On computing shortest external watchman routes for convex polygons

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Abstract

We study the relationship between the interior angles of a convex polygon and the lengths of external watchman routes.

1 Introduction

The *external watchman route* problem is to externally inspect a polygon by finding the shortest closed route from which each point on the boundary of the polygon is visible from at least one point on the route. By a closed route, we mean that the watchman returns to the starting point, so that the route can be repeated indefinitely.

In [5] Ntafos and Gewali give a linear-time algorithm for the external watchman route problem for *convex* polygons. In that case they show that a shortest external watchman route W for a convex polygon P has one of two types: i) W follows the boundary of P to make a *convex hull route*, or ii) W is a *two-leg route* consisting of three parts W_m, which follows the boundary of P, and two legs W_1 and W_2, which join W_m perpendicularly to the extensions of two adjacent edges of P. See Fig 1.

The algorithm of [5] computes the length of the 2-leg route at each vertex and compares it with the length of the convex hull route. Here, motivated primarily by curiosity, we ask 1) whether the best 2-leg route occurs at the smallest internal angle (as is true for triangles), and 2) whether this is so when the convex hull route is not optimal. Our main result is a no answer to both conjectures 1 and 2, and a yes answer to a special case.

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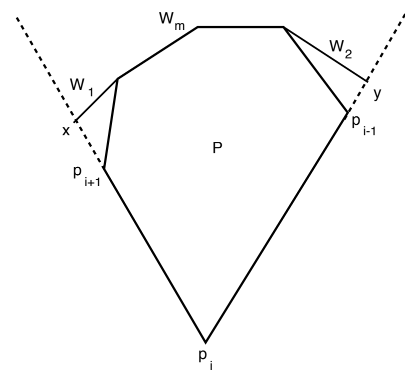


Figure 1: Shortest two-leg route for polygon P

2 Related Work

Guarding problems, especially for the interior of polygons, are often called “art gallery” problems and have an extensive literature [6]. The external guarding problem using mobile guards was mainly addressed in [5], where linear-time algorithms for finding the shortest external watchman route without a specified starting point were provided for convex, star-shaped, monotone and spiral polygons. The algorithm given for simple polygons has the same computational complexity as that for the internal watchman route problem, which is proved in [5] by converting the external problem to a set of internal problems. For the shortest watchman route on simple polygons with n vertices, where a starting point is specified, the algorithm with the lowest running time is $O(n^3 logn)$ [2].

Gewali and Stojmenovic [3] studied the computation of shortest external watchman routes using parallel algorithms. In [4] the same authors studied the problem of finding the shortest external watchman route for a pair of convex polygons, having a total of n vertices, and they gave an $O(n^2)$ time solution.

The problem of external watchman routes on convex and simple polygons is revisited in more detail in the thesis of Absar [1].

3 Routes on convex quadrilaterals

Clearly conjectures 1 and 2 are true for triangles. Next we consider convex quadrilaterals, which are not only the natural next case, but also the shape of many conventional structures.

Theorem 1 *Conjectures 1 and 2 hold for parallelograms.*

Proof. Consider a parallelogram with two acute interior angles. As can be seen in Figure 2, the shortest 2-leg routes W_a and W_c for the wedges with angles greater than 90° have path length equal to $(ad + ab)$ (or $(bc + cd)$). The routes W_d and W_b for the wedges with acute angles have path length equal to $(dx + dy)$, where dx is perpendicular to ab and dy is perpendicular to bc. It can be seen that $(dx + dy) < (ad + ab)$, since $dx < ad$ (ad is the hypotenuse of $\triangle adx$) and $dy < ab$ ($ab = dc$ from the rectangle, and dc is the hypotenuse of $\triangle cdy$). Thus in the case of parallelograms, smaller angle wedges provide shorter routes. $\square$

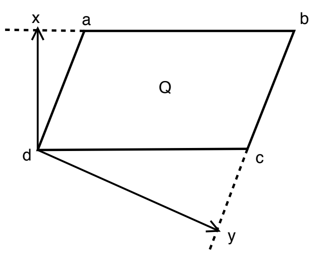
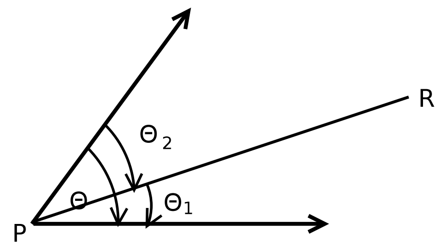


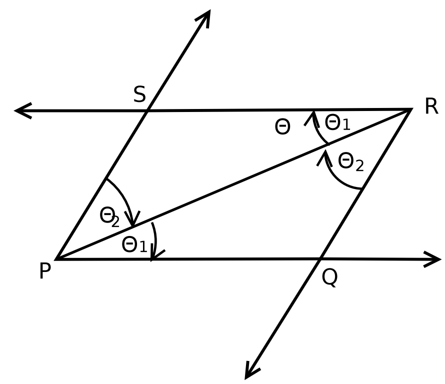
Figure 2: A 2-leg external watchman route xdy for a parallelogram Q

Next we give a method for constructing counterexamples to 1) and 2) by looking at convex quadrilaterals with two equal acute angles placed at opposite vertices.

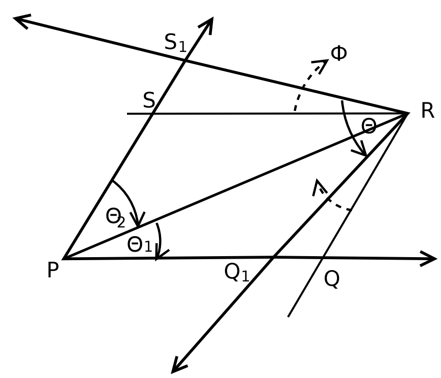
We take a wedge located at a point P and fix the interior angle at θ. We place a point R somewhere within this wedge, so that the line PR divides the angle θ into θ_1 and θ_2 as shown in Figure 3(a). Now we place a wedge of the same angle θ on point R. If the wedge is placed in such away that $\angle SRP = \angle RPQ = \theta_1$ and $\angle QRP = \angle SPR = \theta_2$, we get a parallelogram (shown in Figure 3(b)). However if we keep the wedge at P fixed, together with angles θ_1 and θ_2, by keeping point R fixed, and we only rotate the wedge R about line PR, we can create several convex quadrilaterals with acute angles θ opposite each other. The wedge R can be rotated anti-clockwise at angles within $0 \leq \phi < \theta_2$ and can be rotated clockwise at angles within $-\theta_1 < \phi \leq 0$, where ϕ is the amount by which the wedge is rotated, as can be seen in Figure 3(c).



(a) Wedge P



(b) Wedge P and R



(c) Rotating wedge R about PR

Figure 3: Generating convex quadrilaterals with two equal opposite acute angles

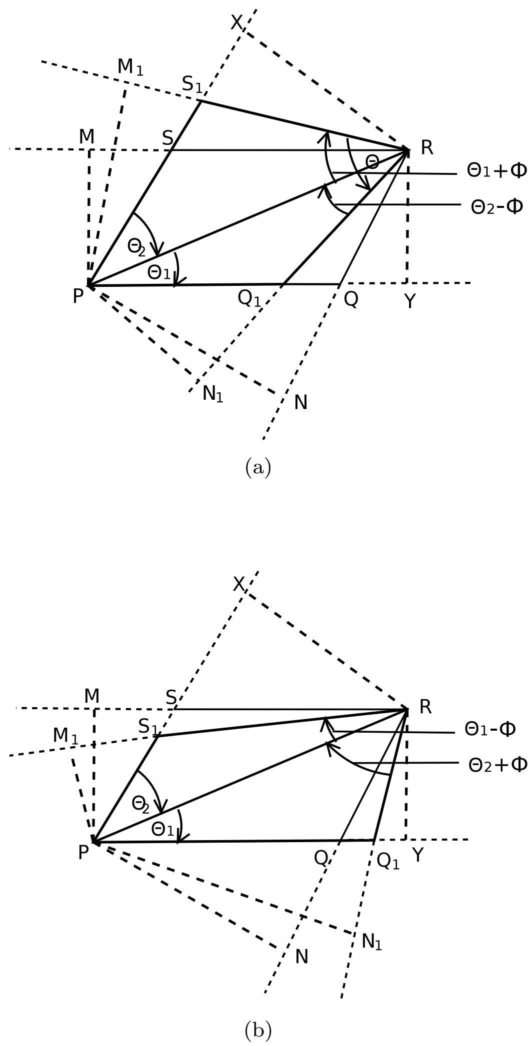


Figure 4: Illustration of rotating wedge R by angle ϕ

Figure 4(a) shows an example of the transformation of the quadrilateral $PQRS$ to the new quadrilateral PQ_1RS_1 when the wedge R is rotated clockwise by an angle ϕ. The corresponding 2-leg route for each wedge is also illustrated. The 2-leg route W_R for wedge R of quadrilateral $PQRS$ is $(PM+PN)$, and W_R for quadrilateral PQ_1RS_1 is $(PM_1 + PN_1)$. The 2-leg route W_P for wedge P remains $(RX+RY)$ even through rotation, as can be seen from the figure. Figure 4(b) similarly shows the formation of the quadrilateral PQ_1RS_1 by rotating the wedge R anti-clockwise by angle ϕ, and its corresponding 2-leg route.

By straightforward analysis, the difference $f(\phi)$ between the path lengths of wedges P and R is:

$f(\phi) = x \cos\phi + y \sin\phi - x$

where $x = (\sin\theta_1 + \sin\theta_2)$ and $y = (\cos\theta_1 - \cos\theta_2)$. There should be a factor PR in the definition of $f(\phi)$ but we leave it out for simplification, since it does not affect the result.

The value of ϕ is restricted between $-\theta_1 < \phi < \theta_2$ so that the sides do not pass the diagonals during rotation. Also, if we do not want the sides to go beyond the perpendicular legs RX and RY of wedge P, the value of ϕ has to also be restricted within $-90 + (\theta_1 + \theta_2) < \phi < 90 - (\theta_1 + \theta_2)$.

Using different data sets, the graph of this function was plotted to see the form of the function within the restricted values of ϕ. A generalized form of the function $f(\phi)$ is shown in Fig 5. As can be seen, the function in this form gives a relative maximum. This can be easily proved using the second derivative test.

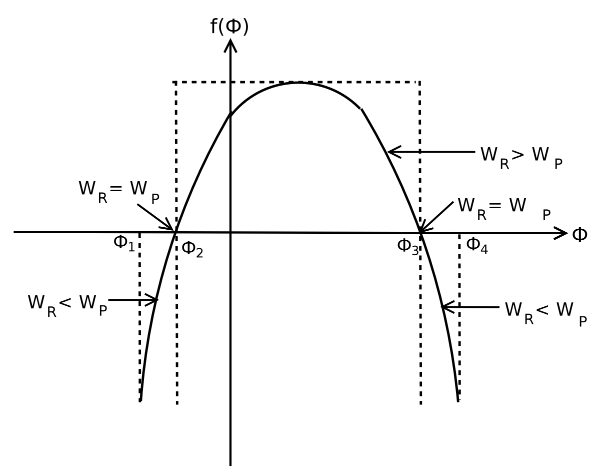


Figure 5: Form of the function $f(\phi)$

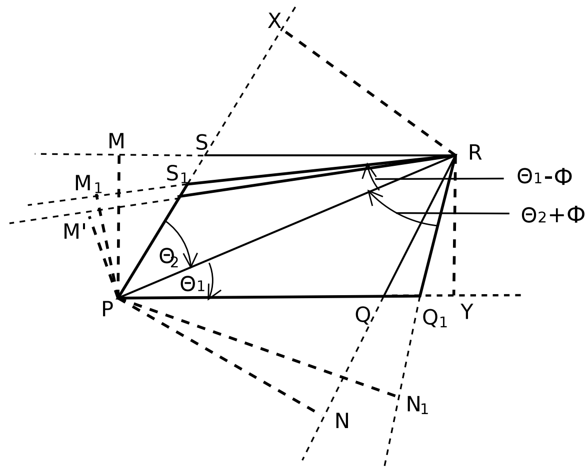


Figure 6: Counterexample to Conjecture 1

As Fig 5 shows, when ϕ lies between ϕ_2 and ϕ_3, then the two-leg route for wedge P is shorter than that for wedge R. Otherwise, when ϕ lies between ϕ_1 and ϕ_2 in Figure 5, or between ϕ_3 and ϕ_4, then the two-leg route for wedge R is shorter than that for wedge P. Exactly when ϕ is equal to ϕ_2 or ϕ_3 are the two-leg route lengths for wedges P and R equal.

Now looking at Figure 5 we can see that at certain points, such as the relative maximum point of $f(\phi)$, the difference between the route lengths is locally maximum. We used this information to create a counterexample for conjecture 1.

Taking a data set where the wedges are of angle $\theta = 60°$, with $\theta_1 = 45°$ and $\theta_2 = 15°$, we found that the relative maximum occurs at $\phi = -15°$. Thus at $\phi = -15°$, the quadrilateral $PQRS$ is such that routes for wedges P and R are significantly unequal even though $\angle P = \angle R$. Now we reduce the angle of wedge R by a small amount, e.g. by reducing $(\theta_1 - \phi)$ by 2°, so that the angle at wedge R is now 58° instead of 60°. The route for wedge P remains the same length, but the route for wedge R consequently reduces in length.

For this particular example, shown in Figure 6, the values found for a quadrilateral with a diagonal $PR =$ 8.4 cm are as follows. When $\angle P = \angle R$, $W_P = RX + RY = 8.114$ and $W_R = PM_1 + PN_1 = 8.4$. However, when $\angle P$ is 2 degrees greater than $\angle R$, $W_P = RX + RY = 8.114$ and $W_R = PM' + PN_1 = 8.144$. Thus wedge P, whose angle is larger than that of wedge R gives the shorter route.

Hence the conjectures are false. (For conjecture 2, note that the convex-hull route is clearly longer than the two 2-leg routes.)

4 Conclusion and an open problem

We investigated other conjectures and found counterexamples for them as well. However, the conjecture below remains open. Neither theory nor our experimental work (see [1]) has produced a counterexample.

Conjecture: All convex obtuse polygons (i.e. convex and with no acute interior angles) have convex-hull routes as their shortest external watchman routes.

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