Assignment 3

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due: Tuesday, 26 February 2019

Name(s): Estimating time spent (per person):

Note: Questions marked with a + are bonus questions. You can receive full marks without doing them.
§1 Natural deduction: iff and negation

A simplified system of natural deduction, omitting both kinds of quantification, is shown in Figure 1. This is the starting point for this question.

atomic formulas \( P, Q \)
formulas \( A, B, C := P \mid A \supset B \mid A \& B \mid A \lor B \mid \text{True} \mid \text{False} \mid \neg A \)

\[
\begin{align*}
\Delta \text{true} & \quad \text{A is true} \\
\vdots & \\
B \text{true} \quad \supset \text{Intro}^x & \quad A \supset B \text{ true} \quad A \text{ true} \quad \supset \text{Elim}^x \\
(A \supset B) \text{ true} & \\
\text{True} \text{ true} \quad \text{TrueIntro} & \quad \text{no elim. for True} & \quad \text{intro. for False: see } \neg \text{Elim below} & \quad C \text{ true} \quad \text{FalseElim}
\end{align*}
\]

\[
\begin{align*}
A \text{ true} & \quad B \text{ true} \quad \& \text{Intro} & \quad A \& B \text{ true} \quad \& \text{Elim}^1 & \quad A \& B \text{ true} \quad \& \text{Elim}^2 \\
A \& B \text{ true} & \\
\text{A true} \quad \text{B true} \quad \lor \text{Intro}^1 & \quad A \lor B \text{ true} \quad B \text{ true} \quad \lor \text{Intro}^2 & \quad A \lor B \text{ true} \quad C \text{ true} \quad C \text{ true} \quad \lor \text{Elim}^x \& \text{B true}
\end{align*}
\]

\[
\begin{align*}
\Delta \text{true} & \quad \Delta \text{true} \\
\vdots & \\
\text{False true} \quad \neg \text{Intro}^x & \quad A \text{ true} \quad (\neg A) \text{ true} \quad \neg \text{Elim} \\
(\neg A) \text{ true} & \\
\end{align*}
\]

\textbf{Figure 1} Natural deduction, without quantifiers

\textbf{Part 1(a).} Derive \((P \& Q) \supset (P \lor Q)\) true.
§1 Natural deduction: iff and negation

Part 1(b). Gentzen mentioned a symbol $\supset\subset$, meaning “if and only if”. But, observing that instead of $A \supset\subset B$ we can use $(A \supset B) \& (B \supset A)$, he didn’t include rules for $\supset\subset$.

Design introduction and elimination rules for $\supset\subset$. 

**Hint:** Think about the shape of derivations involving the formula $(A \supset B) \& (B \supset A)$.

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Part 1(c). Complete the following derivation of $\neg P \supset (P \supset False)$ true.

\[
\begin{align*}
& x [\neg P \text{ true}] \\
& y [P \text{ true}] \\
\hline
& (P \supset False) \text{ true} \quad \supset \text{Intro}^y \\
\hline
& (\neg P) \supset (P \supset False) \text{ true} \quad \supset \text{Intro}^x
\end{align*}
\]

Part 1(d). Derive $(P \supset False) \supset (\neg P)$ true.

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2 Language migration

If we believe the way that sequent calculus represents assumptions is more clear than natural deduction, we might want to *migrate* from natural deduction to sequent calculus.
This creates a problem similar to that of compatibility between different programming languages: how do we convert existing programs to a different language, or to a newer version of the same language? Thus, we will show that we can always convert a natural deduction derivation to a sequent calculus derivation. That is, sequent calculus is complete with respect to natural deduction. Proving this for the full rule systems, or even the somewhat smaller system in Figure 1, would be tedious. Instead, we will prove it for a much smaller system that includes only two connectives: \( \supset \) and \( \lor \).

In natural deduction, we implicitly had a rule \( x \) with conclusion \( A \) true for each assumption \( x[A \text{ true}] \). To ensure that we don't miss that case in the proof, I have written out this rule explicitly.

\[
\begin{array}{c}
A \text{ true} \\
\hline
x[A \text{ true}] \\
\hline
\vdots \\
A \text{ true}
\end{array}
\]

\[
\begin{array}{c}
\vdash B \text{ true} \\
\hline
\vdash (A \supset B) \text{ true} \\
\hline
\supset \text{Intro}^x
\end{array}
\]

\[
\begin{array}{c}
A \supset B \text{ true} \\
\hline
A \text{ true} \\
\hline
\vdash B \text{ true} \\
\hline
\supset \text{Elim}
\end{array}
\]

\[
\begin{array}{c}
A \text{ true} \\
\hline
\vdash A \lor B \text{ true} \\
\hline
\lor \text{Intro}^1
\end{array}
\]

\[
\begin{array}{c}
B \text{ true} \\
\hline
\vdash A \lor B \text{ true} \\
\hline
\lor \text{Intro}^2
\end{array}
\]

\[
\begin{array}{c}
\vdash A \lor B \text{ true} \\
\hline
\vdash C \text{ true} \\
\hline
\lor \text{Elim}^{x,y}
\end{array}
\]

Figure 2 Natural deduction, with only \( \supset \) and \( \lor \)

\[
\begin{array}{c}
\Gamma \vdash A \text{ true} \\
\hline
\vdash A \text{ true} \\
\hline
\sc \text{Assum}
\end{array}
\]

\[
\begin{array}{c}
\vdash x[A \text{ true}] \\
\hline
\Gamma, x[A \text{ true}] \vdash B \text{ true} \\
\hline
\sc \supset \text{Intro}
\end{array}
\]

\[
\begin{array}{c}
\Gamma \vdash A \supset B \text{ true} \\
\hline
\Gamma \vdash A \text{ true} \\
\hline
\Gamma \vdash B \text{ true} \\
\hline
\sc \supset \text{Elim}
\end{array}
\]

\[
\begin{array}{c}
\vdash A \lor B \text{ true} \\
\hline
\Gamma \vdash A \lor B \text{ true} \\
\hline
\sc \lor \text{Intro}^1
\end{array}
\]

\[
\begin{array}{c}
\vdash A \lor B \text{ true} \\
\hline
\Gamma, x[A \text{ true}] \vdash C \text{ true} \\
\hline
\sc \lor \text{Intro}^2
\end{array}
\]

\[
\begin{array}{c}
\Gamma \vdash C \text{ true} \\
\hline
\Gamma, y[B \text{ true}] \vdash C \text{ true} \\
\hline
\sc \lor \text{Elim}
\end{array}
\]

Figure 3 Sequent calculus, with only \( \supset \) and \( \lor \)
Part 2(a). Finish the cases $\lor$Intro2 and $\supset$Elim in the following proof of completeness.

Part 2(b)+. Bonus marks: Do the case for $\lor$Elim.

Conjecture 2.1.

For all natural deduction derivations $D$ of $A$ true where $\Gamma$ is a list of $k$ “floating” assumptions, that is, $\Gamma = x_1[B_1 \text{true}], \ldots, x_k[B_k \text{true}]$ and the derivation $D$ looks like

\[
\begin{align*}
    x_1[B_1 \text{true}] \\
    x_2[B_2 \text{true}] \\
    \vdots \\
    x_k[B_k \text{true}] \\
    \vdots
\end{align*}
\]

$D$ derives $A$ true

there exists a sequent calculus derivation of $\Gamma \vdash A$ true.

Proof. By induction on the derivation $D$.

Thus, the induction hypothesis is:

For all natural deduction derivations $D'$ where $D' \prec D$ (that is, $D'$ is a subderivation of $D$) and $D'$ derives $A'$ true where $\Gamma'$ is a list of $k'$ “floating” assumptions, there exists a sequent calculus derivation of $\Gamma' \vdash A'$ true.

Our goal is to derive $\Gamma \vdash A$ true.

Consider cases of the concluding rule in the derivation $D$.

• Rule $x$:

Rule $x$ requires that $x[A \text{true}]$ is a floating assumption. We know that $x$ is the concluding rule because we are inside that case. It is given that $\Gamma$ is the set of floating assumptions.

Our goal is to derive $\Gamma \vdash A$ true.

\[
\begin{align*}
    x[A \text{true}] & \in \Gamma & \text{By inversion on rule } x \\
    \Gamma \vdash A \text{ true} & \text{By rule assum}
\end{align*}
\]

• Rule $\supset$Intro:

Our $A$ clashes with the $A$ in $\supset$Intro, so when using inversion we need to rename the meta-variables. The safest method is to write out the rule again with the meta-variables renamed. I also renamed $x$ to $y$, because it is not among the given assumptions $x_1, \ldots, x_k$.

\[
\begin{align*}
    & y[A_1 \text{true}] \\
    & \vdots \\
    & A_2 \text{ true} \\
    \frac{(A_1 \supset A_2) \text{ true}}{\supset \text{Intro}_y}
\end{align*}
\]

Now, if we “line up” the known fact $A$ true with the conclusion of $\supset$Intro, we learn that $A = (A_1 \supset A_2)$.

\[
\begin{align*}
    & y[A_1 \text{true}] \\
    & \vdots \\
    & A_2 \text{ true} \\
    \frac{(A_1 \supset A_2) \text{ true}}{A \supset \text{Intro}_y}
\end{align*}
\]
The statement of the conjecture says that the derivation of $A$ true looks like

\[
x_1[B_1 \text{ true}]
x_2[B_2 \text{ true}]
\vdots
x_k[B_k \text{ true}]
\vdots
\]

\[D \text{ derives } A \text{ true}\]

Since we know (within this case) that the concluding rule is $\supset$Intro, we know that the derivation must look like this:

\[
x_1[B_1 \text{ true}]
x_2[B_2 \text{ true}]
\vdots
x_k[B_k \text{ true}]
y[A_1 \text{ true}]
\vdots
\]

\[D \text{ derives } A_2 \text{ true} \quad \supset\text{Intro}^y\]

Let $\Gamma_y = (\Gamma, y[A_1 \text{ true}])$.

\[\begin{align*}
A_2 \text{ true} & \quad \text{[assumptions } x_1, \ldots, x_k, y]\quad \text{By inversion on rule } \supset\text{Intro} \\
\Gamma_y & \vdash A_2 \text{ true} \quad \text{By IH [with } \Gamma_y \text{ as } \Gamma' \text{ and } A_2 \text{ as } A'] \\
\Gamma_y[y[A_1 \text{ true}]] & \vdash A_2 \text{ true} \quad \text{By } \Gamma_y = \ldots \\
\Gamma & \vdash (A_1 \supset A_2) \text{ true} \quad \text{By sc-$\supset$Intro} \\
\Gamma & \vdash A \text{ true} \quad \text{By } A = (A_1 \supset A_2)
\end{align*}\]

- Rule $\lor$Intro1:

\[\begin{align*}
A = A_1 \lor A_2 & \quad \text{By inversion on rule } \lor\text{Intro1} \\
A_1 \text{ true} & \quad " \\
\Gamma & \vdash A_1 \text{ true} \quad \text{By IH [with } \Gamma \text{ as } \Gamma' \text{ and } A_1 \text{ as } A'] \\
\Gamma & \vdash A_1 \lor A_2 \text{ true} \quad \text{By sc-$\lor$Intro1}
\end{align*}\]

- Rule $\lor$Intro2: [Complete this case. It should be very similar to $\lor$Intro1.]

\[\begin{align*}
A = A_1 \lor A_2 & \quad \text{By inversion on rule } \lor\text{Intro2} \\
A_2 \text{ true} & \quad " 
\end{align*}\]
§2 Language migration

• Rule $\supset$Elim: Do this case. **Hints:** (1) The meta-variable $A$ clashes with $\supset$Elim, so you will need to rename. You don’t have to explain the renaming with as much detail as I gave for $\supset$Intro, but it’s a good idea to write out the renamed rule for your own reference. (2) $\supset$Elim has two premises, so you’ll need to use the IH twice. (3) Since $\supset$Elim doesn’t add a floating assumption, you can use $\Gamma$ as the $\Gamma'$ when you use the IH. (4) One of the last steps, perhaps the last step, of the proof will be to apply rule sc-$\supset$Elim.

• Rule $\lor$Elim: [For bonus marks. It should be fairly similar to the case for $\supset$Intro.]
§2 Language migration

**Question 2(c).** In the proof above, we used induction on the derivation $D$ rather than the formula $A$. Inducting on $A$ would *not* have worked. Briefly explain why. **Hint:** think about the $\supset$Elim case, and the obligation $A' \prec A$ that must be met when applying the IH.

**Question 2(d).** Conceivably, in the proof above, we could have used induction on the length of $\Gamma$, but that also doesn't work. Briefly explain why.