Sums of Uncertainty: Refinements go gradual

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Gradual typing

I'm last in the session, so I'll keep this brief.

Traditionally, gradual typing is about

- migrating **incrementally** (gradually)
  from dynamically typed code
  to statically typed code.

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. . . I lost him at "dynamically typed".

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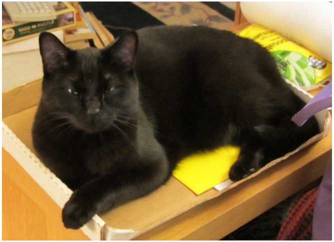

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- migrating **incrementally** (gradually) from **less precisely** statically typed code (like **SML**) to **more precisely** statically typed code

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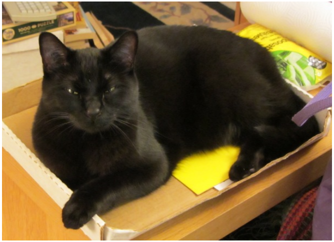

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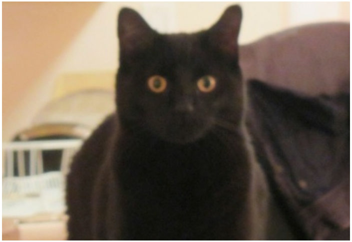

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Our paper has (a simplified form of) what were once called refinement types, which we now call datasort refinements.

Standard ML: dynamically typed?

$\mathsf{datatype}\ \mathsf{nat}\ =\ \mathtt{Zero} \mid \mathtt{Succ}\ \mathsf{of}\ \mathsf{nat}$

$\mathsf{case}\ x : \mathsf{nat}\ \mathsf{of}$
$\quad \mathtt{Zero} \Rightarrow \ldots$
$\quad \mid \mathtt{Succ}\ y \Rightarrow \ldots$

Standard ML: dynamically typed?

```
datatype nat  =  Zero | Succ of nat
```

```
case x : nat of
   Zero ⇒ ...
 | Succ y ⇒ ...
```

But the Definition requires compilers to accept **nonexhaustive** matches:

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case x : nat of
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Standard ML: dynamically typed?

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datatype nat = Zero | Succ of nat
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But the Definition requires compilers to accept **nonexhaustive** matches:

```
case x : nat of
   Succ y ⇒ ...
```

If x = `Zero`, then the exception `Match` is raised.

This nonexhaustive match is fine,
if we know that x will never be `Zero`.

Refined Standard ML

Datasort refinements [Freeman & Pfenning 1991, Davies 2005, . . .] push the knowledge that x is not `Zero` into the type system.

```
case x : nonzero of
   Succ y ⇒ ...
```

This **is** exhaustive, because x has **datasort** nonzero.

Datasorts

Datasorts refine ML datatypes

`datatype nat = Zero | Succ of nat`

- sum type : `Succ` **or** `Zero`
- recursive type: `datatype nat = Zero | Succ of nat`

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datasort zero = Zero

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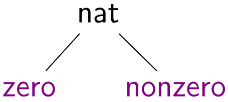

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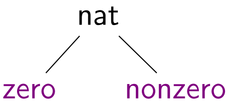

This paper: gradual, refined **sum types**.

Static sums

The usual type-theoretic sum type:

$\texttt{datatype}\ A_1 + A_2 \ =$
$\quad \texttt{inj}_1\ \texttt{of}\ A_1 \mid \texttt{inj}_2\ \texttt{of}\ A_2$

Elimination form: two-armed $\texttt{case}(e, \texttt{inj}_1\, x_1.e_1, \texttt{inj}_2\, x_2.e_2)$

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Subscript sums $A_1 +_1 A_2$ and $A_1 +_2 A_2$, corresponding to datasort refinements:

$$\texttt{datasort}\ A_1 +_{1} A_2 \ = \ \texttt{inj}_{1}\ \texttt{of}\ A_1$$

$$\texttt{datasort}\ A_1 +_{2} A_2 \ = \ \texttt{inj}_{2}\ \texttt{of}\ A_2$$

Elimination form: **one**-armed $\mathsf{case}(e, \mathsf{inj}_k\, x_k.e_k)$.

$$x : (\mathsf{Int} +_1 \mathsf{Bool}) \vdash \mathsf{case}(x, \mathsf{inj}_1\, x_1.x_1) : \mathsf{Int}$$

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Case expressions over $+$, $+_1$, $+_2$ **never** raise Match exceptions.

Dynamic sum

The **dynamic** sum type, corresponding to Standard ML:

$\texttt{datatype}\ A_1 \boxed{+^?} A_2 \ =$
$\quad \texttt{inj}_1\ \texttt{of}\ A_1 \mid \texttt{inj}_2\ \texttt{of}\ A_2$

$+^?$ allows two-armed $\texttt{case}(e, \texttt{inj}_1\, x_1.e_1, \texttt{inj}_2\, x_2.e_2)$.

But $+^?$ **also** allows one-armed $\texttt{case}(e, \texttt{inj}_k\, x_k.e_k)$,
which may raise a `Match` exception.

Gradual sums

					match failures are…
Standard ML				$+^?$	possible
refined SML	$+$	$+_1$	$+_2$		impossible

Gradual sums

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	Standard ML				$+^?$	possible
$+$	refined SML	$+$	$+_1$	$+_2$		impossible
$=$	**Gradual sums**	$+$	$+_1$	$+_2$	$+^?$	**possible** iff $+^?$ used in annotations

Road map

Source
bidirectional
type system

Source
type assignment
system

Target type system
with run-time casts

$$\begin{tikzcd}
e \overset{\Rightarrow}{\Leftarrow} A \arrow[loop left, "\text{varying precision}"'] \arrow[r, shift left] \arrow[d, shift left] & e : A \arrow[l, shift left] \arrow[r, hook, "\text{type-directed translation}"'] & M : T \arrow[r, mapsto, "\text{steps to}"] & M' : T \\
e^{S} \overset{\Rightarrow}{\Leftarrow} A^{S} \arrow[u, shift left] \arrow[rr, hook] & & M : T \arrow[r, mapsto] & M' : T
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matchfail-free

matchfail-free

static sublanguage
(no $+^?$)

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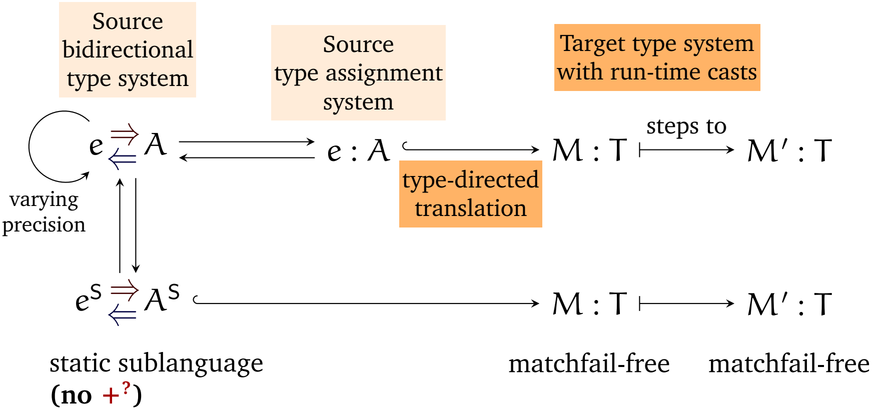

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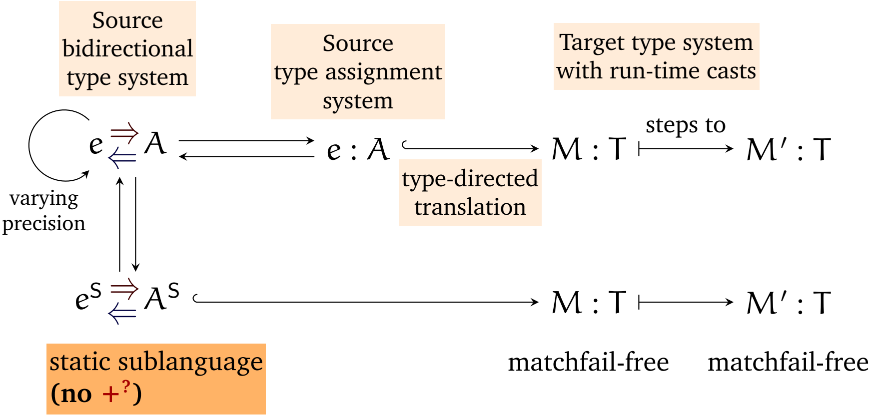

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$e \rightleftarrows A$

$e : A$

type-directed translation

$M : T$

steps to

$M' : T$

varying precision

$e^S \rightleftarrows A^S$

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Source type assignment

Design **introduction** and **elimination** rules:

- How are the static sums $+$, $+_1$, $+_2$ introduced and eliminated?
- How is the dynamic sum $+^?$ introduced and eliminated?

Static sums

Design **introduction** and **elimination** rules for $+_1$, $+_2$:

$$\frac{\Gamma \vdash e : A_k}{\Gamma \vdash (\mathtt{inj}_k\, e) : (A_1 +_k A_2)}\ +_k\text{Intro}$$

$$\frac{\Gamma \vdash e : (A_1 +_k A_2) \qquad \Gamma, x_k : A_k \vdash e_k : B}{\Gamma \vdash \mathtt{case}(e, \mathtt{inj}_k\, x_k.e_k) : B}\ +_k\text{Elim}$$

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Introduction rule for $+$ via subtyping:
$(\mathsf{inj}_k\, e) : (A_1 + A_2)$ because $(A_1 +_k A_2) \leq (A_1 + A_2)$.

$$\frac{\Gamma \vdash e : (A_1 + A_2) \qquad \begin{array}{c}\Gamma, x_1 : A_1 \vdash e_1 : B\\ \Gamma, x_2 : A_2 \vdash e_2 : B\end{array}}{\Gamma \vdash \mathsf{case}(e, \mathsf{inj}_1\, x_1.e_1, \mathsf{inj}_2\, x_2.e_2) : B} +\text{Elim}$$

(two-armed elimination for $+_k$ possible via subtyping)

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Varying precision

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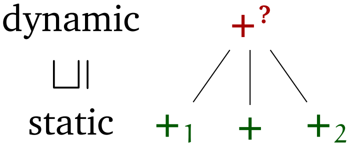
Replacing an annotation $A +_1 B$ with $A +^? B$ preserves typing (varying precision—gradual guarantee)

Replacing an annotation $A +^? B$ with a **more precise** annotation does not always preserve typing.

Defining precision

First, $\sqsubseteq$ on **sum constructors**

$+^?$, $+$, $+_1$, $+_2$:



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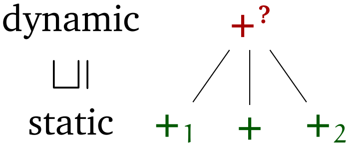


dynamic

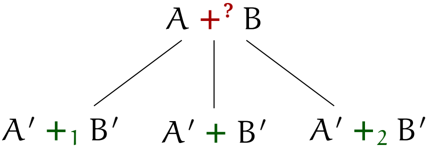
$\sqsupseteq$

static

$+^?$

$+_1$ $\quad$ $+$ $\quad$ $+_2$

Extend $\sqsubseteq$ pointwise: if $A' \sqsubseteq A$ and $B' \sqsubseteq B$ then. . .



$A +^? B$

$A' +_1 B'$ $\quad$ $A' + B'$ $\quad$ $A' +_2 B'$

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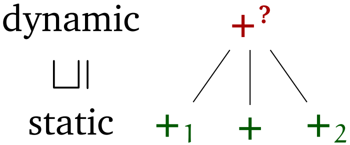

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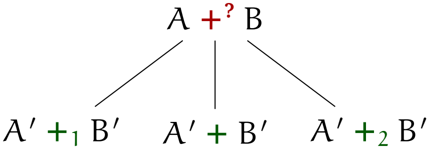
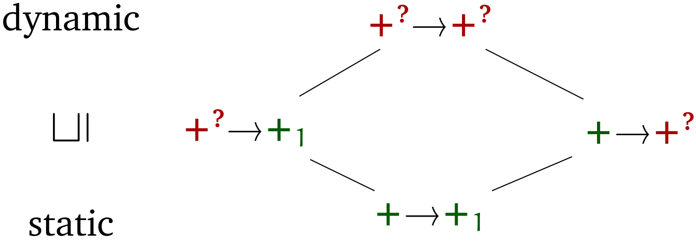


Other constructors **covariant** (similar to $\sqsubset$ in refinement types):



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$$\frac{\Gamma \vdash e : A' \qquad A \sqsubseteq A' \qquad A \leq B \qquad B \sqsubseteq B'}{\Gamma \vdash e : B'}$$

- These 3 premises $=$ **directed consistency** $A' \rightsquigarrow B'$

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$$\frac{\Gamma \vdash e : A' \qquad A \sqsubseteq A' \qquad A \leq B \qquad B \sqsubseteq B'}{\Gamma \vdash e : B'}$$

- These 3 premises = **directed consistency** $A' \rightsquigarrow B'$

$$\begin{array}{ccc} A' & & B' \\ \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} \\ A & \leq & B \end{array}$$

Subsumption

- The usual subsumption rule:

$$\frac{\Gamma \vdash e : A \qquad A \leq B}{\Gamma \vdash e : B}$$

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- Is directed consistency transitive?

Road map

Source bidirectional type system

Source type assignment system

Target type system with run-time casts

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matchfail-free matchfail-free

static sublanguage
(no $+^{?}$)

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Bidirectional typing: why?

Some past answers:

- to handle features beyond Damas–Milner
  (Pierce & Turner 2000;
  Dunfield & Pfenning 2004;
  Dunfield & Krishnaswami 2013; . . .)
- for better (earlier) type error messages

Bidirectional typing: why?

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- for better (earlier) type error messages

Here:

- to make typing **more predictable**, by avoiding **unnecessary imprecision**.

Bidirectional typing in one slide

- **Organize** the flow of information from type annotations:
  - Given Γ, e, and a known type A,
    check e:
    $$\Gamma \vdash e \Leftarrow A$$
  - Given Γ and e,
    synthesize a type for e:
    $$\Gamma \vdash e \Rightarrow A$$
- The type A in the checking judgment $e \Leftarrow A$ is a **goal**.

Bidirectional typing

Frank Pfenning's recipe:
intro rules check, elim rules synthesize.

$$\frac{\Gamma, x : A_1 \vdash e \Leftarrow A_2}{\Gamma \vdash \lambda x.\, e \Leftarrow \boxed{A_1 \to A_2}}\ \textsf{Chk}{\to}\textsf{Intro}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow (A \to B) \qquad \Gamma \vdash e_2 \Leftarrow A}{\Gamma \vdash e_1\, e_2 \Rightarrow B}\ \textsf{Syn}{\to}\textsf{Elim}$$

- Chk→Intro:
  The type $\boxed{A_1 \to A_2}$ must flow from an annotation.
- Syn→Elim: The type $A \to B$ must flow from an annotation,
  perhaps via Γ.

Bidirectional typing

Frank Pfenning's recipe:
intro rules check, elim rules synthesize.

$$\frac{\Gamma, x : A_1 \vdash e \Leftarrow A_2}{\Gamma \vdash \lambda x.\, e \Leftarrow A_1 \rightarrow A_2}\ \mathsf{Chk}{\rightarrow}\mathsf{Intro}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow (A \rightarrow B) \qquad \Gamma \vdash e_2 \Leftarrow A}{\Gamma \vdash e_1\, e_2 \Rightarrow B}\ \mathsf{Syn}{\rightarrow}\mathsf{Elim}$$

- Chk→Intro:
  The type $A_1 \rightarrow A_2$ must flow from an annotation.
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Bidirectional typing

The subsumption rule:

$$\frac{\Gamma \vdash e \Rightarrow A' \qquad A' \rightsquigarrow B'}{\Gamma \vdash e \Leftarrow B'}$$

$$\begin{array}{ccc} A' & \rightsquigarrow & B' \\ \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} \\ A & \leq & B \end{array}$$

Bidirectional typing

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- Subformula property:
  Every type synthesized or checked flows from a type annotation.

Road map

Source bidirectional type system

Source type assignment system

Target type system with run-time casts

$$\begin{tikzcd}
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matchfail-free

matchfail-free

static sublanguage
(no $+^?$)

Road map

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bidirectional
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Source
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Target type system
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type-directed
translation

matchfail-free matchfail-free

static sublanguage
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Target language

- Target sum types include only **static** sums: $+$, $+_1$, $+_2$
- Casts between sums:

$$\langle +_1 \Leftarrow + \rangle(\mathtt{inj}_1\, v) \quad \text{will step to } \mathtt{inj}_1\, v$$

$$\langle +_2 \Leftarrow + \rangle(\mathtt{inj}_1\, v) \quad \text{will step to } \mathtt{matchfail}$$

Type-directed translation: add casts

Where **directed consistency** $\rightsquigarrow$ is used, translation adds a cast from A' to B'

$$\frac{\Gamma \vdash e : A' \hookrightarrow M \qquad A' \rightsquigarrow \mathrm{B}' \hookrightarrow \mathcal{C}}{\Gamma \vdash e : \mathrm{B}' \hookrightarrow \mathcal{C}[\mathrm{M}]}$$

$$\begin{array}{ccc} A' & \rightsquigarrow & \mathrm{B}' \\ \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} \\ A & \leq & \mathrm{B} \end{array}$$

Type-directed translation: add casts

Where **directed consistency** $\leadsto$ is used, translation adds a cast from A' to B'

$$\frac{\Gamma \vdash e : A' \hookrightarrow M \qquad A' \leadsto B' \hookrightarrow \mathcal{C}}{\Gamma \vdash e : B' \hookrightarrow \mathcal{C}[M]}$$

$$\begin{tikzcd}
A' \arrow[r, phantom, "\leadsto"] \arrow[d, phantom, "\rotatebox{90}{$\sqsubseteq$}"] & B' \arrow[d, phantom, "\rotatebox{90}{$\sqsubseteq$}"] \\
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\end{tikzcd}$$

$$\frac{\Gamma \vdash x : (\mathsf{Unit} +^{?} \mathsf{Unit}) \hookrightarrow x \qquad \begin{array}{l} (\mathsf{Unit} +^{?} \mathsf{Unit}) \leadsto (\mathsf{Unit} +_2 \mathsf{Unit}) \\ \hookrightarrow \langle +_2 \Leftarrow + \rangle[\,] \end{array}}{\Gamma \vdash x : B' \hookrightarrow \langle +_2 \Leftarrow + \rangle x}$$

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\mathsf{Unit} +_2 \mathsf{Unit} \arrow[r, phantom, "\leq"] & \mathsf{Unit} +_2 \mathsf{Unit}
\end{tikzcd}$$

Metatheory

Source
bidirectional
type system
Thm. 1

Source
type assignment
system

Target type system
with run-time casts

$$\begin{tikzcd}
e \overset{\Rightarrow}{\Leftarrow} A \arrow[loop left, "\text{varying precision Thm. 4}"'] \arrow[r, shift left, "\text{Thm. 2}"] \arrow[d, shift left, "\text{Thm. 5}"] & e : A \arrow[l, shift left, "\text{Thm. 3}"] \arrow[r, hook, "\text{Thm. 9}", "\text{type-directed translation}"'] & M : T \arrow[r, mapsto, "\text{steps to}"] & M' : T \\
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type safety (Thms. 6,7)

static sublanguage
(no $+^?$)

matchfail-free
Thm. 10

matchfail-free
Thm. 8

Metatheory

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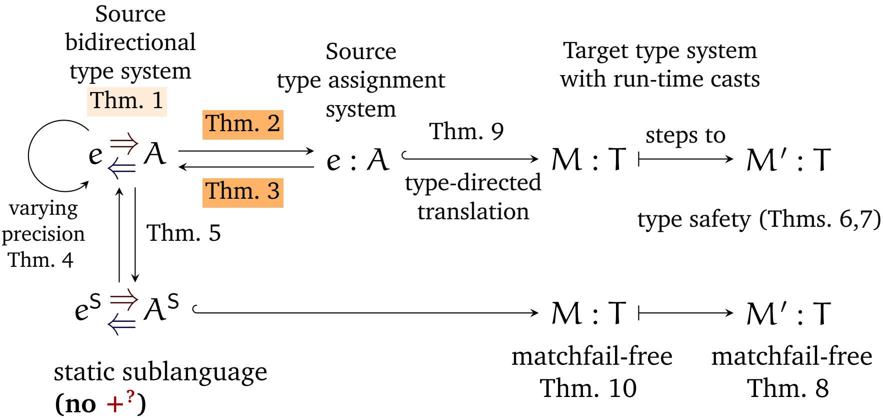

Source bidirectional type system **Thm. 1**

Source type assignment system

Target type system with run-time casts

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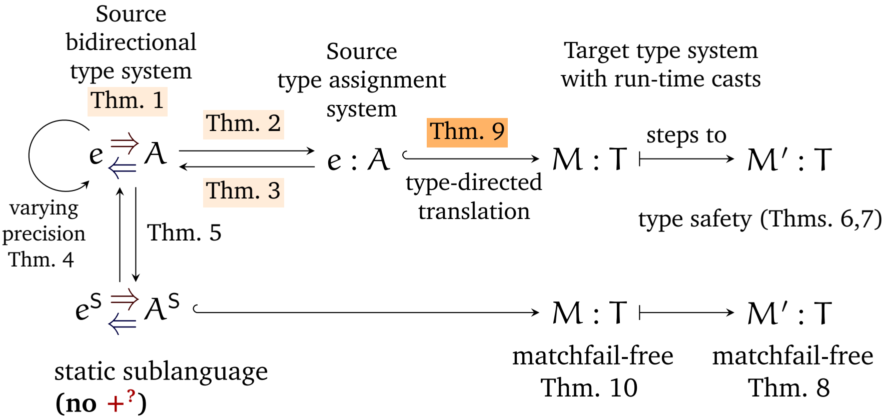

Source bidirectional type system

Thm. 1

Thm. 2

Source type assignment system

Target type system with run-time casts

Thm. 9

steps to

$e \substack{\Rightarrow \\ \Leftarrow} A$

$e : A$

$M : T \longmapsto M' : T$

Thm. 3

type-directed translation

type safety (Thms. 6,7)

varying precision Thm. 4

Thm. 5

$e^{S} \substack{\Rightarrow \\ \Leftarrow} A^{S}$

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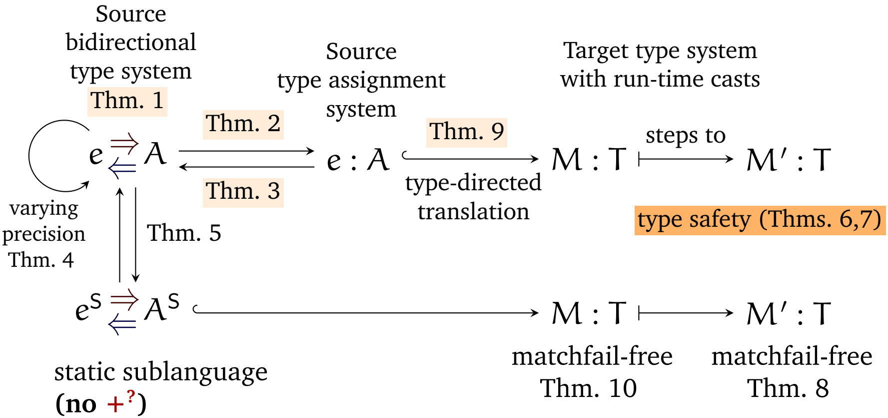

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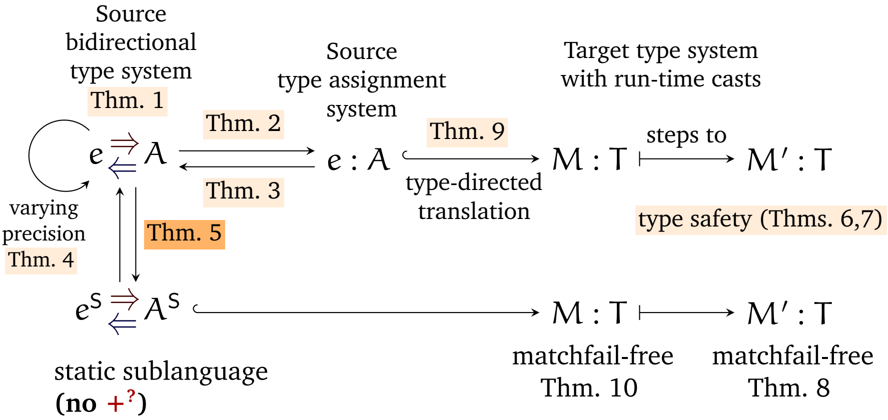

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Metatheory

Gradual guarantee (Siek et al. 2015)

- Thm. 4: Varying precision
- Thm. 5: Static soundness and completeness
- Thm. 15: Dynamic soundness and completeness
- Thm. 11: Translation preserves precision
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- Thm. 13: Precision respects convergence

Metatheory

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Related work

Refinements:

- Datasort refinements:
  Freeman & Pfenning 1991, Davies 2005, …
  $A \sqsubset \tau$ says refinement (**sort**) A refines **type** τ.
  Kind of like $A' \sqsubseteq A$—but sorts and types cannot be mixed:
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- Bidirectionality makes type-checking practical

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Gradual typing:

- Consistency (Siek and Taha 2006, . . .)
- Consistent subtyping (Siek and Taha 2007, . . .)
- Blame (Wadler & Findler 2009, . . .)
- Subformula property (Garcia & Cimini 2015)

What’s next?

- Implement the bidirectional system and translation
- Add more types (intersection, μ, $\forall$)
- Evaluate run-time efficiency

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- Unify and generalize

  (1) classic gradual typing, and
  (2) gradual sums

  through a new type constructor, guided by ideas from **abstracting gradual typing** (Garcia et al. 2016)

Conclusion

- Guided by type-theoretic intuition,
  we combined static sums and dynamic sums
  into a gradual type system
- The subformula property of bidirectional typing
  controls imprecision
- The system enjoys the gradual guarantee

Paper and proofs: `arxiv.org/abs/1611.02392`

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