Spatially referenced methods of processing raster and vector data

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The authors consider a general method of constructing addressing and arithmetic systems for two-dimensional image data using the hierarchy of 'molecular' tilings based on an original isohedral 'atomic' tiling. (Each molecular title at level k is formed from a constant number of tiles at level k - 1; this is termed the 'aperture' property of the hierarchy.) In addition they present 11 objective criteria (which are of significance in cartographic image processing), by which these hierarchies and tilings may be described and compared.

Of the 11 topologically distinct types of isohedral tiling, three ([3], [4] and [6]) are composed of regular polygons, and two of these ([3] and [4]) satisfy the condition that all tiles have the same 'orientation'. In general, although each level in a hierarchy is topologically equivalent, the tiles may differ in shape at different levels and only [6], [4], [4.8] and [4.6.12] are capable of giving rise to hierarchies in which the tiles at all levels are the same shape. The possible apertures of hierarchies obeying this condition are n for any n > 1 in the cases of [6] and [4]; n or 2n in the case of [4.8]; and n or 3n in the case of [4.6.12].

In contrast the only tiling exhibiting the uniform 'adjacency' criterion is [3]. However, hierarchies based on this atomic tiling generate molecular tiles with different shapes at every level. If these disadvantages are accepted, hierarchies based on first-level molecular tiles referred to as the 4-shape, 4-shape, 7-shape and 9-shape are generated. Of these the 4-shape and the 9-shape appear to satisfy many of the cartographically desirable properties in addition to having an atomic tiling which exhibits uniform adjacency.

In recent years the generalized balanced ternary addressing system has been developed to exploit the image processing power of the 7-shape. The authors have generalized and extended this system as 'tesseral addressing and arithmetic', showing how it can be used to render a 4-shape into a spatially correct linear quadtree.

Keywords: cartography, molecular tiling, tesseral addressing and arithmetic

The Thematic Information Service (TIS) of the Natural Environment Research Council undertakes research and development in the fields of digital cartography (within the well established experimental cartography unit (ECU)) and more recently spatial analysis and image analysis. A major area of development within the TIS has resulted from the need to handle not only the conventional vector digitized field survey and map data (approximately 300 Mbytes) belonging to the ECU but also raster digitized satellite data. An initial solution was to provide vector-to-raster conversion software which enables these cartographic datasets to be converted from one form into the other. It is recognized that this solution is far from optimal and that much basic research is necessary if the desired aim of providing smooth and sophisticated data processing and an integrated geo-information system is to be achieved.

DEFINITIONS

A tiling of the plane (or tessellation) is termed isohedral if all the tiles are equivalent under the symmetry group of the tiling and is said to be regular if, in addition, each tile is a regular polygon. If we classify the isohedral tilings according to the action of their symmetry groups, then 81 types are obtained, whereas if we classify them merely by their adjacency structures there are 11 types. If for each of these 11 types we select a representative with the highest possible symmetry for that type, and consider these representatives as nets (their one-skeleton or graph form) we obtain the 11 regular, Subnikov3 or Laves nets. Figure 1 presents the Laves nets and for completeness the (older) 'dual' or archimedean tilings, identifying them by symbols based on the valencies of the vertices of a given tile (for the Laves nets) or the orders of the polygons meeting at a given vertex (for the archimedean tilings). Although in this paper we consider only tilings where the edges are straight lines, most of the 81 isohedral types consist of tiles some of or all of whose edges can be curves.

Numerous authors (eg Klinger and Dye6, Hunter and Steiglitz6 and Rosenfeld and Samet8) have considered how amalgamations of the square tiling [4] may be used for storing and processing image data. The hexagonal tiling [3] has also received some similar attention7 and forms the basis of one of the systems which are commercially available10. Moreover, [3] is seen not only as an efficient...
The allocation of data to a tiling may be based on equal data content or equal areal cover as in the common methods of holding raster data. Rosenfeld has shown how, with an unlimited tiling (see below), it is possible to subdivide a heterogeneous plane until only homogeneous subplanes exist. He also showed that this pyramidal structure may be represented in a number of computationally isomorphic ways. It is now well established that this may be generalized to allow hashing, indexing and tree structures. Similar work on hierarchies based on other 'atomic tilings' is hampered by the complexity of what may be termed 'molecular tiles' formed by these strategies and the difficulties of generating and using the addressing mechanisms which result from such work.

Although much attention has been given to computationally efficient means of addressing raster data, usually with reference to the [44] tiling (e.g. by Woodward and Gargantini), little work appears to have been done to see if these methods are 'spatially correct' in the sense that they can address the entire plane. For example, addressing a square above and to the left of the 'Gargantini addressing space' can be difficult and involves either 'negative or recurring digit addresses', rotation of the addressing system or a complete respecification of the space when required (see the second last section below). These more 'cartographic concerns' have been better examined by Lucas who with the generalized balanced ternary (GBT) addressing structure, for the [36] tiling, provided an addressing mechanism that was both spatially correct and computationally efficient. GBT addressing is based on the observation that the regular hexagonal tiling may be superimposed on the complex plane such that the centroids of two adjacent hexagons occupy the positions 0 and 1. When this is the case the set of all centroids occupy the set of all complex numbers of the form

$$a + bi(-\frac{1}{2} + \frac{\sqrt{3}}{2}),$$

where $a$ and $b$ are integers. Further the sum or product of any two such complex numbers is another complex number of the same form, i.e.

$$a_1 + b_1\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) + a_2 + b_2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = a_1 + a_2 + (b_1 + b_2)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right),$$

$$\left[a_1 + b_1\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)\right]\left[a_2 + b_2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)\right] = a_1a_2 - b_1b_2 + (a_1b_2 + a_2b_1) - b_1b_2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right).$$

GBT arithmetic simply translates these operations into the symbolism of the addressing structure. We have observed that many similar tiling hierarchies may be generated and that if the atomic tiling is [44] or [36], with the centroids of two edge-adjacent tiles occupying the positions 0 and 1, then the set of all tile centroids will be closed under complex addition and multiplication. These opera-
tions, translated into the symbolism of the addressing structure, will (like GBT) obey rules analogous to the 'carry digit' rules of ordinary decimal arithmetic. We propose the terms tesseral addressing and arithmetic to refer to these spatially valid methods of addressing tiles in a hierarchy, and in the second-last section below we present examples of the generation of addressing structures and arithmetic tables for such tiling hierarchies.

PROPERTIES OF ATOMIC AND MOLECULAR TILINGS

To ascertain how satisfactory a tiling is for the image processing and databasing environments we conceive a taxonomy of more or less measurable criteria. These criteria can confer both advantages and disadvantages and in the sections which follow we attempt to name and define these criteria. (It is assumed that the tilings referred to may be either atomic or molecular but that the atomic tiling in each case is one of the 11 in Figure 1.)

Adjacency

Two tiles are considered to be neighbours if they are adjacent either along an edge or at a vertex. A tiling is uniformly adjacent if the distances between the centroid of one tile and the centroids of all its neighbours are the same. More generally the adjacency number of a tiling is the number of different intercentroid distances between any one tile and its neighbours. (If we consider tiles to be neighbours only if they are adjacent along an edge, then an analogous definition can be made.)

In image processing terms, the lower the adjacency number the easier it is to identify those tiles which constitute the same logical object.

Rotation

The rotation number of a tiling is the largest order of a rotational symmetry of the tiling (where a rotation of order \( n \) is a rotation by \( 360\degree/n \)).

It is an advantage to avoid floating point arithmetic and the larger the rotation number the more likely it is that this will be possible.

Aperture

The aperture of a level \( k \) tile is the number of level \( k-1 \) tiles needed to construct it.

The higher the aperture the easier it is to generalize the 'pattern types' which may be generated for any level in the hierarchy. Depending on application this may or may not be an advantage.

Circularity

The circularity of a tile is the difference in area between the smallest circumscribed and the largest inscribed circle which can be drawn, expressed as a fraction of the area of the tile.

Convexity

The convexity of a tile is the difference in area between the tile and its closed convex hull, expressed as a fraction of the area of the tile.

It is an advantage for the points furthest from a tile centroid not to be closer to another tile centroid. (Since all the atomic tiles considered in this paper are convex, the advantage is relevant only for the molecular tiles.)

Orientation

Tiles with the same orientation can be mapped onto each other by translations of the plane which involve no rotation or reflection. The tiling is said to have uniform orientation if all tiles have identical orientation. More generally the orientation number of a tiling is the number of distinct tile orientations which are possible.

If two tiles do not have the same orientation, integer addressing of the tiles will not be possible.

Limit

If a tiling hierarchy has tiles at level \( k+1 \) which are not 'similar' (see below) to those at level \( k \) then that hierarchy is said to be limited. Unlimited tilings do not have a definable atomic tiling because every atomic tile can be subdivided into smaller tiles which then become the atomic tiles of a new hierarchy.

If a tiling is limited no change of scale lower than the limit atomic tiling can be envisaged without difficulty.

Similarity

The tiles at levels \( k \) and 1 are 'similar' if they have identical shape, ie the tiles at level 1 are scaled images of those at level \( k \).

Regularity

A tiling is regular if the atomic tiles are composed of regular polygons.

Isohedrality

An isohedral hierarchy is one where all the molecular tilings are isohedral. Clearly all unlimited tilings are isohedral. Further, Holroyd has shown that if the first three levels of molecular tiling are isohedral then all subsequent levels will also be isohedral.

It is a disadvantage to have separate algorithms to deal with differing tile types as would occur if the hierarchy were not isohedral.
Democracy

Democracy among level $k$ tiles exists when in the level $k + 1$ tile it is impossible to differentiate between the level $k$ tiles.

This is a property which may or may not be an advantage depending on the application. Thus the polygonal coherence afforded to 'pattern types' obtained by having a clear central tile is an advantage of undemocratic molecular tiles (cf. aperture); however, it is an advantage to be able to treat all democratic tiles with the same algorithms. Finally, it is better to avoid the complicated addressing structures which democratic tiles often produce.

UNLIMITED TILING HIERARCHIES

In a tiling hierarchy, each edge of a molecular tile is formed from several atomic tile edges. It follows that an atomic tiling is capable of generating unlimited hierarchies only if each edge of each tile lies on an infinite straight line composed entirely of edges. Thus only $[4^4], [4.6.12], [4.8^2]$ and $[6^3]$ are capable of generating unlimited hierarchies.

Assuming that the relationship between levels $k$ and $k + m$ (for $m$ any positive integer) is independent of $k$ with regard to rotation and/or reflection of tile shapes, hierarchies of the types indicated in Figure 2 are the only unlimited hierarchies which can be constructed. We note that in all four cases there exists a hierarchy which does not require a reflection or rotation on changing levels (Figures 2a, 2b, 2c and 2d). However, two tilings ($[4.6.12]$ and $[4.8^2]$) can generate additional hierarchies if these operations are permitted. In the case of $[4.6.12]$ the operation required is a reflection of the basic tile between levels (Figure 2d2), whereas $[4.8^2]$ requires a rotation of 135° between levels (Figure 2c2). (However, it is important to note that the ability to generate a hierarchy does not imply that the tiling so produced will cover the entire plane.) Where there is no reflection or rotation between levels the number $s$ of atomic tiles in the first molecular tile is $s = n^2 (n > 1)$. In the reflection hierarchy of $[4.6.12]$ the number of atomic tiles in the first molecular tile is $s = 3n^2 (n > 1)$ and in the rotation hierarchy of $[4.8^2]$ $s = 2n^2 (n > 1)$. In all cases the number $t$ of tiles at any level $k$ is $t = s^k$ which implies a constant aperture.

The major property which these four tilings do not exhibit is that of uniform adjacency. In the case of $[4^4]$ there are two adjacency distances, $[4.6.12]$ has 16 adjacency distances and $[4.8^2]$ and $[6^3]$ have eight and three adjacency distances respectively. In the tilings $[4^4]$ and $[6^3]$ display regularity; however, only $[4^4]$ displays uniform orientation. Therefore, ordering these tilings in terms of their usefulness we get the series $[4^4], [6^3], [4.8^2]$ and finally $[4.6.12]$.

LIMITED TILING HIERARCHIES BASED ON HEXAGONAL TILES

Only the hexagonal tiling $[3^6]$ displays the uniform adjacency property and therefore despite being limited is worthy of special attention. However, the hexagonal tiling does give rise to a rich and intriguing collection of hierarchies which do not appear to have been exploited (eg by games programmers) although in the limit the perimeter of such a molecular tile is a fractal curve. To distinguish these tiling hierarchies we classify the shape of the first-level molecular tile on the basis of the number of hexagons it contains and then identify the rotation required to ascend the hierarchy (there are no reflection hierarchies because of the regularity of the atomic tiling).

Figure 3 presents six of these hierarchies grouped according to their shape classification. Four of them are based on first-level molecular tilings composed of four atomic tiles in two shape classifications: a 'regular' 4-shape and a 4°-shape. The two hierarchies based on the 4-shape (Figures 3a1 and 3a2) are of interest and are discussed below. In contrast the two hierarchies based on the 4°-shape (Figures 3a1 and 3a2), although pretty, are of little interest because of the complex tile shapes they generate at higher levels. Figures 3b and 3c contain the 7-hierarchy and zero-rotation 9-hierarchy, which because of their large aperture are drawn at a different scale.

It can be seen that the hierarchies based on the 4- and 9-shapes may be treated similarly. However, unlike the 7-shape these tiles are not symmetric under 60° and 120°...
rotation although they are symmetric under 180° rotation (thus 60° rotation one way has the same effect as 120° rotation the other way). This feature means that there are three ways to build up a hierarchy based on either of these tile shapes:

- no rotation between levels
- 60° anticlockwise rotation (or 120° clockwise rotation) between levels
- 120° anticlockwise rotation (or 60° clockwise rotation) between levels

Figure 3. Tiling hierarchies based on different shape configurations of the first level tile, with in a4 a 120° rotation between levels.
The second and third of these are mirror images of each other and so only the first two need be considered (the 4-hierarchy is Figure 3a, the 4(120)-hierarchy is Figure 3b, and the 9-hierarchy is Figure 3c). The similarity of the molecular 'lozenge-shaped' tiles of the 4- and 9-hierarchy remain fairly constant on ascending the hierarchy. However, the degree of adjacency is peculiar in that, no matter how large the lozenge tile becomes, two such tiles (one above the other) are in contact along only a single edge of a hexagonal atomic tile, while contact in the other two directions is along nearly a quarter of the perimeter of the corresponding molecular tiles and treats the hexagonal pattern as though it were a 'skewed square' pattern. In contrast the 4(120)-hierarchy (Figure 3b) and 9(120)-hierarchy (not illustrated) generate tiles which are more awkwardly shaped but which exhibit better adjacency properties (eg the 4(120)-hierarchy). The 7- and 9-hierarchies suffer in that there is a clear 'inside' and 'periphery' to the first-level molecular tile which results in poor democracy although as outlined earlier this need not be a disadvantage. The circularity and convexity of the 4-hierarchy and 9-hierarchy molecular tiles is better than that for the 7-hierarchy, which in turn is better than that for the 4'-hierarchies. Furthermore, as the 7-hierarchy is ascended, the distance between points on the boundary of a tile and a regular hexagon fitted as closely as possible to the tile edge becomes larger and larger. However, it must be noted that the essential tile shape resembles the hexagon more closely than the lozenge does and therefore exhibits better similarity. Finally, we observe that the rotational hierarchies in general exhibit worse convexity than the unrotated forms.

To place these hierarchies in an order of preference is difficult because of the need to compare properties. However, in general it is possible to dismiss the 4'-hierarchies and to place the 7-hierarchy slightly ahead of the zero-rotation 4- and 9-hierarchies.

**TESSERAL ADDRESSING AND ARITHMETIC**

To simplify further discussion we present in 'colouring book style' the generation of a 'spatially correct' or tesseral addressing system and arithmetic for a molecular 9-shape tile based on the [44] tiling.

- The first step is to examine the atomic tiles of the molecular shape to ascertain if they are democratic. If they are, any tile can be labelled 0. In our example (Figure 4a) the tiling is undemocratic and there is a clear middle tile, which is labelled 0 accordingly.
- Label the remaining tiles from 1 to n, where each label appears only once as in Figure 4b and where label 1 must be on a tile which is the minimum distance from the tile labelled 0. If the centroids of these tiles are joined to the centroid of the tile labelled 0 a series of m vectors is generated. In our example these are 01, 02, 03, 04, 05, 06, 07 and 08.
- Generation of the hierarchy is achieved by nominating the molecular tile generated above, with a label from the series 0, 1, . . . , m, . . . , n. Thus the atomic tiles of the mth molecular tile will have new labels of the form m0, m1, . . . mn. This is illustrated in Figure 5. It may be seen that this example uses a hierarchy which is unrotated. If we had decided to use some rotation, then it would have been necessary to maintain the rotation when ascending every level of the hierarchy.
- Generation of the addition arithmetic begins with the assumption that addition to 0 maps into the number added, ie

\[ 0 + 5 \rightarrow 5 \]

It may be seen that this corresponds to the vector 05 or a move down one square from tile 0 to tile 5. By analogy the addition of 5 + 2 results in moving down one square from tile 2, ie to tile 3 as in Figure 6. Thus

\[ 2 + 5 \rightarrow 3 \]
\[ 3 + 5 \rightarrow 4 \]

Similarly, addition of 2 to 0 results in a move in a direction identical to the vector 02 and thus

\[ 5 + 2 \rightarrow 3 \]

and

\[ 6 + 2 \rightarrow 0 \]
This procedure is repeated until all the positions in the addition table are taken as in Figure 7b.

- To generate the multiplication table begin as before by assuming that multiplication of a number by 1 maps into the number multiplied, eg

1 * 3 → 3

In the tiling example shown in Figure 8, multiplication by 3 results in a clockwise rotation of the vector 01 by 45° resulting in the vector 03. Thus by analogy

5 * 3 → 7

Similarly it may be seen that

1 * 2 → 2

This is a rotation of 45° and a scaling of the vector by 2² (Figure 8). Thus

6 * 2 → 73

(a rotation of 45° and generation of a vector length of 2).

The other entries in the multiplication table may be filled in a similar manner (Figure 7c). It should be noted that multiplication always results in a rotation by a fixed angle and scaling of the vector length.

- The inverse vector may be generated by moving one square to the right, eg

0 + 3 → 3

The inverse operation, moving one square to the left, gives

0 + 7 → 7

and therefore the inverse of 7 is 3. We can think of 7 as -3 (Figure 7d).

Each of the topological types, with the exception of [3.12²], admits tiling hierarchies. For any tiling hierarchy a tesselar addressing structure can be formed by following the first three steps above. If the centroids of the atomic tiles form a subring of the complex numbers (i.e., a set closed under addition and complex multiplication), then the fourth, fifth and sixth steps can also be followed and a tesselar arithmetic constructed; this is true if the atomic tiling is [4] or [3] and the centroids of two edge-adjacent atomic tiles are located at 0 and 1 respectively in the complex plane. However, it is important to note that the arithmetic tables based on one labeling strategy may be different from those based on another; this is illustrated in Figure 9. (The difference is only in the labels, of course; the arithmetic itself is always that of complex addition and multiplication.) Extension of the addressing structure and arithmetic to every point in the plane will be discussed in a subsequent paper.

Addition and multiplication tables exist for the Gargantini addressing system. In Figure 10, the Gargantini addressing space is in the lower right-hand quadrant and an address here can be constructed in the usual way. Thus the address of tile Q0 is 332; however, in reality its address should be 0 recurring 332 where the zeros are ‘understood’. By analogy tile Q1 has the address 332 preceded by 1 recurring. The 1 recurring arises because tile x is left of tile 0 and our arithmetic rules require that

x + 1 → 0

The only possible x which qualifies is 1 recurring, giving the result

1

...11111 1+

...00000 0

In a similar manner it may be seen that the arithmetic table requires all addresses in the quadrant above tile 0 to
be preceded by 2 recurring and all addresses above and to the left of tile 0 to be preceded by 3 recurring

...22222 2+

...22222 3

This results in the generation and rationale of the addition and multiplication tables given in Figures 11b and 11c, although here the zero quadrant is the upper right-hand one (Figure 11a). The mechanism is of considerable interest because of the parallels which may be drawn between 1 recurring and a negative number stored in twos complement form in computers. The extension to handle 2 and 3 recurring depends on the width of the storage field and the number of bits required to store the number which is recurring — 2 bits in this case.

Several points of interest follow from this discovery. The first is that the hierarchy has affinities to the 180° rotation hierarchy (Figure 9b) because of the similar addressing in the leftmost quadrant. (It should be noted that this 180° rotation also generates a spatially correct addressing mechanism as do any similar rotational addressing mechanisms.) Secondly, all the properties described by Gargantini remain true for all the quadrants. Finally, the structure is intuitive (if the diagram and the explanation above are used), although it would be better to have the upper quadrant populated as the zero quadrant as illustrated in Figure 11a.

It is our belief that tesseral addressing and arithmetic as outlined above will have a wide application especially when the storage and processing of recurring digits is solved in a generalized way, preferably in hardware.

**SUMMARY AND GENERAL DISCUSSION**

We have proposed a series of cartographically important criteria for assessing the usefulness of the 11 regular tilings and have concluded that no tiling satisfies all the criteria. The tiling [4^\*] displays most of the desirable properties but fails to exhibit uniform adjacency, in that there are two intercentroid distances to contend with, one through any vertex and the other through any side. In contrast the tiling [3^\*] displays uniform adjacency at the atomic tile level but has unfavourable hierarchical properties and more importantly is limited. It is our impression that the two unexplored tilings [6^\*] and [3.6.3.6] may yet prove to be valuable.

We have found that many limited tiling hierarchies can be generated. However, our tentative conclusion is that those based on the tiling [3^\*] are potentially the most useful when examined in the light of their cartographic properties. We observe that molecular tiles composed of hexagonal atomic tiles may be generated in many ways and some have interesting properties. For example, in the...
limit, certain molecular tile edges of the 4-shape approach zero length when compared with others and these small edges may then exhibit the properties of a vertex. We have also noted that the 4(0)-hierarchy could be treated like a skewed square and could in consequence inherit many of the advantageous properties of the [4\textsuperscript{4}] tiling, including being amenable to quadtree storage and manipulation techniques, while retaining the power of spatially correct or tesseral addressing and arithmetic. In addition, if we adopt the Gargantini method of holding a quadtree as a sorted set of addresses it is clear that her algorithms for cartesian coordinate conversion can also be used. Similarly the 9(0)-hierarchy can be mapped into a skewed square shape which also has a spatially valid addressing structure and arithmetic (Figure 7) and in addition has good aperture properties.

Examination of the tiling hierarchies based on unlimited atomic tilings suggests that it is difficult to improve on [4\textsuperscript{4}]. A tesseral addressing mechanism for this may be based either on the modification of the Gargantini structure using 'recurring digit' addressing as shown in Figure 11 or on a rotating addressing structure. A computationally efficient arithmetic may be conceived for either structure.

Although it is difficult to chart a course for future research and implementation it seems likely that the advantages of quadtree mechanisms must be used, while leaving open the possibility of using the hexagonal tiling for geometrical operations. We believe that the use of recurring digit addressing structures, or of a 180\degree rotation hierarchy with Gargantini addressing which is overlaid on a hexagonal pixellation, offers the most flexible future especially if implemented in hardware and especially if extended to all points in the plane although we accept that this may need fractional addresses if it is to remain truly tesseral. We intend to prototype this approach in software using Landsat 4 data while simultaneously investigating the parallelism afforded by using array processing hardware.

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