lec10: Sequent calculus

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1 The matter with natural deduction

I've complained about natural deduction's notation for assumptions. The original notation was even worse:

- In the rules assuming a Nat-judgment $a : \mathsf{Nat}$, Gentzen did not write any kind of floating assumption.
- In the rules assuming a truth judgment $A\ \mathsf{true}$, Gentzen did not label the assumption.

In his example derivations, Gentzen did write a number next to the name of the rule used, corresponding to our x, y—but then didn't write the content of the assumption ($A\ \mathsf{true}$) at all, so you had to look for the number and determine the content of the assumption from the rule used.

We can characterize the problem with both the original notation and our variant as follows: it is not *locally* clear which assumptions are available. If you're somewhere in the middle of a larger derivation, the only way to know what assumptions are available is to look *up* (for $x[A\ \mathsf{true}]$) and then *down* (for the superscripts on rule names that indicate the lower boundary of where the assumption is available).

Sequent calculus, for our purposes, is an adaptation of natural deduction that clarifies when assumptions are available. We can arrive at this systematically.

First, consider the following example.

$$x\big[P \supset Q\ \mathsf{true}\big]$$
$$y\big[P \supset R\ \mathsf{true}\big]$$
$$z\big[P\ \mathsf{true}\big]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \supset Q}\,x \qquad \dfrac{}{P\ \mathsf{true}}\,z}{Q\ \mathsf{true}}\,{\supset}\text{Elim} \qquad \dfrac{\dfrac{}{P \supset R\ \mathsf{true}}\,y \qquad \dfrac{}{P\ \mathsf{true}}\,z}{R\ \mathsf{true}}\,{\supset}\text{Elim}}{Q \,\&\, R\ \mathsf{true}}\,\&\text{Intro}}{P \supset (Q \,\&\, R)\ \mathsf{true}}\,{\supset}\text{Intro}^{z}}{(P \supset R) \supset \big(P \supset (Q \,\&\, R)\big)\ \mathsf{true}}\,{\supset}\text{Intro}^{y}}{(P \supset Q) \supset (P \supset R) \supset \big(P \supset (Q \,\&\, R)\big)\ \mathsf{true}}\,{\supset}\text{Intro}^{x}$$

We can systematically mark up the derivation to show where each assumption is available. If we write "x" to the left, that means the assumption labelled x—and *only* that assumption—is available to derive the judgment to its right.

$$x\big[P \supset Q\ \mathsf{true}\big]$$
$$y\big[P \supset R\ \mathsf{true}\big]$$
$$z\big[P\ \mathsf{true}\big]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x,y,z\ \ P \supset Q}\,x \qquad \dfrac{}{x,y,z\ \ P\ \mathsf{true}}\,z}{x,y,z\ \ Q\ \mathsf{true}}\,{\supset}\text{Elim} \qquad \dfrac{\dfrac{}{x,y,z\ \ P \supset R\ \mathsf{true}}\,y \qquad \dfrac{}{x,y,z\ \ P\ \mathsf{true}}\,z}{x,y,z\ \ R\ \mathsf{true}}\,{\supset}\text{Elim}}{x,y,z\ \ Q \,\&\, R\ \mathsf{true}}\,\&\text{Intro}}{x,y\ \ P \supset (Q \,\&\, R)\ \mathsf{true}}\,{\supset}\text{Intro}^{z}}{x\ \ (P \supset R) \supset \big(P \supset (Q \,\&\, R)\big)\ \mathsf{true}}\,{\supset}\text{Intro}^{y}}{\emptyset\ \ (P \supset Q) \supset (P \supset R) \supset \big(P \supset (Q \,\&\, R)\big)\ \mathsf{true}}\,{\supset}\text{Intro}^{x}$$

This makes it easier to see which assumptions are available, but we still have to look upwards to see *what* each assumption is. We can solve that, too—but to make the derivation fit, I need to abbreviate "$x\big[P \supset Q\ \mathsf{true}\big], y\big[P \supset R\ \mathsf{true}\big], z\big[P\ \mathsf{true}\big]$". I will also add a "turnstile", $\vdash$, to separate the assumptions from the A true part of the judgment.

$$\Gamma_{xyz} \;=\; x[P \supset Q\ \mathsf{true}], y[P \supset R\ \mathsf{true}], z[P\ \mathsf{true}]$$

$$\begin{array}{r} x[P \supset Q\ \mathsf{true}] \\ y[P \supset R\ \mathsf{true}] \\ z[P\ \mathsf{true}] \end{array}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_{xyz} \vdash P \supset Q}x \qquad \dfrac{}{\Gamma_{xyz} \vdash P\ \mathsf{true}}z}{\Gamma_{xyz} \vdash Q\ \mathsf{true}}{\supset}\mathrm{Elim} \qquad \dfrac{\dfrac{}{\Gamma_{xyz} \vdash P \supset R\ \mathsf{true}}y \qquad \dfrac{}{\Gamma_{xyz} \vdash P\ \mathsf{true}}z}{\Gamma_{xyz} \vdash R\ \mathsf{true}}{\supset}\mathrm{Elim}}{\Gamma_{xyz} \vdash Q \,\&\, R\ \mathsf{true}}\&\mathrm{Intro}}{x[P \supset Q\ \mathsf{true}], y[P \supset R\ \mathsf{true}] \vdash P \supset (Q \,\&\, R)\ \mathsf{true}}{\supset}\mathrm{Intro}^{z}}{x[P \supset Q\ \mathsf{true}] \vdash (P \supset R) \supset \big(P \supset (Q \,\&\, R)\big)\ \mathsf{true}}{\supset}\mathrm{Intro}^{y}}{\emptyset \vdash (P \supset Q) \supset (P \supset R) \supset \big(P \supset (Q \,\&\, R)\big)\ \mathsf{true}}{\supset}\mathrm{Intro}^{x}$$

Now we can look at an individual judgment within the derivation, say $P \supset (Q \,\&\, R)$ true, and immediately see what assumptions are available. If we write derivations this way, the "floating" assumptions and the superscripts by the rule names become redundant.

$$\Gamma_{xyz} \;=\; x[P \supset Q\ \mathsf{true}], y[P \supset R\ \mathsf{true}], z[P\ \mathsf{true}]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_{xyz} \vdash P \supset Q}x \qquad \dfrac{}{\Gamma_{xyz} \vdash P\ \mathsf{true}}z}{\Gamma_{xyz} \vdash Q\ \mathsf{true}}{\supset}\mathrm{Elim} \qquad \dfrac{\dfrac{}{\Gamma_{xyz} \vdash P \supset R\ \mathsf{true}}y \qquad \dfrac{}{\Gamma_{xyz} \vdash P\ \mathsf{true}}z}{\Gamma_{xyz} \vdash R\ \mathsf{true}}{\supset}\mathrm{Elim}}{\Gamma_{xyz} \vdash Q \,\&\, R\ \mathsf{true}}\&\mathrm{Intro}}{x[P \supset Q\ \mathsf{true}], y[P \supset R\ \mathsf{true}] \vdash P \supset (Q \,\&\, R)\ \mathsf{true}}{\supset}\mathrm{Intro}}{x[P \supset Q\ \mathsf{true}] \vdash (P \supset R) \supset \big(P \supset (Q \,\&\, R)\big)\ \mathsf{true}}{\supset}\mathrm{Intro}}{\emptyset \vdash (P \supset Q) \supset (P \supset R) \supset \big(P \supset (Q \,\&\, R)\big)\ \mathsf{true}}{\supset}\mathrm{Intro}$$

At this point, the assumption names also seem redundant and could be removed. Indeed, Gentzen's presentation of sequent calculus does not name the assumptions. However, we will continue to name the assumptions.

1.1 Our sequent calculus

Our grammar used for sequent calculus includes the same definition of formulas as our grammar for natural deduction, but adds *contexts* Γ. By convention, we usually omit the $\emptyset$ symbol and the following comma:

$$x[P\ \mathsf{true}], y[Q\ \mathsf{true}]$$

instead of

$$\emptyset, x[P\ \mathsf{true}], y[Q\ \mathsf{true}]$$

even though this doesn't match the grammar.

atomic formulas	P, Q			
formulas	A, B, C	$::=$	P	atomic formula
			$\mid A \supset B$	implication
			$\mid A \,\&\, B$	conjunction (and)
			$\mid A \vee B$	disjunction (or)
			$\mid \forall a : \mathsf{Nat}.\ A$	universal quantification
			$\mid \exists a : \mathsf{Nat}.\ A$	existential quantification
			$\mid \mathsf{True}$	truth
			$\mid \mathsf{False}$	falsehood
			$\mid \neg A$	negation
contexts	Γ	$::=$	$\emptyset$	empty context (no assumptions)
			$\mid \Gamma, x[A\ \mathsf{true}]$	assumption that A is true
			$\mid \Gamma, y[a : \mathsf{Nat}]$	assumption that a is a natural number

$\boxed{\Gamma \vdash A \text{ true}}$ Under assumptions Γ, formula A is true

$$\frac{x[A \text{ true}] \in \Gamma}{\Gamma \vdash A \text{ true}}\,\text{assum} \qquad \frac{\Gamma, x[A \text{ true}] \vdash B \text{ true}}{\Gamma \vdash (A \supset B) \text{ true}}\,{\supset}\text{Intro} \qquad \frac{\Gamma \vdash A \supset B \text{ true} \qquad \Gamma \vdash A \text{ true}}{\Gamma \vdash B \text{ true}}\,{\supset}\text{Elim}$$

$$\frac{}{\Gamma \vdash \text{True true}}\,\text{TrueIntro} \qquad \text{no elim. for True} \qquad \text{intro. for False: see } \neg\text{Elim below} \qquad \frac{\Gamma \vdash \text{False true}}{\Gamma \vdash C \text{ true}}\,\text{FalseElim}$$

$$\frac{\Gamma \vdash A \text{ true} \qquad \Gamma \vdash B \text{ true}}{\Gamma \vdash A \,\&\, B \text{ true}}\,\&\text{Intro} \qquad \frac{\Gamma \vdash A \,\&\, B \text{ true}}{\Gamma \vdash A \text{ true}}\,\&\text{Elim1} \qquad \frac{\Gamma \vdash A \,\&\, B \text{ true}}{\Gamma \vdash B \text{ true}}\,\&\text{Elim2}$$

$$\frac{\Gamma, x[a : \text{Nat}] \vdash B \text{ true}}{\Gamma \vdash (\forall a : \text{Nat}.\, B) \text{ true}}\,\forall\text{Intro} \qquad \frac{\Gamma \vdash (\forall a : \text{Nat}.\, B) \text{ true} \qquad \Gamma \vdash n : \text{Nat}}{\Gamma \vdash [n/a]B \text{ true}}\,\forall\text{Elim}$$

$$\frac{\Gamma \vdash A \text{ true}}{\Gamma \vdash A \vee B \text{ true}}\,\vee\text{Intro1} \qquad \frac{\Gamma \vdash B \text{ true}}{\Gamma \vdash A \vee B \text{ true}}\,\vee\text{Intro2}$$

$$\frac{\Gamma \vdash A \vee B \text{ true} \qquad \Gamma, x[A \text{ true}] \vdash C \text{ true} \qquad \Gamma, y[B \text{ true}] \vdash C \text{ true}}{\Gamma \vdash C \text{ true}}\,\vee\text{Elim}$$

$$\frac{\Gamma \vdash n : \text{Nat} \qquad \Gamma \vdash \big([n/a]B\big) \text{ true}}{\Gamma \vdash (\exists a : \text{Nat}.\, B) \text{ true}}\,\exists\text{Intro} \qquad \frac{\Gamma \vdash (\exists a : \text{Nat}.\, B) \text{ true} \qquad \Gamma, x[a : \text{Nat}], y[B \text{ true}] \vdash C \text{ true}}{\Gamma \vdash C \text{ true}}\,\exists\text{Elim}$$

$$\frac{\Gamma, x[A \text{ true}] \vdash \text{False true}}{\Gamma \vdash (\neg A) \text{ true}}\,\neg\text{Intro} \qquad \frac{\Gamma \vdash A \text{ true} \qquad \Gamma \vdash (\neg A) \text{ true}}{\Gamma \vdash \text{False true}}\,\neg\text{Elim (LoC)}$$

1.2 Gentzen's sequent calculus

In addition to some superficial differences, discussed below, Gentzen's sequent calculus differs from ours in four major respects.

First, Gentzen's rule for assumptions required that the assumption used be the *only* assumption in Γ:

$$\frac{}{x[A \text{ true}] \vdash A \text{ true}}$$

Partly because of this, Gentzen included several rules to rearrange assumptions:

$$\frac{\Gamma, \Gamma' \vdash C \text{ true}}{\Gamma, x[A \text{ true}], \Gamma' \vdash C \text{ true}}\,\text{thinning} \qquad \frac{\Gamma, x[A \text{ true}], x[A \text{ true}], \Gamma' \vdash C \text{ true}}{\Gamma, x[A \text{ true}], \Gamma' \vdash C \text{ true}}\,\text{duplication}$$

$$\frac{\Gamma, y[B \text{ true}], x[A \text{ true}], \Gamma' \vdash C \text{ true}}{\Gamma, x[A \text{ true}], y[B \text{ true}], \Gamma' \vdash C \text{ true}}\,\text{exchange}$$

Second, in addition to introduction and elimination rules that manipulate the *succedent formula* to the right of the turnstile ⊢, Gentzen included rules to manipulate the formulas *inside* the assumptions (the antecedent). One such rule was

$$\frac{\Gamma, x[A \text{ true}], \Gamma' \vdash C \text{ true} \qquad \Gamma, x[B \text{ true}], \Gamma' \vdash C \text{ true}}{\Gamma, x[A \vee B \text{ true}], \Gamma' \vdash C \text{ true}}\,\vee\text{-Intro-Antecedent}$$

In effect, Gentzen's $\vee$-Intro-Antecedent *replaced* the $\vee$-Elim rule.

Third, Gentzen allowed multiple right-hand formulas (succedents), which were interpreted disjunctively rather than conjunctively. For example, the sequent

$$x\left[\mathsf{P}\ \mathsf{true}\right], y\left[\mathsf{Q}\ \mathsf{true}\right] \vdash \mathsf{P}\ \mathsf{true}, \mathsf{Q}\ \mathsf{true}, \neg\mathsf{Q}\ \mathsf{true}$$

should be read

assuming P is true and Q is true, then P is true or Q is true or ¬Q is true

This formulation led to interesting symmetries. As these symmetries are more clear in Gentzen's notation, which omitted the labels and brackets in assumptions (antecedents) and omitted the word true throughout, we temporarily adopt his notation. For example, Gentzen's rules for negation looked like

$$\frac{A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta, \neg A}\ \neg\text{-Intro-Succedent} \qquad \frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta}\ \neg\text{-Intro-Antecedent}$$

(In addition to Γ, Gentzen used Δ, Θ, and Λ for lists of formulas.)

Fourth, Gentzen included a rule called *cut*:

$$\frac{\Gamma \vdash \Theta, A \qquad A, \Delta \vdash \Lambda}{\Gamma, \Delta \vdash \Theta, \Lambda}\ \text{Cut}$$

To understand this rule, it helps to simplify it by considering the special case when Θ is empty:

$$\frac{\Gamma \vdash A \qquad A, \Delta \vdash \Lambda}{\Gamma, \Delta \vdash \Lambda}\ \text{Cut (restricted to } \Theta = \emptyset)$$

We can then read the rule as follows:

- We want to prove Λ, under assumptions Γ, Δ.
- First we prove A under assumptions Γ.
- Since A holds under Γ, it is true, and we can safely add it as an assumption to prove Λ.

The situation is somewhat like realizing that proving something requires a separate lemma that has not yet been proved. The cut rule says that we can separately prove the lemma (represented by $\Gamma \vdash A$), then use the fact that the lemma holds

It may help to restrict the rule further, to the situation where Γ is empty.

$$\frac{\emptyset \vdash A \qquad A, \Delta \vdash \Lambda}{\Delta \vdash \Lambda}\ \text{Cut (restricted to } \Theta = \Gamma = \emptyset)$$

We would find ourselves in this situation whenever the lemma can be proved from no assumptions. That is, the premise $\emptyset \vdash A$ says that, from no assumptions, A is true; the premise $A, \Delta \vdash \Lambda$ says that, from our starting assumptions Δ (I call these the starting assumptions because they are the assumptions available in the conclusion $\Delta \vdash \Lambda$) along with our lemma A, we can show Λ.

1.3 Historical notes

To separate the antecedent formulas (the assumptions) of a sequent from its succedent formulas, Gentzen used an arrow $\rightarrow$ rather than a turnstile $\vdash$.

The original German for sequent calculus was *Sequenzenkalkül*, "calculus of sequences". The word "sequent" avoids the misleading suggestion that it is a calculus of Taylor series or some other numerical sequence. I had thought "sequent" was an invention of Stephen Kleene, but it was an existing (though archaic) English word with several meanings, including "consequence" and "an element of a sequence":

> There bee others that delight in figures, and their words fall in, one after another like sequents. (William Cavendish, *Horæ subseciuæ. Obseruations and discourses*, 1620)