A_0] \leq [\Omega]B \]  
\[ [\Omega, \alpha : \kappa] \rightarrow [\Omega, \gamma : \kappa] \rightarrow \Omega_\delta' \]  
\[ \Gamma, \alpha : \kappa \vdash A \leq [\Gamma](\forall \beta : \kappa, B_0) \rightarrow \Delta' \]  
\[ \Omega, \alpha : \kappa \rightarrow \Omega' \]  
\[ \Delta' \rightarrow \Omega_\delta' \]  
\[ \Gamma, \alpha : \kappa \rightarrow \Delta' \]  
By Lemma **43** (Instantiation Extension)
\[ (\exists \alpha : \kappa, \Theta) \rightarrow \Omega_\delta' \]  
By Lemma **22** (Extension Inversion) (i)
\[ \Gamma \rightarrow \Delta \]  
\[ \Delta, \alpha : \kappa, \Theta \rightarrow \Omega_\delta' \]  
By \( \Delta' \rightarrow \Omega_\delta' \) and above equality
\[ \Omega_\delta' = (\exists \alpha : \kappa, \Omega_R) \]  
By Lemma **22** (Extension Inversion) (i)
Proof of Theorem 10 (Completeness of Subtyping)  thm:subtyping-completeness

\[ \Gamma, \alpha : \kappa \vdash [\Gamma]A_0 \triangleleft^+ [\Gamma]B \vdash \Delta, \alpha : \kappa, \Theta \]
\[ \Omega, \alpha : \kappa \rightarrow \Omega', \alpha : \kappa, \Omega_R \]
\[ \rightarrow \Omega \rightarrow \Omega' \]

By above equality

\[ \Gamma \vdash \exists \alpha : \kappa. [\Gamma]A_0 \triangleleft^+ [\Gamma]B \vdash \Delta \]

By [6] (Extension Transitivity)

We consider whether \( [\Omega]A \) is headed by an existential.

If \( [\Omega]A = \exists \alpha : \kappa'. A' \):

\[ [\Omega] \Gamma, \alpha : \kappa \vdash A' \triangleleft^+ [\Omega]B \]

By Lemma [5] (Subtyping Inversion)

The remaining steps are similar to the \( \triangleleft^+ \) case.

If \( [\Omega]A \) not headed by \( \exists \):

\[ [\Omega] \Gamma \vdash \tau : \kappa \]
\[ [\Omega] \Gamma \vdash [\Omega]A \triangleleft^+ \frac{[\tau/\beta]B_0}{[\Omega]B} \]

Subderivation

Given

By Marker

By Solve

By definition of context application (lines 16, 13)

\[ [\Omega] \Gamma \vdash [\Omega]A \triangleleft^+ \frac{[\tau/\beta]B_0}{[\Omega]B} \]

Subderivation

By above equality

By definition of substitution

By definition of substitution

By distributivity of substitution

\[ \Gamma, \triangleright \alpha, \hat{\alpha} : \kappa \vdash [\Gamma, \triangleright \alpha, \hat{\alpha} : \kappa] [\hat{\alpha}/\beta]B_0 \vdash \Delta_0 \]

By i.h. (\( B \) lost a quantifier)

By definition of substitution

By Lemma [50] (Subtyping Extension)

By Lemma [22] (Extension Inversion) (ii)

By above equalities

By Lemma [22] (Extension Inversion) (ii)
\[ \Gamma, \Gamma \vdash_\alpha : \kappa \vdash \Gamma \widehat{\alpha} \vdash \Gamma \mid \beta \Gamma \vdash \Delta, \Gamma \vdash_\alpha, \Theta \quad \text{By above equality } \Delta_0 = (\Delta, \Gamma \vdash_\alpha, \Theta) \]
\[ \Gamma, \Gamma \vdash_\alpha : \kappa \vdash \Gamma \mid \beta \Gamma \vdash \Delta, \Gamma \vdash_\alpha, \Theta \quad \text{By def. of subst. } (\Gamma \mid \beta = \beta) \]
\[ \Gamma \vdash \Gamma \mid \beta \not \vdash \exists \gamma : \kappa, \Gamma \beta_0 \vdash \Delta \quad \text{From the case hypothesis} \]
\[ \Gamma \vdash \Gamma \mid \beta \vdash \Gamma (\exists \beta : \kappa, \beta_0 \vdash \Delta) \quad \text{By definition of substitution} \]

K'.3 Completeness of Typing

Lemma 99 (Variable Decomposition). If \( \Pi \vdash_\alpha \Pi', \) then

1. if \( \Pi \vdash_\alpha \Pi'' \) then \( \Pi'' = \Pi' \).
2. if \( \Pi \not \vdash_\alpha \Pi'' \) then there exists \( \Pi'' \) such that \( \Pi'' \vdash_\alpha \Pi'' \) and \( \Pi'' \vdash_\alpha \Pi' \).
3. if \( \Pi \vdash_\alpha \Pi_L \parallel \Pi_R \vdash_\alpha \Pi' \) then \( \Pi_L \vdash_\varnothing \Pi' \) and \( \Pi_R \vdash_\varnothing \Pi' \).
4. if \( \Pi \vdash_\varnothing \Pi_L \parallel \Pi_R \vdash_\varnothing \Pi' \) then \( \Pi' = \Pi_L \parallel \Pi_R \).

Proof. Each of these follows by induction on \( \Pi \) and decomposition of the two input derivations.

Lemma 100 (Pattern Decomposition and Substitution).

1. If \( \Pi \vdash_\alpha \Pi' \) then \( \varnothing \mid \Pi \vdash_\alpha \varnothing \mid \Pi' \).
2. If \( \Pi \vdash_\alpha \Pi' \) then \( \varnothing \mid \Pi \vdash_\alpha \varnothing \mid \Pi' \).
3. If \( \Pi \vdash_\alpha \Pi' \) then \( \varnothing \mid \Pi \vdash_\alpha \varnothing \mid \Pi' \).
4. If \( \Pi \vdash_\alpha \Pi_L \parallel \Pi_R \vdash_\alpha \Pi' \) then \( \varnothing \mid \Pi_L \parallel \Pi_R \vdash_\alpha \varnothing \mid \Pi' \).
5. If \( \Pi \vdash_\alpha \Pi_L \parallel \Pi_R \vdash_\alpha \Pi' \) then \( \varnothing \mid \Pi_L \parallel \Pi_R \vdash_\alpha \varnothing \mid \Pi' \).
6. If \( \varnothing \mid \Pi \vdash_\alpha \varnothing \mid \Pi' \) then there is \( \Pi'' \) such that \( \varnothing \mid \Pi'' = \Pi' \) and \( \Pi \vdash_\alpha \Pi'' \).
7. If \( \varnothing \mid \Pi \vdash_\alpha \varnothing \mid \Pi' \) then there is \( \Pi'' \) such that \( \varnothing \mid \Pi'' = \Pi' \) and \( \Pi \vdash_\alpha \Pi'' \).
8. If \( \varnothing \mid \Pi \vdash_\alpha \varnothing \mid \Pi' \) then there is \( \Pi'' \) such that \( \varnothing \mid \Pi'' = \Pi' \) and \( \Pi \vdash_\alpha \Pi'' \).
9. If \( \varnothing \mid \Pi \vdash_\alpha \varnothing \mid \Pi' \) then there are \( \Pi_1 \) and \( \Pi_2 \) such that \( \varnothing \mid \Pi_1 = \Pi_1' \) and \( \varnothing \mid \Pi_2 = \Pi_2' \) and \( \Pi \vdash_\alpha \varnothing \mid \Pi_1 \parallel \Pi_2 \).
10. If \( \varnothing \mid \Pi \vdash_\alpha \varnothing \mid \Pi' \) then there are \( \Pi_1 \) and \( \Pi_2 \) such that \( \varnothing \mid \Pi_1 = \Pi_1' \) and \( \varnothing \mid \Pi_2 = \Pi_2' \) and \( \Pi \vdash_\alpha \varnothing \mid \Pi_1 \parallel \Pi_2 \).

Proof. Each case is proved by induction on the relevant derivation.

Lemma 101 (Pattern Decomposition Functionality).

1. If \( \Pi \vdash_\alpha \Pi' \) and \( \Pi \vdash_\alpha \Pi'' \) then \( \Pi' = \Pi'' \).
2. If \( \Pi \vdash_\alpha \Pi' \) and \( \Pi \vdash_\alpha \Pi'' \) then \( \Pi' = \Pi'' \).
3. If \( \Pi \vdash_\alpha \Pi' \) and \( \Pi \vdash_\alpha \Pi'' \) then \( \Pi' = \Pi'' \).
4. If \( \Pi \vdash_\alpha \Pi_L \parallel \Pi_R \vdash_\alpha \Pi' \) then \( \Pi_1 = \Pi_1' \) and \( \Pi_2 = \Pi_2' \).
5. If \( \Pi \vdash_\varnothing \Pi_L \parallel \Pi_R \vdash_\varnothing \Pi' \) then \( \Pi_1 = \Pi_1' \) and \( \Pi_2 = \Pi_2' \).

Proof. By induction on the derivation of \( \Pi \vdash_\alpha \Pi' \).
Lemma 102 (Decidability of Variable Removal). For all \( \Pi \), either there exists a \( \Pi' \) such that \( \Pi \xrightarrow{\text{var}} \Pi' \) or there does not.

Proof. This follows from an induction on the structure of \( \Pi \).

Lemma 103 (Variable Inversion).

1. If \( \Pi \xrightarrow{\text{var}} \Pi' \) and \( \Psi \vdash \Pi \) covers \( \bar{A}, \bar{q} \) then \( \Psi \vdash \Pi' \) covers \( \bar{A}, \bar{q} \).

2. If \( \Pi \xrightarrow{\text{var}} \Pi' \) and \( \Gamma \vdash \Pi \) covers \( \bar{A}, \bar{q} \) then \( \Gamma \vdash \Pi' \) covers \( \bar{A}, \bar{q} \).

Proof. This follows by induction on the relevant derivations.

Theorem 11 (Completeness of Match Coverage).

1. If \( \Gamma \vdash \bar{A}, \bar{q} \) types and \( |\Gamma|\bar{A} = \bar{A} \) and (for all \( \Omega \) such that \( \Gamma \rightarrow \Omega \), we have \( |\Omega|\Gamma \vdash (|\Omega|\Pi \) covers \( (|\Omega|\bar{A}, \bar{q}) \))

2. If \( |\Gamma|\bar{A} = \bar{A} \) and \( |\Gamma|\bar{P} = \bar{P} \) and \( \Gamma \vdash \bar{A} \) types and (for all \( \Omega \) such that \( \Gamma \rightarrow \Omega \), we have \( |\Omega|\Gamma / (|\Omega|\bar{A} \) covers \( (|\Omega|\bar{P} \))

Proof. By mutual induction, with the induction metric lexicographically ordered on the number of pattern constructor symbols in the branches of \( \Pi \), the number of connectives in \( \bar{A} \), and 1 if \( \bar{P} \) is present/0 if it is absent.

1. Assume \( \Gamma \vdash \bar{A}, \bar{q} \) types and \( |\Gamma|\bar{A} = \bar{A} \) and (for all \( \Omega \) such that \( \Gamma \rightarrow \Omega \), we have \( |\Omega|\Gamma \vdash (|\Omega|\Pi \) covers \( (|\Omega|\bar{A}, \bar{q}) \))

   - Case \( \bar{A} = : \):
     Choose a completing substitution \( \Omega \).
     Then we have \( |\Omega|\Gamma \vdash (|\Omega|\Pi \) covers \( \bar{q} \).
     By inversion, we see that \( \text{DeclCoversEmpty} \) was the last rule, and that we have \( |\Omega|\Gamma \vdash |\Omega| \Rightarrow e_1 \ldots \Rightarrow \text{covers} \cdot \bar{q} \).
     Hence by \( \text{CoversEmpty} \), we have \( \Gamma \vdash \Rightarrow e_1 \ldots \Rightarrow \text{covers} \cdot \bar{q} \).

   - Case \( \bar{A} = \bar{A}, \bar{B} \):
     By Lemma 102 (Decidability of Variable Removal) either
     - Case \( \Pi \xrightarrow{\text{var}} \Pi' \):
       Assume \( \Omega \) such that \( \Gamma \rightarrow \Omega \).
       By assumption, \( |\Omega|\Gamma \vdash (|\Omega|\Pi \) covers \( (|\Omega|\bar{A}, \bar{B}) \).
       By Lemma 100 (Pattern Decomposition and Substitution), \( |\Omega|\Pi \xrightarrow{\text{var}} (|\Omega|\Pi' \).
       By Lemma 102 (Variable Inversion), \( |\Omega|\Gamma \vdash (|\Omega|\Pi' \) covers \( (|\Omega|\bar{B} \).
       So for all \( \Omega \) such that \( \Gamma \rightarrow \Omega \), \( |\Omega|\Gamma \vdash (|\Omega|\Pi' \) covers \( (|\Omega|\bar{B} \).
       By induction, \( \Gamma \vdash \Pi' \) covers \( \bar{B} \).
       \( \Rightarrow \) By rule \( \text{CoversVar} \), \( \Gamma \vdash \Pi \) covers \( \bar{A}, \bar{B} \).

     - Case \( \forall \Pi' \rightarrow (\Pi \xrightarrow{\text{var}} \Pi') \):
       * Case \( \hat{\alpha}, \bar{B} \):
         This case is impossible. Choose a completing substitution \( \Omega \) such that \( |\Omega|\hat{\alpha} = 1 \rightarrow 1 \), and then by assumption we have \( |\Omega|\Gamma \vdash (|\Omega|\Pi \) covers \( 1 \rightarrow 1, |\Omega|\bar{B} \).
         By inversion we have that \( |\Omega|\Pi \xrightarrow{\text{var}} \Pi' \). By Lemma 100 (Pattern Decomposition and Substitution), we have a \( \Pi'' \) such that \( |\Omega|\Pi'' = \Pi' \), and \( \Pi \xrightarrow{\text{var}} \Pi'' \). This yields the contradiction.
       * Case \( C \rightarrow D, \bar{B} \):
       * Case \( \forall \alpha : \kappa, \bar{A}, \bar{B} \):
       * Case \( \alpha, \bar{B} \):
         Similar to the \( \hat{\alpha} \) case.
* Case $\tilde{A} = 1, \tilde{B}$:

Choose an arbitrary $\Omega$ such that $\Gamma \rightarrow \Omega$.

By assumption, $[\Omega] \Gamma \vdash [\Omega] \Pi \text{ covers } [\Omega](1, \tilde{B}) q$.

By inversion, we know the rule $\text{DeclCovers}$ applies (since the variable case has been ruled out).

Hence $[\Omega] \Pi \vdash \Pi''$ and $[\Omega] \Gamma \vdash \Pi'' \text{ covers } [\Omega]\tilde{B} q$.

By Lemma 100 (Pattern Decomposition and Substitution), there is a $\Pi'$ such that $[\Omega] \Pi' = \Pi''$ and $\Pi \vdash \Pi'$.

Assume $\Omega$ such that $\Gamma \rightarrow \Omega$.

By assumption, $[\Omega] \Gamma \vdash [\Omega] \Pi \text{ covers } [\Omega](1, \tilde{B}) q$.

By inversion, we know the rule $\text{DeclCovers}$ applies (since the variable case has been ruled out).

Hence $[\Omega] \Pi \vdash \Pi''$ and $[\Omega] \Gamma \vdash \Pi'' \text{ covers } [\Omega]\tilde{B} q$.

By Lemma 100 (Pattern Decomposition and Substitution),

there is a $\hat{\Pi}'$ such that $\Pi'' = [\Omega]\hat{\Pi}'$ and $\Pi \vdash \hat{\Pi}'$.

By Lemma 101 (Pattern Decomposition Functionality), we know $\hat{\Pi}' = \Pi'$.

So for all $\Omega$ such that $\Gamma \rightarrow \Omega$, $[\Omega] \Gamma \vdash [\Omega] \Pi' \text{ covers } [\Omega]\tilde{B} q$.

By induction, $\Gamma \vdash \Pi' \text{ covers } \tilde{B} q$.

By rule $\text{Covers}$, $\Gamma \vdash \Pi \text{ covers } A, \tilde{B} q$.

* Case $C \times D, \tilde{B}$:

Choose an arbitrary $\Omega$ such that $\Gamma \rightarrow \Omega$.

By assumption, $[\Omega] \Gamma \vdash [\Omega] \Pi \text{ covers } [\Omega](C \times D, \tilde{B}) q$.

By inversion, we know the rule $\text{DeclCovers}$ applies (since the variable case has been ruled out).

Hence $[\Omega] \Pi \vdash \Pi''$ and $[\Omega] \Gamma \vdash \Pi'' \text{ covers } [\Omega](C, D, \tilde{B}) q$.

By Lemma 100 (Pattern Decomposition and Substitution), there is a $\Pi'$ such that $[\Omega] \Pi' = \Pi''$ and $\Pi \vdash \Pi'$.

Assume $\Omega$ such that $\Gamma \rightarrow \Omega$.

By assumption, $[\Omega] \Gamma \vdash [\Omega] \Pi \text{ covers } [\Omega](C \times D, \tilde{B}) q$.

By inversion, we know the rule $\text{DeclCovers}$ applies (since the variable case has been ruled out).

Hence $[\Omega] \Pi \vdash \Pi''$ and $[\Omega] \Gamma \vdash \Pi'' \text{ covers } [\Omega](C, D, \tilde{B}) q$.

By Lemma 100 (Pattern Decomposition and Substitution),

there is a $\hat{\Pi}'$ such that $\Pi'' = [\Omega]\hat{\Pi}'$ and $\Pi \vdash \hat{\Pi}'$.

By Lemma 101 (Pattern Decomposition Functionality), we know $\hat{\Pi}' = \Pi'$.

So for all $\Omega$ such that $\Gamma \rightarrow \Omega$, $[\Omega] \Gamma \vdash [\Omega] \Pi' \text{ covers } [\Omega](C, D, \tilde{B}) q$.

By induction, $\Gamma \vdash \Pi' \text{ covers } C, D, \tilde{B} q$.

By rule $\text{Covers}$, $\Gamma \vdash \Pi \text{ covers } C \times D, \tilde{B} q$.

* Case $C + D, \tilde{B}$:

Choose an arbitrary $\Omega$ such that $\Gamma \rightarrow \Omega$.

By assumption, $[\Omega] \Gamma \vdash [\Omega] \Pi \text{ covers } [\Omega](C \times D, \tilde{B}) q$.

By inversion, we know the rule $\text{DeclCovers}$ applies (since the variable case has been ruled out).

Hence $[\Omega] \Pi \vdash \Pi''$ and $[\Omega] \Gamma \vdash \Pi'' \text{ covers } [\Omega](C, D, \tilde{B}) q$.

By Lemma 100 (Pattern Decomposition and Substitution), there is a $\Pi_1$ and $\Pi_2$ such that $[\Omega] \Pi_1 = \Pi''$ and $[\Omega] \Pi_2 = \Pi' \text{ and } \Pi \vdash \Pi_1 \parallel \Pi_2$.

Assume $\Omega$ such that $\Gamma \rightarrow \Omega$.

By assumption, $[\Omega] \Gamma \vdash [\Omega] \Pi \text{ covers } [\Omega](C \times D, \tilde{B}) q$.

By inversion, we know the rule $\text{DeclCovers}$ applies (since the variable case has been ruled out).

Hence $[\Omega] \Pi \vdash \hat{\Pi}'$ and $[\Omega] \Gamma \vdash \hat{\Pi}' \text{ covers } [\Omega](C, \tilde{B}) q$ and $[\Omega] \Gamma \vdash \hat{\Pi}' \text{ covers } [\Omega](D, \tilde{B}) q$. 
2. Assume \([\Gamma]\bar{A} = \bar{A}\) and \([\Gamma]P = P\) and \(\Gamma \vdash \bar{A} \uparrow \) types and (for all \(\Omega\) such that \(\Gamma \rightarrow \Omega\), we have \([\Omega]\bar{A}\). Let \((t_1, t_2)\) be \(P\).
Consider whether the mgu\((t_1, t_2)\) exists

- Case \(\emptyset = \text{mgu}(t_1, t_2)\):
  \[
  \text{mgu}(t_1, t_2) = \emptyset \quad \text{Premise}
  \]
  \[
  \Gamma / t_1 \triangleq t_2 : \kappa \vdash \Gamma, \Theta \quad \text{By Lemma 94 [Completeness of ElimEq] (1)}
  \]
  \[
  \Gamma / [\Gamma]t_1 \triangleq [\Gamma]t_2 : \kappa \vdash \Gamma, \Theta \quad \text{Follows from given assumption}
  \]

Assume \(\Omega\) such that \(\Gamma, \Theta \rightarrow \Omega\).
By Lemma 559 [Canonical Completion], there is a \(\Omega'\) such that \([\Omega]\Gamma = [\Omega']\Gamma\) and \(\text{dom}(\Gamma) = \text{dom}(\Gamma')\).
Moreover, by Lemma 222 [Extension Inversion], we can construct a \(\Omega''\) such that \(\Omega' = \Omega'', \Theta\) and \(\Gamma \rightarrow \Omega''\).
By assumption, \([\Omega'']\Gamma / [\Omega''](t_1 = t_2) \vdash [\Omega'']\Pi \text{ covers } \bar{A}\).
There is only one way this derivation could be constructed:
Proof of Theorem 12 (Completeness of Algorithmic Typing). Given \( \Gamma \rightarrow \Omega \) such that \( \text{dom}(\Gamma) = \text{dom}(\Omega) \):

(i) If \( \Gamma \vdash A \ p \ \text{type} \) and \( [\Omega] \vdash [\Omega] e \leftrightarrow [\Omega] A \ p \) and \( p' \sqsubseteq p \)
then there exist \( \Delta \) and \( \Omega' \) such that \( \Delta \rightarrow \Omega' \) and \( \text{dom}(\Delta) = \text{dom}(\Omega') \) and \( \Omega \rightarrow \Omega' \) and \( \Gamma \vdash e \leftrightarrow [\Gamma] A \ p' \rightarrow \Delta \).

(ii) If \( \Gamma \vdash A \ p \ \text{type} \) and \( [\Omega] \vdash [\Omega] e \Rightarrow A \ p \)
then there exist \( \Delta, \Omega', A', \) and \( p' \sqsubseteq p \)
such that \( \Delta \rightarrow \Omega' \) and \( \text{dom}(\Delta) = \text{dom}(\Omega') \) and \( \Omega \rightarrow \Omega' \) and \( \Gamma \vdash e \Rightarrow A' \ p' \rightarrow \Delta \) and \( A' = [\Delta] A' \) and \( A = [\Omega'] A' \).

(iii) If \( \Gamma \vdash A \ p \ \text{type} \) and \( [\Omega] \vdash [\Omega] s : [\Omega] A \ p' \gg B \ q \) and \( p' \sqsubseteq p \)
then there exist \( \Delta, \Omega', B' \) and \( q' \sqsubseteq q \)
such that \( \Delta \rightarrow \Omega' \) and \( \text{dom}(\Delta) = \text{dom}(\Omega') \) and \( \Omega \rightarrow \Omega' \) and \( \Gamma \vdash s : [\Gamma] A \ p' \gg B' \ [q'] \rightarrow \Delta \) and \( B' = [\Delta] B' \) and \( B = [\Omega'] B' \).

(iv) If \( \Gamma \vdash A \ p \ \text{type} \) and \( [\Omega] \vdash [\Omega] s : [\Omega] A \ p \gg B \ q \) and \( p' \sqsubseteq p \)
then there exist \( \Delta, \Omega', B' \) and \( q' \sqsubseteq q \)
such that \( \Delta \rightarrow \Omega' \) and \( \text{dom}(\Delta) = \text{dom}(\Omega') \) and \( \Omega \rightarrow \Omega' \) and \( \Gamma \vdash s : [\Gamma] A \ p' \gg B' \ [q'] \rightarrow \Delta \) and \( B' = [\Delta] B' \) and \( B = [\Omega'] B' \).

(v) If \( \Gamma \vdash A \ p \ \text{type} \) and \( \Gamma \vdash C \ p \ \text{type} \) and \( [\Omega] \vdash [\Omega] A \ q \leftrightarrow [\Omega] C \ p \) and \( p' \sqsubseteq p \)
then there exist \( \Delta, \Omega', \) and \( C \) such that \( \Delta \rightarrow \Omega' \) and \( \text{dom}(\Delta) = \text{dom}(\Omega') \) and \( \Omega \rightarrow \Omega' \) and \( \Gamma \vdash \Pi : [\Gamma] A \ q \leftrightarrow [\Gamma] C \ p' \rightarrow \Delta \).
(vi) If \( \Gamma \vdash A! \) types and \( \Gamma \vdash P \) prop and \( \text{FEV}(P) = \emptyset \) and \( \Gamma \vdash C \, p \) type 
and \( [\Omega] \Gamma / [\Omega]P \vdash [\Omega] \Pi : [\Omega]A! \leftrightarrow [\Omega]C \, p \) 
and \( p' \subseteq p \) 
then there exist \( \Delta, \Omega' \), and \( C \) 
such that \( \Delta \rightarrow \Omega' \) and \( \text{dom}(\Delta) = \text{dom}(\Omega') \) and \( \Omega \rightarrow \Omega' \) 
and \( \Gamma / [\Gamma]P \vdash \Pi : [\Gamma]A! \leftrightarrow [\Gamma]C \, p' \vdash \Delta \).

**Proof.** By induction, using the measure in Definition 7.

- **Case** \( \{x : A \} \in [\Omega] \Gamma \quad \text{DeclVar} \)
  \( [\Omega] \Gamma \vdash x \Rightarrow A \, p \)
  \( \{x : A \} \in [\Omega] \Gamma \) \hspace{1cm} Premise
  \( \Gamma \rightarrow [\Omega] \) \hspace{1cm} Given
  \( [\Omega] \Gamma \vdash x \Rightarrow A \, p \) \hspace{1cm} From definition of context application
  Let \( \Delta = \Gamma \).
  Let \( \Omega' = \Omega \).

  \( \text{Given} \quad \Gamma \rightarrow [\Omega] \)
  \( \Omega \rightarrow [\Omega] \quad \text{By Lemma 32 (Extension Reflexivity)} \)
  \( \Gamma \vdash x \Rightarrow [\Gamma]A' \, \, p \vdash \Gamma \quad \text{By Var} \)
  \( [\Gamma]A' = [\Gamma] \, [\Gamma]A' \quad \text{By idempotence of substitution} \)
  \( \text{Given} \quad \text{dom}(\Gamma) = \text{dom}(\Omega) \)
  \( \Gamma \rightarrow [\Omega] \quad \text{Given} \)
  \( [\Omega]/[\Gamma]A' = [\Omega]A' \quad \text{By Lemma 29 (Substitution Monotonicity)} \) (iii)

  \( = A \quad \text{By above equality} \)

- **Case** \( [\Omega] \Gamma \vdash [\Omega]e \Rightarrow B \quad [\Omega] \Gamma \vdash B \leq \text{join}(\text{pol}(A), \text{pol}(B)) \, [\Omega]A \quad \text{DeclSub} \)
  \( [\Omega] \Gamma \vdash [\Omega]e \Leftarrow A \, p \)
  \( [\Omega] \Gamma \vdash [\Omega]e \Rightarrow B \) \hspace{1cm} Subderivation
  \( \Gamma \vdash e \Rightarrow B \, q \vdash \Theta \quad \text{By i.h.} \)
  \( B = [\Omega]B' \)
  \( \Theta \rightarrow [\Omega]_0 \)
  \( [\Omega]_0 \rightarrow [\Omega] \)
  \( \text{dom}(\Theta) = \text{dom}(\Omega_0) \)

  \( \Gamma \rightarrow [\Omega] \quad \text{By Lemma 33 (Extension Transitivity)} \)
  \( \Gamma \rightarrow [\Omega]_0 \quad \text{Subderivation} \)
  \( [\Omega] \Gamma \vdash B \leq \text{join}(\text{pol}(A), \text{pol}(B)) \, [\Omega]A \quad \text{By Lemma 56 (Confluence of Completeness)} \)
  \( [\Omega] \Theta \vdash B \leq \text{join}(\text{pol}(A), \text{pol}(B)) \, [\Omega]A \quad \text{By above equalities} \)
  \( \Theta \rightarrow [\Omega]_0 \quad \text{Above} \)
  \( [\Omega]_0 \vdash B' \leq \text{join}(\text{pol}(A), \text{pol}(B)) \, A \vdash \Delta \quad \text{By Theorem 10} \)
  \( [\Omega]_0 \rightarrow [\Omega]' \quad \text{"} \)

  \( \text{dom}(\Delta) = \text{dom}(\Omega') \quad \text{By Lemma 33 (Extension Transitivity)} \)
  \( \Delta \rightarrow [\Omega]' \quad \text{By Lemma 33 (Extension Transitivity)} \)
  \( \Omega \rightarrow [\Omega]' \quad \text{By Lemma 33 (Extension Transitivity)} \)

  \( \Gamma \vdash e \Leftarrow A \, p \vdash \Delta \quad \text{By Sub} \)
Proof of Theorem 12 (Completeness of Algorithmic Typing) thm:typing-completeness

• Case  
\[ \Omega \mid \Gamma \vdash \Omega \mid \Gamma \text{ type} \]
\[ \Omega \mid \Gamma \vdash \Omega \mid \Gamma \text{ e}_0 \leftarrow \Omega \mid \Gamma \text{ A} ! \]
\[ \Omega \mid \Gamma \vdash \Omega \mid \Gamma \text{ e}_0 : \text{ A} \Rightarrow \text{ A} ! \]

- **DeclAnno**
- **Subderivation**
- **By Lemma 29 (Substitution Monotonicity)**
- **By above equality**
- **By i.h.**

\[ \Delta \rightarrow \Omega \]
\[ \Omega \rightarrow \Omega ' \]
\[ \text{dom}(\Delta) = \text{dom}(\Omega ') \]
\[ \Delta \rightarrow \Omega ' \]

- **By Lemma 33 (Extension Transitivity)**
- **Given**

\[ \Gamma \vdash \text{ A} ! \text{ type} \]

- **By Anno**

\[ [\Delta] \text{ A} = [\Delta][\Delta] \text{ A} \]
\[ \text{ A} = [\Omega] \text{ A} \]
\[ = [\Omega '] \text{ A} \]
\[ = [\Omega '] [\Delta] \text{ A} \]

- **By Lemma 55 (Completing Completeness) (ii)**
- **By Lemma 29 (Substitution Monotonicity)**

• Case  
\[ [\Omega] \Gamma \vdash () \leftarrow \text{ 1 p} \]

We have \[ [\Omega] \text{ A} = 1 \]. Either \[ [\Gamma] \text{ A} = 1 \], or \[ [\Gamma] \text{ A} = \hat{\alpha} \] where \( \hat{\alpha} \in \text{ unsolved}(\Gamma) \).

In the former case:

\[ \text{ Let } \Delta = \Gamma. \]
\[ \text{ Let } \Omega ' = \Omega. \]
\[ \Gamma \rightarrow \Omega \]
\[ \Omega \rightarrow \Omega ' \]
\[ \text{ dom}(\Gamma) = \text{ dom}(\Omega) \]
\[ \Gamma \vdash () \leftarrow \text{ 1 p } \Gamma \]
\[ 1 = [\Gamma]1 \]

In the latter case, since \( \text{ A} = \hat{\alpha} \) and \( \Gamma \vdash \hat{\alpha} \text{ p} \text{ type} \) is given, it must be the case that \( \text{ p} = \not{\hat{\alpha}} \).

\[ \Gamma_0[\hat{\alpha} : \ast] \vdash () \leftarrow \hat{\alpha} \not{\rightarrow} \Gamma_0[\hat{\alpha} : \ast] = 1 \]
\[ \Gamma_0[\hat{\alpha} : \ast] \vdash () \leftarrow [\Gamma_0[\hat{\alpha} : \ast]] \hat{\alpha} \not{\rightarrow} \Gamma_0[\hat{\alpha} : \ast] = 1 \]

\[ \Gamma_0[\hat{\alpha} : \ast] \rightarrow \Omega \]
\[ \Gamma_0[\hat{\alpha} : \ast] = 1 \rightarrow \Omega \]
\[ \Omega \rightarrow \Omega \]

- **By Lemma 27 (Parallel Extension Solution)**
- **By Lemma 32 (Extension Reflexivity)**

• Case  
\[ \nu \text{ chk-1} \]
\[ [\Omega] \Gamma, \alpha : \kappa \vdash [\Omega] \nu \leftarrow \text{ A}_0 \text{ p} \]
\[ [\Omega] \Gamma \vdash [\Omega] \nu \leftarrow \forall \alpha : \kappa. \text{ A}_0 \text{ p} \]

- **DeclAnno**
Proof of Theorem 12 (Completeness of Algorithmic Typing)

$\forall \alpha : \kappa. A_0 \Rightarrow [\Omega] A = \forall \alpha : \kappa. A_0'$

Given

By def. of subst. and predicativity of $\Omega$

$A_0 = [\Omega] A'$

Follows from above equality

$[\Omega][\Gamma, \alpha : \kappa] \vdash [\Omega] v \iff [\Omega] A' \ p$

Subderivation and above equality

$\Gamma \rightarrow \Omega$

By $\text{Uvar}$

$\Gamma, \alpha : \kappa \rightarrow \Omega, \alpha : \kappa$

By definition of context substitution

$[\Omega][\Gamma, \alpha : \kappa] = [\Omega, \alpha : \kappa](\Gamma, \alpha : \kappa)$

By definition of context substitution

$[\Omega, \alpha : \kappa](\Gamma, \alpha : \kappa) \vdash [\Omega] v \iff [\Omega] A' \ p$

By above equality

$[\Omega, \alpha : \kappa](\Gamma, \alpha : \kappa) \vdash [\Omega] v \iff [\Omega, \alpha : \kappa] A' \ p$

By definition of substitution

$\Gamma, \alpha : \kappa \vdash v \iff [\Gamma, \alpha : \kappa] A' \ p \vdash \Delta'$

By i.h.

$\Delta' \rightarrow \Omega_0'$

By Lemma 22 (Extension Inversion) (i)

$\Omega, \alpha : \kappa \rightarrow \Omega_0'$

By above equality

$\Delta' = (\Delta, \alpha : \kappa, \Theta)$

By definition of context application

$\Delta, \alpha : \kappa, \Theta \rightarrow \Omega_0'$(i)

By definition of substitution

$\Omega' \rightarrow \Delta'$

By above equality

$\text{dom}(\Delta) = \text{dom}(\Omega')$

By definition of context substitution

$\Omega \rightarrow \Omega'$

By Lemma 22 (Extension Inversion) on $\Omega, \alpha : \kappa \rightarrow \Omega_0'$

$\Gamma, \alpha : \kappa \vdash v \iff [\Gamma, \alpha : \kappa] A' \ p \vdash \Delta, \alpha : \kappa, \Theta$

By above equality

$\Gamma, \alpha : \kappa \vdash v \iff [\Gamma] A' \ p \vdash \Delta, \alpha : \kappa, \Theta$

By definition of substitution

$\Gamma \vdash v \iff \forall \alpha : \kappa. [\Gamma] A' \ p \vdash \Delta$

By i.h.

$\Gamma \vdash v \iff [\Gamma](\forall \alpha : \kappa. A') \ p \vdash \Delta$

By definition of substitution

Case

$[\Omega][\Gamma] \vdash \tau : \kappa$

Subderivation

$[\Omega][\Gamma] \vdash [\Omega](e \ s_0) : [\tau / \alpha][\Omega] A_0 \ f \gg B \ q$

DeclSpine

$[\Omega][\Gamma] \vdash [\Omega](e \ s_0) : \forall \alpha : \kappa. [\Omega] A_0 \ f \gg B \ q$

$\Gamma \rightarrow \Omega$

Given

$\Gamma, \hat{\alpha} : \kappa \rightarrow \Omega, \hat{\alpha} : \kappa = \tau$

By $\text{Solve}$

$[\Omega][\Gamma] \vdash [\Omega](e \ s_0) : [\tau / \alpha][\Omega] A_0 \ f \gg B \ q$

Subderivation

$\tau = [\Omega] \tau$

FEV($\tau$) = 0

$[\tau / \alpha][\Omega] A_0 = [\tau / \alpha][\Omega, \hat{\alpha} : \kappa = \tau] A_0$

By def. of subst.

By above equality

$[\Omega, \hat{\alpha} : \kappa = \tau] [\hat{\alpha} / \alpha] A_0$

By distributivity of substitution

$[\Omega][\Gamma] = [\Omega, \hat{\alpha} : \kappa = \tau](\Gamma, \hat{\alpha} : \kappa)$

By definition of context application
Proof of Theorem 12 (Completeness of Algorithmic Typing)  thm:typing-completeness  173

\[ [\Omega, \alpha : \kappa = \tau](\Gamma, \alpha : \kappa) \vdash [\Omega](e s_0) : [\Omega, \alpha : \kappa = \tau][\alpha/\alpha]A_0 \not\succ B q \]  By above equalities
\[ \Gamma, \alpha : \kappa \vdash e s_0 : [\alpha/\alpha]A_0 \not\succ B q \dashv \Delta \]  By i.h.
\[ B = [\Omega, \alpha : \kappa = \tau]B' \]

\[ \Delta \rightarrow \Omega' \]
\[ \text{dom}(\Delta) = \text{dom}(\Omega') \]
\[ \Omega \rightarrow \Omega' \]
\[ B' \rightarrow [\Delta]B' \]
\[ B \rightarrow [\Omega']B' \]

\[ [\Gamma, \alpha : \kappa][\alpha/\alpha]A_0 = [\Gamma][\alpha/\alpha]A_0 \]  By def. of context application
\[ \Gamma, \alpha : \kappa \vdash e s_0 : [\alpha/\alpha]A_0 \not\succ B' q \dashv \Delta \]  By above equality
\[ \Gamma \vdash e s_0 : \forall \alpha : \kappa. [\Gamma]A_0 \not\succ B' q \dashv \Delta \]  By \text{vSpine}
\[ \Gamma \vdash e s_0 : [\Gamma](\forall \alpha : \kappa. A_0) \not\succ B' q \dashv \Delta \]  By def. of subst.

- Case \( v \text{ chk-I} \)

\[
\frac{\Omega \vdash [\Omega]v \equiv [\Omega]A_0 !}{\Omega \vdash [\Omega]v \equiv [(\Omega)P \cup [\Omega]A_0 !} \quad \text{Decl<->}
\]

\[ [\Omega]P \vdash [\Omega]v \equiv [\Omega]A_0 ! \quad \text{Subderivation} \]

The concluding rule in this subderivation must be \text{DeclCheck} or \text{DeclCheckUnify}. In either case, \([\Omega]P\) has the form \((\sigma' = \tau')\) where \(\sigma' = [\Omega]\sigma\) and \(\tau' = [\Omega]\tau\).

- Case \( mgu([\Omega]\sigma, [\Omega]\tau) = \bot \)

\[
\frac{[\Omega]P \vdash [\Omega]v \equiv [\Omega]A_0 !}{[\Omega]v' \equiv [\Omega]A_0 !} \quad \text{DeclCheck}\_\text{I}
\]

We have \(mgu([\Omega]\sigma, [\Omega]\tau) = \bot\). To apply Lemma 94 (Completeness of \text{Elimeq}) (2), we need to show conditions 1–5.

***
\[ \Gamma \vdash (\sigma = \tau) \supset A_0 ! \quad \text{type} \]
\[ [\Omega](\sigma = \tau) \supset A_0 = [\Gamma](\sigma = \tau) \supset A_0 \]  By Lemma 39 (Principal Agreement) (i)
\[ [\Omega]\sigma = [\Gamma]\sigma \]  By a property of subst.
\[ [\Omega]\tau = [\Gamma]\tau \]  Similar

\[ \Gamma \vdash \sigma : \kappa \]  By inversion

3 \[ \Gamma \vdash [\Gamma]\sigma : \kappa \]  By Lemma 11 (Right-Hand Substitution for Sorting)

4 \[ \Gamma \vdash [\Gamma]\tau : \kappa \]  Similar

\[ mgu([\Omega]\sigma, [\Omega]\tau) = \bot \quad \text{Given} \]
\[ mgu([\Gamma]\sigma, [\Gamma]\tau) = \bot \quad \text{By above equalities} \]
\[ \text{FEV}(\sigma) \cup \text{FEV}(\tau) = \emptyset \]  By inversion on ***
\[ \text{FEV}([\Omega]\sigma) \cup \text{FEV}([\Omega]\tau) = \emptyset \]  By a property of complete contexts
\[ \text{FEV}([\Gamma]\sigma) \cup \text{FEV}([\Gamma]\tau) = \emptyset \]  By above equalities
\[ 1 \quad [\Gamma]\sigma = [\Gamma]\sigma \]  By idempotence of subst.
\[ 2 \quad [\Gamma]\tau = [\Gamma]\tau \]  By idempotence of subst.
Proof of Theorem 12 (Completeness of Algorithmic Typing)

\[ \Gamma \vdash \nu \iff (\Gamma \sigma = [\Gamma \tau : \kappa \vdash \bot]) \quad \text{By Lemma 94 (Completeness of Elimeq) (2)} \]

\[ \Gamma, \triangleright_p \vdash [\Gamma \sigma = [\Gamma \tau \vdash \bot]] \quad \text{By \texttt{ElimpropEq}} \]

\[ \Gamma \vdash \nu \iff ([\Gamma \sigma = [\Gamma \tau \vdash \bot]) \supset \Gamma \theta \vdash \Gamma \oslash \Gamma \quad \text{By def. of subst.} \]

\[ \Gamma \longrightarrow \Omega \quad \text{Given} \]

\[ \Omega \longrightarrow \Omega \quad \text{By Lemma 32 (Extension Reflexivity)} \]

\[ \text{dom} (\Gamma) = \text{dom} (\Omega) \quad \text{Given} \]

\[ \text{Case } \quad - \text{ Fig. 174: } \mu (\Omega \sigma, \Omega) = \mu (\sigma, \tau) = \theta \quad \text{By Lemma 94 (Completeness of Elimeq) (1)} \]

\[ \theta (\mu (\Omega \sigma, \Omega)) = \theta (\mu (\sigma, \tau)) \quad \text{By def. of subst.} \]

\[ \text{We have } \mu (\Omega \sigma, \Omega) = \theta, \text{ and will need to apply Lemma 94 (Completeness of Elimeq) (1). That lemma has five side conditions, which can be shown exactly as in the (DeclCheck Unify) case above.} \]

\[ \mu (\sigma, \tau) = \theta \quad \text{Premise} \]

\[ \Omega_0 = \{ \Omega_0, \triangleright_p \} \quad \text{Let} \]

\[ \Gamma \longrightarrow \Omega \quad \text{Given} \]

\[ \Gamma, \triangleright_p \longrightarrow \Omega_0 \quad \text{By \texttt{Marker}} \]

\[ \text{dom} (\Gamma) = \text{dom} (\Omega) \quad \text{Given} \]

\[ \text{dom} (\Gamma, \triangleright_p) = \text{dom} (\Omega_0) \quad \text{By def. of dom (–)} \]

\[ \Gamma, \triangleright_p \vdash [\Gamma \sigma = [\Gamma \tau : \kappa \vdash \bot, \Gamma, \triangleright_p, \theta] \quad \text{By Lemma 94 (Completeness of Elimeq) (1)} \]

\[ \Gamma, \triangleright_p \vdash [\Gamma \tau : \kappa \vdash \bot, \Gamma, \triangleright_p, \theta] \quad \text{By \texttt{ElimpropEq}} \]

\[ \Omega_0 \rightarrow \Omega \quad \text{for all } \Gamma, \triangleright_p \vdash u : \kappa. \quad [\Gamma, \triangleright_p, \theta]u = \theta ([\Gamma, \triangleright_p]u) \quad \text{‘} \]

\[ \Gamma \vdash P \supset A_0 \quad \text{type} \quad \text{Given} \]

\[ \Gamma \vdash A_0 \quad \text{type} \quad \text{By inversion} \]

\[ \Gamma \longrightarrow \Omega \quad \text{Given} \]

\[ \text{EQa } \quad [\Gamma]A_0 = [\Omega]A_0 \quad \text{By Lemma 39 (Principal Agreement) (i)} \]

\[ \text{Let } \Omega_1 = (\Omega_0, \triangleright_p, \theta). \]

\[ \theta (\mu (\Omega) \Gamma) = \theta (\mu (\nu \theta) \Gamma A_0) \quad \text{Subderivation} \]

\[ \Gamma, \triangleright_p, \theta \longrightarrow \Omega_1 \quad \text{By induction on } \theta \]

\[ \theta ([\Omega]A_0) = \theta ([\Gamma]A_0) \quad \text{By above equality EQa} \]

\[ = [\Gamma, \triangleright_p, \theta]A_0 \quad \text{By Lemma 95 (Substitution Upgrade) (i) (with EQ0)} \]

\[ = [\Omega_0]A_0 \quad \text{By Lemma 39 (Principal Agreement) (i)} \]

\[ = [\Omega_1] [\Gamma, \triangleright_p, \theta]A_0 \quad \text{By Lemma 39 (Substitution Monotonicity) (iii)} \]

\[ \theta ([\Delta]e) = [\Delta]e \quad \text{By Lemma 95 (Substitution Upgrade) (iv)} \]

\[ [\Omega_1] [\Gamma, \triangleright_p, \theta] = [\Omega_1]e \quad \text{By above equalities} \]

\[ \text{dom} (\Gamma, \triangleright_p, \theta) = \text{dom} (\Omega_1) \quad \text{dom} (\Gamma) = \text{dom} (\Omega) \]
Proof of Theorem 12 (Completeness of Algorithmic Typing)

Case \( \text{dom} \ / \ \text{PointingHand} \)

\[ \begin{align*}
\Gamma, \text{p}, \Theta & \vdash e \iff [\Gamma, \text{p}, \Theta]A_0 \vdash \Delta' \quad \text{By i.h.} \\
\Delta' & \rightarrow \Omega_1' \\
\Omega & \rightarrow \Omega_1 \\
\end{align*} \]

\begin{itemize}
\item \( \text{dom}(\Delta') = \text{dom}(\Omega_1') \)
\item \( \Delta' = (\Delta, \text{p}, \Delta'') \) \quad By Lemma 22 (Extension Inversion) (ii)
\item \( \Omega_1' = (\Omega', \text{p}, \Omega_Z) \) \quad By Lemma 22 (Extension Inversion) (ii)
\end{itemize}

\[ \begin{align*}
\Delta & \rightarrow \Omega' \\
\Omega_0 & \rightarrow \Omega_1' \\
\end{align*} \]

By Lemma 33 (Extension Transitivity)

By above equalities

\[ \begin{align*}
\text{dom}(\Delta) & = \text{dom}(\Omega') \\
\end{align*} \]

Proof of Theorem 12

\[ \begin{align*}
\Gamma, \text{p}, \Theta & \vdash e \iff [\Gamma, \text{p}, \Theta]A_0 \vdash \Delta, \text{p}, \Delta'' \quad \text{By above equality} \\
\Gamma & \vdash e \iff ([\Gamma] \sigma = [\Gamma] \tau) \vdash [\Gamma]A_0 \vdash \Delta \quad \text{By } \sqsubseteq \\
\Gamma & \vdash e \iff [\Gamma](\text{p} \supset A_0) \vdash \Delta \quad \text{By def. of subst.}
\end{align*} \]
Proof of Theorem 12 (Completeness of Algorithmic Typing)

• Case \([\Omega]\Gamma \vdash [\Omega]e_0 \leftarrow A'_k \ p\)
  \([\Omega]\Gamma \vdash \text{inj}_k [\Omega]e_0 \leftarrow A'_1 + A'_2 \ p\)
  \(\text{Decl+I}_k\)

Either \(\Gamma A = A_1 + A_2\) (where \(\Omega A_k = A'_k\)) or \(\Gamma A = \hat{\alpha} \in \text{unsolved}(\Gamma)\).
In the former case:
  \([\Omega]\Gamma \vdash [\Omega]e_0 \leftarrow A'_k \ p\)
  Subderivation
  \([\Omega]\Gamma \vdash [\Omega]e_0 \leftarrow [\Omega]A_k \ p\)
  \([\Omega]A_k = A'_k\)
  \(\Gamma \vdash e_0 \leftarrow [\Gamma]A_k \ p \vdash \Delta\)
  By i.h.
  \(\Rightarrow \Delta \rightarrow \Omega\)
  "
  \(\Rightarrow \text{dom}(\Delta) = \text{dom}(\Omega')\)
  "
  \(\Rightarrow \Omega \rightarrow \Omega'\)
  "
  \(\Gamma \vdash \text{inj}_k e_0 \leftarrow ([\Gamma]A_1) + ([\Gamma]A_2) \ p \vdash \Delta\)
  By \(+I_k\)
  \(\Rightarrow \Gamma \vdash \text{inj}_k e_0 \leftarrow [\Gamma](A_1 + A_2) \ p \vdash \Delta\)
  By def. of subst.

In the latter case, \(A = \hat{\alpha}\) and \(\Omega A = [\Omega]\hat{\alpha} = A'_1 + A'_2 = \tau'_1 + \tau'_2\).
By inversion on \(\Gamma \vdash \hat{\alpha} \ p\) type, it must be the case that \(p = \text{f}\).
\(\Gamma \rightarrow \Omega\) Given
\(\Gamma = \Gamma_0[\alpha : \tau]\) \(\hat{\alpha} \in \text{unsolved}(\Gamma)\)
\(\Omega = \Omega_0[\hat{\alpha} : \star = \tau_0]\) By Lemma 22 (Extension Inversion) (vi)

Let \(\Omega_2 = \Omega_0[\hat{\alpha}_1 : \star = \tau'_1, \hat{\alpha}_2 : \star = \tau'_2, \hat{\alpha} : \star = \hat{\alpha}_1 + \hat{\alpha}_2]\).

Let \(\Gamma_2 = \Gamma_0[\hat{\alpha}_1 : \star, \hat{\alpha}_2 : \star, \hat{\alpha} : \star = \hat{\alpha}_1 + \hat{\alpha}_2]\).

\(\Gamma \rightarrow \Gamma_2\) By Lemma 23 (Deep Evar Introduction) (iii) twice
  and Lemma 26 (Parallel Admissibility) (ii)
\(\Omega \rightarrow \Omega_2\) By Lemma 23 (Deep Evar Introduction) (iii) twice
  and Lemma 26 (Parallel Admissibility) (iii)
\(\Gamma_2 \rightarrow \Omega_2\) By Lemma 26 (Parallel Admissibility) (ii), (ii), (iii)

\(\Rightarrow \Delta \rightarrow \Omega'\) "
\(\Rightarrow \text{dom}(\Delta) = \text{dom}(\Omega')\) "
\(\Rightarrow \Omega_2 \rightarrow \Omega'\) "
\(\Rightarrow \Omega \rightarrow \Omega'\) By Lemma 33 (Extension Transitivity)
\(\Rightarrow \Gamma \vdash \text{inj}_k e_0 \Rightarrow \hat{\alpha} \ f \vdash \Delta\) By \(+I\hat{\alpha}_k\)
\(\Rightarrow \Gamma \vdash \text{inj}_k e_0 \Rightarrow [\Gamma]\hat{\alpha} \ f \vdash \Delta\) \(\hat{\alpha} \in \text{unsolved}(\Gamma)\)

• Case \([\Omega]\Gamma, x : A'_1 \ p \vdash [\Omega]e_0 \leftarrow A'_2 \ p\)
  \([\Omega]\Gamma \vdash \lambda x. [\Omega]e_0 \leftarrow A'_1 \rightarrow A'_2 \ p\)
  \(\text{Decl} \rightarrow I\)

We have \(\Omega A = A'_1 \rightarrow A'_2\). Either \(\Gamma A = A_1 \rightarrow A_2\) where \(A'_1 = [\Omega]A_1\) and \(A'_2 = [\Omega]A_2\)—or \(\Gamma A = \hat{\alpha}\) and \(\Omega \hat{\alpha} = A'_1 \rightarrow A'_2\).
Proof of Theorem 12 (Completeness of Algorithmic Typing)

In the former case:

\[ [\Omega] \Gamma, x : A_1 p \vdash [\Omega] e_0 \iff A_2 p \]

Subderivation

\[ \begin{align*}
A_1 & = [\Omega] A_1 \\
& = [\Omega] [\Gamma] A_1 \\
\end{align*} \]

By inversion on \( \Gamma \)

\[ [\Omega] A_1 = [\Omega] [\Omega] [\Gamma] A_1 \]

Applying \( \Omega \) on both sides

\[ \begin{align*}
[\Omega] \Gamma, x : A_1 p & = [\Omega, x : A_1 p] [\Gamma, x : [\Gamma] A_1 p] \\
& \text{By definition of context application}
\end{align*} \]

\[ [\Omega, x : A_1 p] [\Gamma, x : [\Gamma] A_1 p] \vdash [\Omega] e_0 \iff A_2 p \]

By above equality

\[ \begin{align*}
\Gamma & \longrightarrow \Omega \\
\Gamma, x : [\Gamma] A_1 p & \longrightarrow \Omega, x : A_1 p \\
\text{dom}(\Gamma) & = \text{dom}(\Omega) \\
\text{dom}(\Gamma, x : [\Gamma] A_1 p) & = \Omega, x : A_1 p \\
\Gamma, x : [\Gamma] A_1 p & \vdash e_0 \iff A_2 p \vdash \Delta' \\
\Delta' & \longrightarrow \Omega_0' \\
\text{dom}(\Delta') & = \text{dom}(\Omega_0') \\
\Omega, x : A_1 p & \longrightarrow \Omega_0' \\
\Omega_0' & = (\Omega', x : A_1 p, \Omega_Z) \\
\Omega & \longrightarrow \Omega'
\end{align*} \]

By definition of context application

\[ \begin{align*}
\Gamma, x : [\Gamma] A_1 p & \longrightarrow \Delta' \\
\Delta' & = (\Delta, x : \cdots, \Theta) \\
\Delta, x : \cdots, \Theta & \longrightarrow \Omega', x : A_1 p, \Omega_Z \\
\Delta & \longrightarrow \Omega' \\
\text{dom}(\Delta) & = \text{dom}(\Omega')
\end{align*} \]

By above equalities

\[ \begin{align*}
\Gamma, x : [\Gamma] A_1 p & \vdash e_0 \iff [\Gamma] A_2 p \vdash \Delta, x : \cdots, p, \Theta \\
\Gamma & \vdash \lambda x. e_0 \iff ([\Gamma] A_1) \to ([\Gamma] A_2) p \vdash \Delta \\
\Gamma & \vdash \lambda x. e_0 \iff [\Gamma] (A_1 \to A_2) p \vdash \Delta \\
\end{align*} \]

By definition of substitution

In the latter case \(([\Gamma] A = \hat{\alpha} \in \text{unsolved}(\Gamma)) \) and \([\Omega] \hat{\alpha} = A_1' \to A_2' = \tau_1' \to \tau_2'\): By inversion on \( \Gamma \vdash \hat{\alpha} \) and \( \hat{\alpha} \) for type \( \hat{\alpha} \).

Since \( \hat{\alpha} \in \text{unsolved}(\Gamma) \), the context \( \Gamma \) must have the form \( \Gamma_0[\hat{\alpha} : \star] \).

Let \( \Gamma_2 = \Gamma_0[\hat{\alpha}_1 : \star, \hat{\alpha}_2 : \star, \hat{\alpha} : \star = \hat{\alpha}_1 \to \hat{\alpha}_2] \).

\[ \begin{align*}
\Gamma & \longrightarrow \Gamma_2 \\
\text{By Lemma 23 (Deep Evar Introduction)} (\text{iii}) \text{ twice} \\
\text{and Lemma 26 (Parallel Admissibility)} (\text{ii})
\end{align*} \]

\[ [\Omega] \hat{\alpha} = \tau_1' \to \tau_2' \]

Known in this subcase

\[ \begin{align*}
\Gamma & \longrightarrow \Omega \\
\Omega & = \Omega_0[\hat{\alpha} : \star = \tau_0] \\
\text{By Lemma 22 (Extension Inversion)} (\text{vi})
\end{align*} \]

Let \( \Omega_2 = \Omega_0[\hat{\alpha}_1 : \star = \tau_1', \hat{\alpha}_1 : \star = \tau_2', \hat{\alpha} : \star = \hat{\alpha}_1 \to \hat{\alpha}_2] \).
Proof of Theorem 12 (Completeness of Algorithmic Typing)

\[ \Gamma \longrightarrow \Gamma_2 \]

By Lemma 23 (Deep Evar Introduction) (iii) twice and Lemma 26 (Parallel Admissibility) (ii)

\[ \Omega \longrightarrow \Omega_2 \]

By Lemma 23 (Deep Evar Introduction) (iii) twice and Lemma 26 (Parallel Admissibility) (iii)

\[ \Gamma_2 \longrightarrow \Omega_2 \]

By Lemma 26 (Parallel Admissibility) (ii), (ii), (iii)

\[ \Omega[\Gamma, x : \tau'] \eta \vdash \Omega[e_0] \triangleq \tau' \eta \]

Subderivation

\[ \Omega[\Gamma] = \Omega[\Omega_2]\Gamma_2 \]

By Lemma 57 (Multiple Confluence)

\[ \tau' = \Omega[\Delta_2] \]

From above equality

\[ = [\Omega_2] \Delta_2 \]

By Lemma 55 (Completing Completeness) (i)

\[ \tau'_1 = [\Omega_2] \Delta_1 \]

Similar

\[ [\Omega_2 \Gamma_2, x : \tau'_1] \eta \vdash [\Omega_2, x : \tau'_1] \eta(\Gamma_2, x : \Delta_1 \eta) \]

By def. of context application

\[ [\Omega_2, x : \tau'_1] \eta(\Gamma_2, x : \Delta_1 \eta) \vdash \Omega[e_0] \triangleq [\Omega_2] \Delta_2 \eta \]

By above equalities

\[ \text{dom}(\Gamma) = \text{dom}(\Omega) \]

Given

\[ \text{dom}(\Gamma_2, x : \Delta_1 \eta) = \text{dom}(\Omega_2, x : \tau'_1) \eta) \]

By def. of \( \Gamma_2 \) and \( \Omega_2 \)

\[ \Gamma_2, x : \Delta_1 \eta \vdash e_0 \triangleq [\Gamma_2, x : \Delta_1 \eta] \Delta_2 \eta \rightarrow \Delta^+ \]

By i.h.

\[ \Delta^+ \longrightarrow \Omega^+ \]

By above equality

\[ \Omega^+ = (\Omega^+, x : \ldots, \eta, \Omega_x) \]

By Lemma 22 (Extension Inversion) (v)

\[ \Delta \longrightarrow \Omega' \]

By Lemma 51 (Typing Extension)

\[ \Delta^+ = (\Delta, x : \Delta_1 \eta, \Delta_2) \]

By Lemma 22 (Extension Inversion) (v)

\[ \text{dom}(\Delta) = \text{dom}(\Omega') \]

By above equality

\[ \Omega \longrightarrow \Omega_2 \]

Above

\[ \Omega \longrightarrow \Omega^+ \]

By Lemma 33 (Extension Transitivity)

\[ \Omega \longrightarrow \Omega' \]

By Lemma 22 (Extension Inversion) (v)

\[ \Gamma \vdash \lambda x. e_0 \triangleq [\Gamma] \Delta \eta \]

By \(-\lambda x\)

\[ \Delta = [\Gamma] \Delta \]

\[ \Delta \in \text{unsolved}(\Gamma) \]

\[ \Gamma \vdash \lambda x. e_0 \triangleq \Gamma[\Delta] \eta \rightarrow \Delta \]

By above equality
Proof of Theorem 12 (Completeness of Algorithmic Typing)

- Case $[\Omega][\Gamma, x : [\Omega]A p \vdash [\Omega]v \iff [\Omega]A p] \frac{\text{DeclRec}}{[\Omega][\Gamma] \vdash \text{rec} x. [\Omega]v \iff [\Omega]A p}$


  \[
  \begin{align*}
  \Gamma & \longrightarrow [\Omega] \\
  \text{dom}(\Gamma) &= \text{dom}(\Omega) \\
  \Gamma, x : [\Gamma]A p \vdash v & \iff [\Gamma]A p \vdash \Delta' \\
  \Delta' & \longrightarrow [\Omega]' \\
  \text{dom}(\Delta') &= \text{dom}(\Omega)' \\
  \Omega, x : [\Omega]A p & \longrightarrow [\Omega]' \\
  \Omega' &= (\Omega', x : [\Omega]A p, \Theta) \\
  \Omega & \longrightarrow [\Omega]' \\
  \end{align*}
  \]

  By Lemma 22 (Extension Inversion) (v)

- Case $[\Omega][\Gamma] \vdash [\Omega][e_0 : A q] \Rightarrow C p \frac{\text{DeclE}}{[\Omega][\Gamma] \vdash [\Omega][e_0 s_0] \Rightarrow C p}$

  $[\Omega][\Gamma] \vdash [\Omega][e_0] A q$ Subderivation

  $\Gamma \vdash e_0 \Rightarrow A' q \uparrow \Theta$ By i.h.

  $\Theta \longrightarrow [\Omega]_{\Theta}$ "

  dom(\Theta) = dom(\Omega) "

  $\Omega \longrightarrow [\Omega]_{\Theta}$ "

  $A = [\Omega]_{\Theta} A'$ "

  $A' = [\Theta] A'$ "

  By above equality

  $[\Omega][\Gamma] \vdash \text{rec} x. [\Omega]v \iff [\Omega]A p$ By Rec
\begin{align*}
\Gamma \rightarrow \Omega \\
[\Omega] \Gamma = [\Omega_\Theta] \Theta \\
[\Omega] \Gamma \vdash [\Omega] s_0 : A \gg C [p] \\
[\Omega_\Theta] \Theta \vdash [\Omega] s_0 : [\Omega_\Theta] A' \gg C [p] \\
\Theta \vdash s_0 : [\Theta] A' \gg C' [p] \dashv \Delta
\end{align*}

By Lemma \ref{lem:multi-confluence} (Multiple Confluence)
Subderivation
By above equalities
By i.h.

\begin{align*}
C' &= [\Delta] C' \\
\Delta &\rightarrow \Omega' \\
\text{dom}(\Delta) &= \text{dom}(\Omega') \\
\Omega_\Theta &\rightarrow \Omega' \\
C &= [\Omega'] C' \\
\Theta \vdash s_0 : A' q \gg C' [p] \dashv \Delta
\end{align*}

By above equality
By Lemma \ref{lem:extension-transitivity} (Extension Transitivity)

\begin{align*}
\Gamma \vdash e_0 s_0 \Rightarrow C' p \dashv \Delta
\end{align*}

By $\Rightarrow$E
Proof of **Theorem 12** (Completeness of Algorithmic Typing) thm:typing-completeness

- **Case**

  \[
  \begin{align*}
  \{\Omega\} \Gamma \vdash [\Omega] s : [\Omega] A & \Rightarrow C \ \Leftrightarrow \ [\Omega] \Gamma \vdash [\Omega] s : [\Omega] A \Rightarrow C_2 \ \text{if} \ [\Omega] \Gamma \vdash [\Omega] s : [\Omega] A \Rightarrow C \ [\Omega] \Gamma \vdash [\Omega] s : [\Omega] A \Rightarrow C_1
  \end{align*}
  \]

  Given
  \[
  \begin{align*}
  \Gamma \rightarrow \Omega & \quad \text{Subderivation} \\
  [\Omega] \Gamma \vdash [\Omega] s : [\Omega] A ! 
  \end{align*}
  \]

  \[
  \begin{align*}
  \Delta \rightarrow \Omega' & \quad \text{By Lemma 60 (Split Solutions)} \\
  \Omega \rightarrow \Omega' & \quad \text{"} \\
  \text{dom}(\Delta) = \text{dom}(\Omega') & \quad \text{"} \\
  C = [\Omega'] C' & \quad \text{"} \\
  C' = [\Delta] C' & \quad \text{"}
  \end{align*}
  \]

  Suppose, for a contradiction, that \( \text{FEV}([\Delta] C') \neq \emptyset \).
  That is, there exists some \( \hat{\alpha} \in \text{FEV}([\Delta] C') \).

  \[
  \Delta \rightarrow \Omega_2
  \]

  \[
  \begin{align*}
  \Omega_2 \rightarrow \Omega'_2 & \quad \text{By Lemma 60 (Split Solutions)} \\
  \Omega_2 = \Omega'_2 [\hat{\alpha} : \kappa = t_2] & \quad \text{"} \\
  t_2 \neq t_1 & \quad \text{"}
  \end{align*}
  \]

  (NEQ) \( [\Omega_2] \hat{\alpha} \neq [\Omega_1'] \hat{\alpha} \)

  By def. of subst. \( t_2 \neq t_1 \)

  (EQ) \( [\Omega_2] \hat{\beta} = [\Omega_1'] \hat{\beta} \) for all \( \hat{\beta} \neq \hat{\alpha} \)

  By construction of \( \Omega_2 \)

  and \( \Omega_2 \) canonical

Choose \( \hat{\alpha}_R \) such that \( \hat{\alpha}_R \in \text{FEV}(C') \) and either \( \hat{\alpha}_R = \hat{\alpha} \) or \( \hat{\alpha}_R \in \text{FEV}([\Delta] \hat{\alpha}_R) \).

Then either \( \hat{\alpha}_R = \hat{\alpha} \), or \( \hat{\alpha}_R \) is declared to the right of \( \hat{\alpha} \) in \( \Delta \).

\[
[\Omega_2] C' \neq [\Omega'] C'
\]

From (NEQ) and (EQ)

\[
\begin{align*}
\Gamma \vdash s : [\Gamma] A ! \Rightarrow C' \ \& \Delta \\
\Gamma \vdash [\Omega_2] s : [\Omega_2] [\Gamma] A ! \Rightarrow [\Omega_2] C' \ \& \Delta
\end{align*}
\]

By Theorem 9

\[
\begin{align*}
\Gamma \vdash s : [\Gamma] A ! \Rightarrow C' & \quad \text{Above} \\
\Gamma \vdash A ! \text{ type} & \quad \text{Given} \\
\Gamma \vdash [\Gamma] A ! \text{ type} & \quad \text{By Lemma 13 (Right-Hand Substitution for Typing)}
\end{align*}
\]

FEV([\Gamma] A) = \emptyset

FEV([\Gamma] A) \subseteq \text{dom}(\cdot)

\[
\Delta = (\Delta_L \ast \Delta_R)
\]

FEV(C') \subseteq \text{dom}(\Delta_R)

\[
\hat{\alpha}_R \in \text{FEV}(C')
\]

\[
\begin{align*}
\hat{\alpha}_R \in \text{dom}(\Delta_R) & \quad \text{Above} \\
\text{dom}(\Delta_L) \cap \text{dom}(\Delta_R) = \emptyset & \quad \text{Property of} \subseteq \\
\hat{\alpha}_R \notin \text{dom}(\Delta_L) & \quad \Delta \text{ well-formed} \\
\hat{\alpha}_R \notin \text{dom}(\Gamma) & \quad \text{By Definition 5}
\end{align*}
\]

Proof of **Theorem 12** (Completeness of Algorithmic Typing) thm:typing-completeness
Proof of **Theorem 12** (Completeness of Algorithmic Typing)  \[ \text{thm:typing-completeness} \]

\[
[\Omega_2]\Gamma \vdash [\Omega_2]s : [\Omega_2][\Gamma]A \implies [\Omega_2]C' f
\]

\(\Omega_2\) and \(\Omega_1\) differ only at \(\hat{\alpha}\)  
\(\text{FEV}([\Gamma]A) = \emptyset\)  
\([\Omega_2][\Gamma]A = [\Omega_1][\Gamma]A\)  
\(\Gamma \vdash [\Gamma]A \text{ type}\)  
\(\Gamma \rightarrow \Omega_2\)  
\(\Omega_2 \vdash [\Gamma]A \text{ type}\)  
\(\dom(\Omega_2) = \dom(\Omega_1)\)  
\(\Omega_1 \vdash [\Gamma]A \text{ type}\)  
\(\Omega_1\) and \(\Omega_2\) differ only at \(\hat{\alpha}\)  
By preceding two lines

\(\Gamma \vdash [\Gamma]A \text{ type}\)  
By Lemma 33 (Extension Transitivity)  
\([\Omega_1][\Gamma]A = [\Omega'][\Gamma]A = [\Omega][\Gamma]A\)  
By Lemma 55 (Completing Completeness) (ii) twice  
\(= [\Omega]A\)  
By Lemma 29 (Substitution Monotonicity) (iii)

\([\Omega][\Gamma] = [\Omega'][\Gamma]\)  
By Lemma 57 (Multiple Confluence)  
\(= [\Omega_1][\Gamma]\)  
By Lemma 57 (Multiple Confluence)  
\(= [\Omega_2][\Gamma]\)  
Follows from \(\hat{\alpha}_R \notin \dom(\Gamma)\)

\([\Omega_2]s = [\Omega]s\)  
\(\Omega_2\) and \(\Omega\) differ only in \(\hat{\alpha}\)

\([\Omega][\Gamma] \vdash [\Omega]s : [\Omega]A ! \implies [\Omega_2]C' f\)  
By above equalities

\(C = [\Omega']C'\)  
By def. of subst.  
\([\Omega']C' \neq [\Omega_2]C'\)  
By above equality  
\(C \neq [\Omega_2]C'\)  
Instantiating “for all \(C_2\)” with \(C_2 = [\Omega_2]C'\)

\(\Rightarrow \Leftarrow\)  
\(\text{FEV}([\Delta]C') = \emptyset\)  
By contradiction

\(\Gamma \vdash s : [\Gamma]A ! \implies C' [t] \rightarrow \Delta\)  
By SpineRecover

**Case**  
\([\Omega][\Gamma] \vdash [\Omega]s : [\Omega]A p \implies C q\)  
\([\Omega][\Gamma] \vdash [\Omega]s : [\Omega]A p \implies C [q]\)  
DeclSpinePass

\([\Omega][\Gamma] \vdash [\Omega]s : [\Omega]A p \implies C q\)  
Subderivation  
\(\Gamma \vdash s : [\Gamma]A p \implies C' q \rightarrow \Delta\)  
By i.h.

\(\Delta \rightarrow [\Omega']\)  
"  
\(\text{dom}(\Delta) = \dom(\Omega')\)  
"  
\(\Omega \rightarrow [\Omega']\)  
"  
\(C' = [\Delta]C'\)  
"  
\(C = [\Omega']C'\)  
"

We distinguish cases as follows:

- If \(p = f\) or \(q = t\), then we can just apply SpinePass

---

\(\text{thm:typing-completeness}\)
Proof of Theorem 12 (Completeness of Algorithmic Typing) thm:typing-completeness

\( \Gamma \vdash s : [\Gamma]A \triangleright\triangleright A \vdash \Delta \) By SpinePass

Otherwise, \( p = ! \) and \( q = \gamma \). If \( \text{FEV}(C) \neq \emptyset \), we can apply SpinePass as above. If \( \text{FEV}(C) = \emptyset \), then we instead apply SpineRecover.

\( \Gamma \vdash s : [\Gamma]A \triangleright\triangleright C \vdash \Delta \) By SpineRecover

Here, \( q' = ! \) and \( q = \gamma \), so \( q' \subseteq q \).

**Case**

\[ \Omega | \Gamma \vdash : [\Omega]A \triangleright\triangleright [\Omega]A \] DeclEmptySpine

\( \Gamma \vdash : [\Gamma]A \triangleright\triangleright [\Gamma]A \vdash \Gamma \) By EmptySpine

\( [\Gamma]A = [\Gamma][\Gamma]A \) By idempotence of substitution

\( \Gamma \rightarrow \Omega \) Given

\( \text{dom}(\Gamma) = \text{dom}(\Omega) \) Given

\( [\Omega][\Gamma]A = [\Omega]A \) By Lemma 29 (Substitution Monotonicity) (iii)

\( \Omega \rightarrow \Omega \) By Lemma 32 (Extension Reflexivity)

**Case**

\[ [\Omega]\Gamma \vdash [\Omega]e_0 \leftarrow [\Omega]A_1 \ q \quad [\Omega]\Gamma \vdash [\Omega]s_0 : [\Omega]A_2 \ q \triangleright\triangleright B \] Decl→Spine

\[ [\Omega]\Gamma \vdash [\Omega](e_0 s_0) : ([\Omega]A_1) \rightarrow ([\Omega]A_2) \ q \triangleright\triangleright B \]Decl→Spine

\( [\Omega]\Gamma \vdash [\Omega]e_0 \leftarrow [\Omega]A_1 \ q \) Subderivation

\( \Gamma \vdash e_0 \leftarrow A' \ q \vdash \Theta \) By i.h.

\( \Theta \rightarrow \Omega_{\Theta} \) "

\( \Omega \rightarrow \Omega_{\Theta} \) "

\( A = [\Omega_{\Theta}]A' \) "

\( A' = [\Theta]A' \) "

\( [\Omega]\Gamma \vdash [\Omega]s_0 : [\Omega]A_2 \ q \triangleright\triangleright B \) Subderivation

\( \Gamma \vdash s_0 : A_2 \ q \triangleright\triangleright B \vdash \Delta \) By i.h.

\( \Delta \rightarrow \Omega' \) "

\( \text{dom}(\Delta) = \text{dom}(\Omega') \) "

\( \Omega \rightarrow \Omega' \) "

\( B' = [\Delta]B' \) "

\( B = [\Omega']B' \) "

\( \Gamma \vdash e_0 s_0 : A_1 \triangleright\triangleright A_2 \ q \triangleright\triangleright B \ p \vdash \Delta \) By →Spine
Proof of Theorem 12 (Completeness of Algorithmic Typing) thm:typing-completeness

- Case $\sqsubseteq \Gamma \vdash [\Omega]P$ true

If $e$ not a case, then:

\[
\begin{align*}
\Gamma \vdash P &\quad \text{Subderivation} \\
\Theta &\rightarrow \Omega_0' & \text{By Lemma 97} \text{(Completeness of Checkprop)} \\
\Omega &\rightarrow \Omega_0' & \" \\
\Gamma &\rightarrow \Omega & \text{Given} \\
\Gamma &\rightarrow \Omega_0' & \text{By Lemma 33} \text{(Extension Transitivity)} \\
[\Omega|\Gamma] &\vdash [\Omega]\Omega & \text{By Lemma 54} \text{(Completing Stability)} \\
&= [\Omega_0'|\Omega_0'] & \text{By Lemma 55} \text{(Completing Completeness) (iii)} \\
&= [\Omega_0'|\Theta] & \text{By Lemma 56} \text{(Confluence of Completeness)} \\
\Gamma &\vdash A_0 \land P \vdash \text{p type} & \text{Given} \\
\Gamma &\vdash A_0 \land P \vdash \text{p type} & \text{By inversion} \\
[\Omega|A_0] &\vdash [\Omega_0]|A_0 & \text{By Lemma 55} \text{(Completing Completeness) (ii)} \\
[\Omega|\Gamma] &\vdash [\Omega]e &\Leftarrow [\Omega|A_0] \vdash \text{p} & \text{Subderivation} \\
[\Omega_0'|\Theta] &\vdash [\Omega]e &\Leftarrow [\Omega_0]|A_0 \vdash \text{p} & \text{By above equalities} \\
\Theta &\vdash e &\Leftarrow [\Theta|A_0] \vdash \text{p} & \text{By i.h.} \\
\Delta &\rightarrow \Omega' & \" \\
\text{dom}(\Delta) &\rightarrow \text{dom}(\Omega') & \" \\
\Omega_0' &\rightarrow \Omega' & \text{By Lemma 33} \text{(Extension Transitivity)} \\
\Gamma &\vdash e &\Leftarrow A_0 \land P \vdash \text{p} & \text{By } [\land] \\
\end{align*}
\]

Otherwise, we have $e = \text{case}(e_0, \Pi)$. Let $n$ be the height of the given derivation.

\[
\begin{align*}
n - 1 &\quad [\Omega|\Gamma] \vdash [\Omega](\text{case}(e_0, \Pi)) &\Leftarrow [\Omega|A_0] \vdash \text{p} & \text{Subderivation} \\
&\quad \text{By Lemma 52} \text{(Case Invertibility)} \\
n - 2 &\quad [\Omega|\Gamma] \vdash [\Omega]e_0 &\Rightarrow B & \text{!} & \text{By i.h.} \\
&\quad \Theta &\rightarrow \Omega_0' & \" \\
&\quad \Omega &\rightarrow \Omega_0' & \" \\
&\quad B &\Rightarrow [\Omega_0]|B' & \text{By Lemma 30} \text{(Substitution Invariance)} \\
&\quad = [\Omega_0]|(\Theta)B' & \text{By Lemma 30} \text{(Substitution Invariance)} \\
[\Omega|\Gamma] &\vdash [\Omega_0]|\Theta & \text{By Lemma 57} \text{(Multiple Confluence)} \\
[\Omega](A_0 \land P) &\vdash [\Omega_0]|(A_0 \land P) & \text{By Lemma 55} \text{(Completing Completeness) (ii)} \\
\end{align*}
\]
Proof of Theorem 12 \textbf{(Completeness of Algorithmic Typing)}

\[ n - 1 \quad [\Omega_0'] \Theta \vdash [\Omega] [\Pi] :: [\Omega_0'] [\Theta] B' \leftarrow [\Omega_0'] (A_0 \land P) \quad p \quad \text{By above equalities} \]
\[ \Theta \vdash \Pi :: [\Theta] B' \leftarrow A_0 \land P \quad p \vdash \Delta \quad \text{By i.h.} \]
\[ \Delta \rightarrow \Omega' \quad " \]
\[ \text{dom}(\Delta) = \text{dom}(\Omega') \quad " \]
\[ \Omega_0' \rightarrow \Omega' \quad " \]

\[ \Theta \vdash \Pi \quad \text{covers} \quad [\Theta] B' \quad \text{By Theorem 11} \]
\[ \Omega' \quad \text{By Lemma 33 (Extension Transitivity)} \]
\[ \Gamma \vdash \text{case}(e_0, \Pi) \leftarrow A_0 \land P \quad p \vdash \Delta \quad \text{By Case} \]

- **Case**
  \[ [\Omega][\Gamma] \vdash \tau : \kappa \quad \quad [\Omega][\Gamma] \vdash e \leftarrow [\tau/\alpha][\Omega] A_0 \quad f \quad \quad \text{Subderivation} \]
  \[ [\Omega][\Gamma] \vdash e \leftarrow [\tau/\alpha][\Omega] A_0 \quad f \quad \text{Subderivation} \]
  \[ \text{Let } \Omega_0 = ([\Omega, \alpha : * = \tau]). \]
  \[ [\Omega][\Gamma] = [\Omega_0]([\Gamma, \alpha : *]) \quad \text{By def. of context substitution} \]
  \[ [\Omega_0][\Gamma, \alpha : *] \vdash e \leftarrow [\tau/\alpha][\Omega] A_0 \quad f \quad \text{By above equality} \]
  \[ [\tau/\alpha][\Omega] A_0 = ([\Omega, \alpha : * = \tau][\alpha/\alpha] A_0) \quad \text{By a property of substitution} \]
  \[ [\Omega_0][\Gamma, \alpha : *] \vdash e \leftarrow [\Omega_0][\alpha/\alpha] A_0 \quad f \quad \text{By above equality} \]
  \[ \Gamma, \alpha : * \vdash e \leftarrow [\alpha/\alpha] A_0 \quad f \vdash \Delta \quad \text{By i.h.} \]
  \[ \Delta \rightarrow \Omega' \quad " \]
  \[ \text{dom}(\Delta) = \text{dom}(\Omega') \quad " \]
  \[ \Omega_0 \rightarrow \Omega' \quad " \]
  \[ \Omega \rightarrow \Omega_0 \quad \text{By AddSolved} \]
  \[ \Omega \rightarrow \Omega' \quad \text{By Lemma 33 (Extension Transitivity)} \]
  \[ \Gamma \vdash e \leftarrow \exists \alpha : \kappa. A_0 \quad p \vdash \Delta \quad \text{By i.h} \]

- **Case**
  \[ \boxed{\text{DeclNil}} \quad \text{Similar to the first part of the Decl/\land case.} \]

- **Case**
  \[ [\Omega][\Gamma] \vdash ([\Omega] t) = \text{succ}(t_2) \quad \quad [\Omega][\Gamma] \vdash [\Omega] e_1 \leftarrow [\Omega] A_0 \quad p \quad \boxed{\text{DeclCons}} \]
  \[ [\Omega][\Gamma] \vdash ([\Omega] e_2) \leftarrow ([\Omega] e_2) \leftarrow (\text{Vec } t_2 [\Omega] A_0) \quad \]

Let \[ \Omega^+ = ([\Omega, \alpha : N = t_2]). \]
\[ [\Omega][\Gamma] = ([\Omega] t) = \text{succ}(t_2) \quad \text{true} \quad \text{Subderivation} \]
\[ [\Omega^+][\Gamma, \alpha : N = t] = ([\Omega] t) = [\Omega^+] \text{succ}(\alpha) \quad \text{true} \quad \text{Defs. of extension and subst.} \]
\[ 1 \quad \Gamma, \alpha : N \vdash t = \text{succ}(\alpha) \quad \text{true} \quad p \quad \text{By Lemma 97 (Completeness of Checkprop)} \]
\[ \Gamma' \rightarrow \Omega_0' \quad \text{true} \quad \text{true} \]
\[ \Omega^+ \rightarrow \Omega_0' \quad \text{true} \quad \text{true} \]
Proof of Theorem 12 (Completeness of Algorithmic Typing) thm:typing-completeness

\[
\Gamma, \triangleright \alpha, \beta : \mathbb{N} \rightarrow \Gamma' \\
\Gamma, \triangleright \alpha, \beta : \mathbb{N} \rightarrow \Omega'_0
\]

By Lemma 47 (Checkprop Extension)

\[
[\Omega]\Gamma = [\Omega]\Omega
= [\Omega^+]\Omega^+
= [\Omega'_0]^{\Omega'_0}
= [\Omega'_0]\Gamma'
\]

By def. of context application

\[
[\Omega]\Gamma_0 = [\Omega^+]A_0
= [\Omega'_0]A_0
\]

By def. of context application

\[
[\Omega]\Gamma \vdash [\Omega]e_1 \leftarrow [\Omega]A_0 p
\]

Subderivation

\[
[\Omega'_0]\Gamma' \vdash [\Omega]e_1 \leftarrow [\Omega'_0]A_0 p
\]

By above equalities

\[
2 \Gamma' \vdash e_1 \leftarrow [\Gamma']A_0 p \triangleright \Theta
\]

By i.h.

\[
\Theta \rightarrow \Omega''
\]

By above equalities

\[
\Omega'_0 \rightarrow \Omega''
\]

By above equalities

\[
[\Omega]\Gamma \vdash [\Omega]e_2 \leftarrow (\text{Vec} t_2 [\Omega]A_0) f
\]

Subderivation

\[
[\Omega]\Gamma \vdash [\Omega]e_2 \leftarrow (\text{Vec} ([\Omega^+]\alpha) [\Omega]A_0) f
\]

By def. of substitution

\[
[\Omega'_0]\Theta \vdash [\Omega]e_2 \leftarrow (\text{Vec} ([\Omega'_0]\alpha) [\Omega'_0]A_0) f
\]

By lemmas

\[
[\Omega'_0]\Theta \vdash [\Omega]e_2 \leftarrow [\Omega'_0]^{\text{Vec} \alpha A_0} f
\]

By def. of subst.

\[
3 \Theta \vdash e_2 \leftarrow [\Theta]A_0 p \triangleright \Delta, \triangleright \alpha, \Delta'
\]

By i.h.

\[
\Delta, \triangleright \alpha, \Delta' \rightarrow \Omega''
\]

By above equalities

\[
\text{dom}(\Delta, \triangleright \alpha, \Delta') = \text{dom}(\Omega'')
\]

By above equalities

\[
\Omega'' = (\Omega, \triangleright \alpha, \ldots)
\]

By Lemma 22 (Extension Inversion) (ii)

\[
\Delta \rightarrow \Omega'
\]

By above equalities

\[
\text{dom}(\Delta) = \text{dom}(\Omega')
\]

By above equalities

\[
(\Gamma', \triangleright \alpha, \ldots) \rightarrow \Omega'
\]

By Lemma 33 (Extension Transitivity)

\[
\Omega \rightarrow \Omega'
\]

By Lemma 22 (Extension Inversion) (ii)

\[
\Gamma \vdash e_1 :: e_2 \leftarrow (\text{Vec} t A_0) p \triangleright \Delta
\]

By Cons

\[\text{• Case } \frac{[\Omega]\Gamma \vdash [\Omega]e_1 \leftarrow A'_1 p \quad [\Omega]\Gamma \vdash [\Omega]e_2 \leftarrow A'_2 p}{[\Omega]\Gamma \vdash ([\Omega]e_1, [\Omega]e_2) \leftarrow A'_1 \times A'_2 p} \quad \text{Decl}\times\]

Either \([\Gamma]A = A_1 \times A_2\) or \([\Gamma]A = \hat{\alpha} \in \text{unsolved}(\Gamma)\).

- In the first case \((\Gamma]A = A_1 \times A_2)\), we have \(A'_1 = [\Omega]A_1\) and \(A'_2 = [\Omega]A_2\).
Proof of Theorem 12 (Completeness of Algorithmic Typing)

\[ \Omega \vdash \Gamma \vdash [\Omega] e_1 \Leftarrow A_1 \]  
Subderivation

\[ \Gamma \vdash e_1 \Leftarrow [\Gamma] A_1 \vdash \Theta \]  
By i.h.

\[ \Theta \rightarrow \Omega_\Theta \]  
By Lemma [51] (Typing Extension)

\[ \Delta = \text{dom}(\Theta) \]  
By Lemma [57] (Multiple Confluence)

\[ \Omega_\Theta \rightarrow \Omega \]  
By Lemma [55] (Completing Completeness) (ii)

\[ \Gamma \vdash \langle e_1, e_2 \rangle \Leftarrow ([\Gamma] A_1 \times [\Gamma] A_2) \vdash \Delta \]  
By def. of subst.

– In the second case, where \( [\Gamma] A = \alpha \), combine the corresponding subcase for [Decl+I/k] with some straightforward additional reasoning about contexts (because here we have two subderivations, rather than one).

• Case

\[ \Gamma \vdash [\Omega] e_0 \Rightarrow C \]  
Subderivation

\[ \Gamma \vdash e_0 \Rightarrow C' \vdash \Theta \]  
By i.h.

\[ \Delta = \text{dom}(\Theta) \]  
By Lemma [33] (Extension Transitivity)

\[ \Omega_\Theta \rightarrow \Omega \]  
By Lemma [63] (Well-Formed Outputs of Typing)

\[ \Theta \vdash C' \vdash \text{type} \]  
By a property of substitution
Proof of **Theorem 12** *(Completeness of Algorithmic Typing)*

\[ \begin{align*}
\Gamma & \rightarrow \Omega \\
\Delta & \rightarrow \Omega \\
\Theta & \rightarrow \Omega \\
[\Omega]\Gamma &= [\Omega]\Theta = [\Omega]\Delta \\
\Gamma & \rightarrow \Theta \\
\Gamma & \rightarrow \Omega_\Theta \\
[\Omega]\Gamma &= [\Omega_\Theta]\Theta \\
\Gamma & \vdash A \text{ type} \\
\Omega & \vdash A \text{ type} \\
[\Omega]A &= [\Omega_\Theta]A \\
[\Omega]\Gamma & \vdash [\Omega]\Pi \vdash C \Leftrightarrow [\Omega]A \vdash p \rightarrow \Delta \\
\theta \vdash \Pi :: C' \Leftrightarrow [\Theta]A \vdash p \rightarrow \Delta & \text{ By above equalities} \\
\theta & \vdash C' \Leftrightarrow [\Theta]A \vdash p \rightarrow \Delta & \text{ By i.h. (v)} \\
\Delta & \rightarrow \Omega' \\
\Omega & \rightarrow \Omega' & \text{ By Lemma 33 (Extension Transitivity)} \\
[\Omega]\Gamma & \vdash \Omega & \text{ Instantiation of quantifier} \\
[\Omega]\Gamma &= [\Omega]\Delta \\
[\Omega'\Delta] &= [\Omega\prime]\Delta \\
[\Omega'\Delta] & \vdash [\Omega]\Pi \vdash C' & \text{ By above equalities} \\
\Delta & \rightarrow \Omega' & \text{ By Lemma 33 (Extension Transitivity)} \\
\Gamma & \vdash C' \text{ type} \\
\Gamma & \rightarrow \Delta & \text{ By Lemma 51 (Typing Extension) & 33} \\
\Delta & \vdash C' \text{ type} & \text{ By Lemma 41 (Extension Weakening for Principal Typing)} \\
[\Delta]C' &= C' & \text{ By FEV} (C') = \emptyset \text{ and a property of subst.} \\
\Delta & \vdash \Pi \text{ covers } C' & \text{ By Theorem 11} \\
\Gamma & \vdash \text{case} (e_0, \Pi) \Leftrightarrow [\Gamma]A \vdash p \rightarrow \Delta & \text{ By Case} \\
\end{align*} \]

- **Case**

\[ \begin{align*}
[\Omega] & \vdash [\Omega]e_1 \Leftrightarrow A_1 \vdash p \\
[\Omega] & \vdash [\Omega]e_2 \Leftrightarrow A_2 \vdash p \\
[\Omega] & \vdash ([\Omega]e_1, [\Omega]e_2) \Leftrightarrow A_1 \times A_2 \vdash p & \text{ **DeclX**} \\
[\Omega] & \vdash [\Omega]A' \vdash p & \text{ Subderivation} \\
\end{align*} \]

Either \( A = \hat{\alpha} \) where \([\Omega]\hat{\alpha} = A_1 \times A_2\), or \( A = A_1' \times A_2' \) where \( A_1 = [\Omega]A_1'\) and \( A_2 = [\Omega]A_2'\).

In the former case \( A = \hat{\alpha} \):

We have \([\Omega]\hat{\alpha} = A_1 \times A_2\). Therefore \( A_1 = [\Omega]A_1' \) and \( A_2 = [\Omega]A_2' \). Moreover, \( \Gamma = \Gamma_0[\hat{\alpha} : \kappa] \).

\[ [\Omega] \vdash [\Omega]e_1 \Leftrightarrow [\Omega]A'_1 \vdash p \]

Let \( \Gamma' = \Gamma_0[\hat{\alpha}_1 : \kappa, \hat{\alpha}_2 : \kappa, \hat{\alpha} : \kappa = \hat{\alpha}_1 + \hat{\alpha}_2] \).
Proof of Theorem 12 (Completeness of Algorithmic Typing)

Now we turn to parts (v) and (vi), completeness of matching.

In the latter case \((A = A'_1 \times A'_2)\):

\[
\begin{align*}
[\Omega]\Gamma & = [\Omega]\Gamma' & \text{By def. of context substitution} \\
[\Omega]\Gamma' + [\Omega]e_1 & \leftarrow [\Omega]A'_1 p & \text{By above equality} \\
\Gamma' + e_1 & \leftarrow [\Gamma']A'_1 p \vdash \Theta & \text{By i.h.} \\
\Theta & \rightarrow \Omega_1 & \text{”} \\
\Omega & \rightarrow \Omega_1 & \text{”} \\
\text{dom}(\Theta) & = \text{dom}(\Omega_1) & \text{”}
\end{align*}
\]

\[
\begin{align*}
[\Omega]\Gamma & = [\Omega_1]\Theta & \text{By Lemma 57 (Multiple Confluence)} \\
[\Omega]A'_2 & = [\Omega_1]A'_2' & \text{By Lemma 55 (Completing Completeness) (ii)} \\
[\Omega_1]\Theta + [\Omega]e_2 & \leftarrow [\Omega_1]A'_2 p & \text{By above equalities} \\
\Theta + e_2 & \leftarrow [\Theta]A'_2 p \vdash \Delta & \text{By i.h.} \\
\text{dom}(\Delta) & = \text{dom}(\Omega') & \text{”} \\
\Delta & \rightarrow \Omega' & \text{”} \\
\Omega_1 & \rightarrow \Omega' & \text{”} \\
\Omega & \rightarrow \Omega' & \text{By Lemma 33 (Extension Transitivity)}
\end{align*}
\]

\[
\begin{align*}
\Gamma + (e_1, e_2) & \leftarrow \Theta p \vdash \Delta & \text{By } \times \times I
\end{align*}
\]

In the latter case \((A = A'_1 \times A'_2)\):

\[
\begin{align*}
[\Omega]\Gamma + [\Omega]e_1 & \leftarrow A_1 p & \text{Subderivation} \\
[\Omega]\Gamma + [\Omega]e_1 & \leftarrow [\Omega]A'_1 p & A_1 = [\Omega]A'_1 \\
\Gamma + e_1 & \leftarrow [\Gamma]A'_1 p \vdash \Theta & \text{By i.h.} \\
\Theta & \rightarrow \Omega_0 & \text{”} \\
\text{dom}(\Theta) & = \text{dom}(\Omega_0) & \text{”} \\
\Omega & \rightarrow \Omega_0 & \text{”}
\end{align*}
\]

\[
\begin{align*}
[\Omega]\Gamma + [\Omega]e_1 & \leftarrow A_2 p & \text{Subderivation} \\
[\Omega]\Gamma + [\Omega]e_1 & \leftarrow [\Omega]A'_2 p & A_2 = [\Omega]A'_2 \\
\Gamma + A'_1 \times A'_2 p & \text{type} & \text{Given } (A = A'_1 \times A'_2) \\
\Gamma + A'_2 \text{ type} & \text{By inversion} \\
\Gamma & \rightarrow \Omega & \text{Given} \\
\Gamma & \rightarrow \Omega_0 & \text{By Lemma 33 (Extension Transitivity)} \\
\Omega_0 + A'_2 \text{ type} & \text{By Lemma 38 (Extension Weakening (Types))} \\
[\Omega]\Gamma + [\Omega]e_2 & \leftarrow [\Omega]A'_2 p & \text{By Lemma 55 (Completing Completeness)} \\
[\Omega]\Gamma + [\Omega]e_2 & \leftarrow [\Omega_0]A'_2 p & \text{By Lemma 29 (Substitution Monotonicity) (iii)} \\
[\Omega]\Theta + [\Omega]e_2 & \leftarrow [\Theta]A'_2 p & \text{By Lemma 57 (Multiple Confluence) (ii)} \\
\Theta + e_2 & \leftarrow [\Theta]A'_2 p \vdash \Delta & \text{By i.h.} \\
\text{dom}(\Delta) & = \text{dom}(\Omega') & \text{”} \\
\Omega_0 & \rightarrow \Omega' & \text{”} \\
\Omega & \rightarrow \Omega' & \text{By Lemma 33 (Extension Transitivity)}
\end{align*}
\]

\[
\begin{align*}
\Gamma + (e_1, e_2) & \leftarrow ([\Omega]A_1) \times ([\Omega]A_2) p \vdash \Delta & \text{By } \times \times I \\
\Gamma + (e_1, e_2) & \leftarrow [\Omega](A_1 \times A_2) p \vdash \Delta & \text{By def. of substitution}
\end{align*}
\]
• Case **DeclMatchEmpty**: Apply rule **MatchEmpty**

• Case **DeclMatchSeq**: Apply the i.h. twice, along with standard lemmas.

• Case **DeclMatchBase**: Apply the i.h. (i) and rule **MatchBase**

• Case **DeclMatchUnit**: Apply the i.h. and rule **MatchUnit**

• Case **DeclMatch=**: By i.h. and rule **Match=**

• Case **DeclMatch×**: By i.h. and rule **Match×**

• Case **DeclMatch+×**: By i.h. and rule **Match+×**

• Case **DeclMatch\∧**: By i.h. (vi), we will show (1) $\Gamma \vdash (A, \bar{A}) ! \text{types}$, (2) $\Gamma \vdash P \text{ prop}$, (3) $\text{FEV}(P) = \emptyset$, (4) $\Gamma \vdash C p \text{ type}$, (5) $[\Omega] \Gamma / [\Omega] P \vdash \bar{p} \Rightarrow [\Omega] e :: [\Omega] A ! \leftarrow [\Omega] C p$, and (6) $p' \subseteq p$.

  \[
  \Gamma \vdash (A \land P, \bar{A}) ! \text{types} \quad \text{Given}
  \]
  \[
  \Gamma \vdash (A \land P) ! \text{type} \quad \text{By inversion on PrincipalTypevecWF}
  \]
  \[
  \Gamma \vdash A ! \text{type} \quad \text{By Lemma 42 (Inversion of Principal Typing)} (3)
  \]

  (2) $\Gamma \vdash P \text{ prop}$

  (3) $\text{FEV}(P) = \emptyset$ \quad By inversion

  (1) $\Gamma \vdash (A, \bar{A}) ! \text{types}$ \quad By inversion and PrincipalTypevecWF

  (4) $\Gamma \vdash C p \text{ type}$ \quad Given

  (5) $[\Omega] \Gamma / P \vdash \bar{p} \Rightarrow [\Omega] e :: [\Omega] A, [\Omega] \bar{A} \leftarrow [\Omega] C p$ \quad Subderivation

  (6) $p' \subseteq p$ \quad Given

  \[
  \Gamma / [\Gamma] P \vdash \bar{p} \Rightarrow e :: [\Gamma](A, \bar{A}) \leftarrow [\Gamma] C p' \rightarrow \Delta \quad \text{By i.h. (vi)}
  \]

  \[
  \Delta \rightarrow \Omega' \quad ""\]

  \[
  \text{dom}(\Delta) = \text{dom}(\Omega') \quad ""
  \]

  \[
  \Omega \rightarrow \Omega' \quad ""
  \]

  \[
  \Gamma / [\Gamma] P \vdash \bar{p} \Rightarrow e :: [\Gamma](A \land P, \bar{A}) \leftarrow [\Gamma] C p' \rightarrow \Delta \quad \text{By def. of subst.}
  \]

  \[
  \Gamma \vdash \bar{p} \Rightarrow e :: (\Gamma A \land \Gamma P), \Gamma \bar{A} \leftarrow [\Gamma] C p' \rightarrow \Delta \quad \text{By Match\∧}
  \]

  \[
  \Gamma \vdash \bar{p} \Rightarrow e :: [\Gamma](A \land P, \bar{A}) \leftarrow [\Gamma] C p' \rightarrow \Delta \quad \text{By def. of subst.}
  \]

• Case **DeclMatchNeg**: By i.h. and rule **MatchNeg**

• Case **DeclMatchWild**: By i.h. and rule **MatchWild**

• Case **DeclMatchNil**: Similar to the DeclMatch\∧ case.

• Case **DeclMatchCons**: Similar to the DeclMatch\∧ and DeclMatch\× cases.

• Case **DeclMatch\∃**: By i.h. and rule **Match\∃**

\[
[\Omega] \Gamma \vdash [\Omega] \sigma = [\Omega] \tau \vdash \bar{p} \Rightarrow e :: [\Omega] \bar{A} ! \leftarrow [\Omega] C p \quad \text{DeclMatch\⊥}
\]
Proof of Theorem 12 (Completeness of Algorithmic Typing)

\[ \text{Given} \]
\[ \text{FEV}(\sigma = \tau) = \emptyset \]
\[ |\Omega|\sigma = |\Gamma|\sigma \quad \text{By Lemma 39 (Principal Agreement) (i)} \]
\[ |\Omega|\tau = |\Gamma|\tau \quad \text{Similar} \]
\[ \text{mgu}(|\Omega|\sigma, |\Omega|\tau) = \perp \quad \text{Given} \]
\[ \text{mgu}(|\Gamma|\sigma, |\Gamma|\tau) = \perp \quad \text{By above equalities} \]
\[ \Gamma / \sigma \Rightarrow \tau : \kappa \vdash \perp \quad \text{By Lemma 94 (Completeness of Elimeq) (2)} \]

\[ \Omega / \rho \Rightarrow e : |\Gamma|\bar{A} \leftarrow |\Gamma|C \rho \vdash \Gamma \]
\[ \text{dom}(\Gamma) = \text{dom}(\Omega) \quad \text{By Lemma 32 (Extension Reflexivity)} \]

**Case** \[ \text{mgu}(|\Omega|\sigma, |\Omega|\tau) = \emptyset \quad \theta([\Omega]|\Gamma) + \theta(\rho \Rightarrow [\Omega]e) : \theta([\Omega]|\bar{A}) \leftarrow \theta([\Omega]|C) p \quad \text{DeclMatchUnify} \]
\[ [\Omega|\Gamma] / [\Omega]|\sigma = [\Gamma|\tau] \quad \text{As in DeclMatchUnify case} \]
\[ \text{mgu}(|\Omega|\sigma, |\Omega|\tau) = \emptyset \quad \text{Given} \]
\[ \text{mgu}(|\Gamma|\sigma, |\Gamma|\tau) = \emptyset \quad \text{By above equalities} \]
\[ \Gamma / \sigma \Rightarrow \tau : \kappa \vdash (\Gamma, \Theta) \quad \text{By Lemma 94 (Completeness of Elimeq) (1)} \]
\[ \Theta = (\alpha_1 = t_1, \ldots, \alpha_n = t_n) \quad " \]
\[ [\Gamma, \Theta] u = \theta([\Gamma] u) \quad " \text{for all } \Gamma \vdash u : \kappa \]
\[ \theta([\Omega]|\Gamma) + \theta(\rho \Rightarrow [\Omega]e) : \theta([\Omega]|\bar{A}) \leftarrow \theta([\Omega]|C) p \quad \text{Subderivation} \]
\[ \theta([\Omega]|\Gamma) = [\Omega, \rhd_{\gamma}, \Theta][\Gamma, \rhd_{\gamma}, \Theta] \quad \text{By Lemma 95 (Substitution Upgrade) (iii)} \]
\[ \theta([\Omega]|\bar{A}) = [\Omega, \rhd_{\gamma}, \Theta][\bar{A}] \quad \text{By Lemma 95 (Substitution Upgrade) (i) (over } \bar{A}) \]
\[ \theta([\Omega]|C) = [\Omega, \rhd_{\gamma}, \Theta][C] \quad \text{By Lemma 95 (Substitution Upgrade) (i)} \]
\[ \theta(\rho \Rightarrow [\Omega]e) = [\Omega, \rhd_{\gamma}, \Theta](\rho \Rightarrow e) \quad \text{By Lemma 95 (Substitution Upgrade) (iv)} \]

\[ [\Omega, \rhd_{\gamma}, \Theta][\Gamma, \rhd_{\gamma}, \Theta] / [\Omega, \rhd_{\gamma}, \Theta](\rho \Rightarrow e) : [\Omega, \rhd_{\gamma}, \Theta][\bar{A}] \leftarrow [\Omega, \rhd_{\gamma}, \Theta][\bar{C}] p \vdash \Delta, \rhd_{\gamma}, \Delta' \quad \text{By above equalities} \]
\[ \Gamma / \rho \Rightarrow e : [\Gamma, \rhd_{\gamma}, \Theta][\bar{A}] \leftarrow [\Gamma, \rhd_{\gamma}, \Theta][\bar{C}] p \vdash \Delta, \rhd_{\gamma}, \Delta' \quad \text{By i.h.} \]
\[ \Delta, \rhd_{\gamma}, \Delta' \rightarrow [\Omega, \rhd_{\gamma}, \Theta][\bar{A}] \leftarrow [\Omega, \rhd_{\gamma}, \Theta][\bar{C}] p \vdash \Delta, \rhd_{\gamma}, \Delta' \quad " \]
\[ \Omega, \rhd_{\gamma}, \Theta \rightarrow [\Omega', \rhd_{\gamma}, \Theta][\bar{A}] \leftarrow [\Omega', \rhd_{\gamma}, \Theta][\bar{C}] p \vdash \Delta, \rhd_{\gamma}, \Delta' \quad " \]
\[ \text{dom}(\Delta, \rhd_{\gamma}, \Delta') = \text{dom}(\Omega', \rhd_{\gamma}, \Omega') \quad " \]

\[ \Delta \rightarrow \Omega' \quad \text{By Lemma 22 (Extension Inversion) (ii)} \]
\[ \text{dom}(\Delta) = \text{dom}(\Omega') \quad " \]
\[ \Omega \rightarrow \Omega' \quad \text{By Lemma 22 (Extension Inversion) (ii)} \]
\[ \Gamma / \rho \Rightarrow e : [\Gamma, \rhd_{\gamma}, \Theta][\bar{A}] \leftarrow [\Gamma, \rhd_{\gamma}, \Theta][\bar{C}] p \vdash \Delta \quad \text{By MatchUnify} \]