1 Subtyping

1.1 Introduction

Many real programming languages include some form of subtyping. You may be most familiar with subtyping in object-oriented languages, where the primary form of subtyping is achieved through inheritance: if class C2 inherits from class C1, then C2 is a subclass of C1, and therefore C2 is a subtype of C1.

However, we can interpret subtyping more broadly:

\[ S \text{ is a subtype of } T \text{ if every } S \text{ is a } T \]
or

\[ S \text{ is a subtype of } T \text{ if } S \subseteq T \]

We cannot really say that, because types S and T are defined by a grammar; what does it mean for one string of symbols to be a subset of another?

We can say

\[ S \text{ is a subtype of } T \text{ if, for every value } v \text{ such that } \emptyset \vdash v : S, \]
we can also derive \( \emptyset \vdash v : T \)

Most subtyping systems—sets of rules deriving a judgment \( S <: T \)—do not quite reflect this idea. Instead, they approximate it, by being sound with respect to it:

If \( S <: T \) then, for every value \( v \) such that... but not complete, that is, the following does not hold:

If, for every value \( v \) such that..., we can derive \( S <: T \)

An example of a “sound subtyping” that many subtyping systems cannot derive is

\( (\bot \times \text{int}) <: \bot \)

It is true that every value of type \( (\bot \times \text{int}) \) also has type \( \bot \), but only because there are no values of type \( (\bot \times \text{int}) \)—because there are no values of type \( \bot \).

On the other hand, subtyping systems can derive many useful subtypings. For example, if we add a type \text{nat} of integers that are greater than or equal to zero, with a typing rule

\[
\frac{n \geq 0}{\Gamma \vdash n : \text{nat}} \text{natIntro}
\]
then every value of type nat also has type int (because we can use our existing rule intIntro), making the following subtyping rule sound.

\[
\text{nat} <: \text{int} \\
\text{sub-nat-int}
\]

In the remainder of these notes, we design sound subtyping rules for other types in our language, including \(\times\), \(\rightarrow\) and \(+\).

### 1.2 Reflexivity

A rule that doesn't say anything interesting is the **reflexivity rule**:

\[
S <: S \\
\text{sub-refl}
\]

It says that every type is a subtype of itself. Intuitively, this says that, if a value has type \(S\) then it has type \(S\), which is certainly sound.

### 1.3 Subtyping for pairs

\[
\frac{S_1 <: T_1 \quad S_2 <: T_2}{(S_1 \times S_2) <: (T_1 \times T_2)} \quad \text{sub-pair}
\]

You can gain some intuition for this rule by drawing the Cartesian plane, interpreting \((\text{Pair} \times y)\) as the point \((x, y)\) where \(x\) and \(y\) are integers, and considering the types

- nat \(\times\) nat,
- nat \(\times\) int,
- int \(\times\) nat, and
- int \(\times\) int.

Then the rule sub-pair says that the upper-right quadrant (nat \(\times\) nat) is a subtype of the three other types, that the right-hand half (nat \(\times\) int) is a subtype of the entire plane (int \(\times\) int), and that the upper half (int \(\times\) nat) is a subtype of the entire plane (int \(\times\) int).

**Exercise 1.** Add a type neg, like pos but negative. Design an appropriate subtyping rule. Design appropriate subtyping rule(s).

**Exercise 2.** Add a type zero, whose only value is 0. Design an appropriate typing rule. Design appropriate subtyping rule(s).
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1.4 Substitutability

The visual intuition of the Cartesian plane may be enough to figure out subtyping for $\times$, but subtyping for some other types will be tricky. We need another source of guidance.

A useful way to approach subtyping is substitutability, which asks: If I expect something of type $T$, when should I allow something of type $S$ instead? If I expect $T$ but allow $S$, then values of type $S$ are substitutable for values of type $T$, and it is okay for $S$ to be a subtype of $T$. (See the Liskov–Wing principle. Aside: I was a TA for Jeannette Wing in 2001.)

For example, if I expect something of type $\text{nat} \times \text{int}$, I should allow you to give me something of type $\text{nat} \times \text{nat}$: I expect something from the right-hand half of the Cartesian plane, and you are giving me something from the upper-right quadrant, which is contained within the right-hand half.

\[
\begin{array}{c}
\text{nat} <: \text{nat} \quad \text{sub-refl} \\
\text{nat} <: \text{int} \quad \text{sub-nat-int} \\
\hline
(\text{nat} \times \text{nat}) <: (\text{nat} \times \text{int}) \quad \text{sub-pair}
\end{array}
\]

1.5 Subtyping for functions

It’s tempting to write a rule

\[
S_1 <: T_1 \quad S_2 <: T_2 \quad (S_1 \rightarrow S_2) <: (T_1 \rightarrow T_2) \quad \text{sub-\rightarrow-UNSOUND}
\]

Unfortunately, only one of these two premises is okay.

The okay premise is the second one. For example, we need the second premise to show

\[
(\text{unit} \rightarrow \text{nat}) <: (\text{unit} \rightarrow \text{int})
\]

Under substitutability, if I expect something of type $\text{unit} \rightarrow \text{int}$—that is, a function that takes $()$ and returns an integer—I should accept your offer of a function that takes $()$ and returns a natural number, because $\text{nat} <: \text{int}$ (every natural number is an integer).

However, as John C. Reynolds\textsuperscript{1} once said, “something funny happens to the left of the arrow”. The premise $S_1 <: T_1$ allows us to derive

\[
\text{nat} <: \text{int} \quad \text{nat} <: \text{nat} \quad \text{sub-\rightarrow-UNSOUND}
\]

That is, if I expect a function of type $\text{int} \rightarrow \text{nat}$, I should accept a function of type $\text{nat} \rightarrow \text{nat}$.

An example of a function of type $\text{int} \rightarrow \text{nat}$ is

\[
\text{absf} = (\text{Lam x (Abs x)})
\]

If I call absf, I will always get a natural number, even when I pass a negative number. This function also has type $\text{nat} \rightarrow \text{nat}$.

\textsuperscript{1}His last student, Neel Krishnaswami, wrote about him shortly after his death. When I met John for the first time, I was impressed that he seemed genuinely interested in what I thought about Java, even though I was an undergraduate student and he was one of the greatest researchers in the field.
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However, another example of a function of type \( \text{nat} \to \text{nat} \) is the identity function:

\[
\text{idf} = (\text{Lam } x \ x)
\]

If I pass a negative number like \(-5\) to idf, I will get \(-5\).

Therefore, if I expect a function like absf of type \( \text{int} \to \text{nat} \), and you give me idf of type \( \text{nat} \to \text{nat} \), I will be unhappy.

To fix the subtyping rule and make it sound, we could require the argument types, \( S_1 \) and \( S_2 \), to be the same:

\[
S_1 = T_1 \quad S_2 \ll T_2 \quad \text{sub-\to-sound-but-weak}
\]

This rule properly disallows \( (\text{nat} \to \text{nat}) \ll (\text{int} \to \text{nat}) \). But it is not as strong as it could be. It turns out that \( S_1 \) and \( T_1 \) don’t have to be the same; rather, \( T_1 \)—the type from the right-hand side of the conclusion—must be a subtype of \( S_1 \)—which is from the left-hand side of the conclusion. This “swapping” is called contravariance.

\[
T_1 \ll S_1 \quad S_2 \ll T_2 \quad \text{sub-\to}
\]

Let’s say that I expect a function of type \( \text{nat} \to \text{nat} \). Maybe I expect something like idf. If you give me a function of type \( \text{int} \to \text{nat} \), you are giving me a more powerful tool—a function that can take any integer, not only a positive integer. I will only pass natural numbers to the function, because I think it has type \( \text{nat} \to \text{nat} \); I won’t use the extra power, but it does no harm. Our correct rule sub-\to says that’s okay:

\[
\text{nat} \ll \text{int} \quad \text{nat} \ll \text{nat} \quad \text{sub-refl} \quad \text{sub-\to}
\]

Exercise 3. Complete the following derivation. (Yes, this is possible! Rule sub-\to swaps the argument types, and you need to use sub-\to twice, so the types get swapped twice.)

\[
(\text{nat} \to \text{int}) \to \text{unit} \ll (\text{int} \to \text{int}) \to \text{unit}
\]

1.6 Subtyping for sums

A value of type \( T_1 + T_2 \) is either

1. \( (\text{Inj}_1 \ v_1) \) where \( v_1 \) has type \( T_1 \), or
2. \( (\text{Inj}_2 \ v_2) \) where \( v_2 \) has type \( T_2 \).

If I expect a value of type \( T_1 + T_2 \), and you give me a value of type \( S_1 + S_2 \), I should accept it as long as every value of type \( S_1 \) is also a value of type \( T_1 \), and the same for \( S_2 \) and \( T_2 \). When
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I eliminate $T_1 + T_2$ using a Case, I expect the variable $x_1$ to have type $T_1$ in one branch, and the variable $x_2$ will have type $T_2$ in the other branch. If you give me an $x_1$ of type $S_1$, that's okay as long as $S_1 <: T_1$.

$$
\frac{S_1 <: T_1 \quad S_2 <: T_2}{(S_1 + S_2) <: (T_1 + T_2)}_{\text{sub}+}
$$

For example, if I expect a value $v$ to have type $\text{int} + \text{unit}$, then I expect either

1. $v = (\text{Inj}_1 n)$ where $n$ is an integer, or
2. $v = (\text{Inj}_2 ())$.

If you give me a $v$ of type $\text{nat} + \text{unit}$, then you are guaranteeing that either

1. $v = (\text{Inj}_1 n)$ where $n$ is an integer and $n \geq 0$, or
2. $v = (\text{Inj}_2 ())$.

The first part of your guarantee is stronger than what I need, because I only need to know that $n$ is an integer, but that's okay.

$$
\frac{\text{nat} <: \text{int} \quad \text{unit} <: \text{unit}}{(\text{nat} + \text{unit}) <: (\text{nat} + \text{unit})}_{\text{sub}+}
$$

1.7 Subsumption rule

Defining subtyping rules is only of theoretical interest unless we incorporate subtyping into our type system. We can add a rule known as subsumption.

$$
\frac{\Gamma \vdash e : S \quad S <: T}{\Gamma \vdash e : T}_{\text{type-subsume}}
$$

Adding this rule has some interesting consequences: if we know that, say, an expression $e$ has the form $(\text{Call } e_1 e_2)$, we no longer know that the rule concluding a derivation $\Gamma \vdash e : T$ has to be $\rightarrow{\text{Elim}}$, because type-subsume could have been used instead.