Problems

1. Illustrate DeMorgan’s Law \((A \cap B)^c = A^c \cup B^c\) using Venn diagrams.

![Venn Diagrams](image)

Figure 1: \((A \cap B)\) is shown in (a), and (c) and (d) illustrate \(B^c\) and \(A^c\) respectively. Finally (b) shows that \((A \cap B)^c = A^c \cup B^c\).

2. Let \(A_i = \{1, 2, 3, \ldots, i\}\) for all \(i \in \mathbb{N}\). For example \(A_4 = \{1, 2, 3, 4\}\).

   (a) What are the elements of the set \(\bigcup_{i=1}^{n} A_i\) ?

   \[
   \bigcup_{i=1}^{n} A_i = A_n
   \]

   (b) What are the elements of the set \(\bigcap_{i=1}^{n} A_i\) ?

   \[
   \bigcap_{i=1}^{n} A_i = A_1
   \]
3. Observe that $A \subseteq B$ has the same meaning as $A \cap B = A$. Draw a Venn diagram to illustrate this fact.

See Figure 2. If $A \subseteq B$ then every element $x \in A$ is also and element in $B$, which in turn implies that $A \cap B = A$.

4. Use a Venn diagram to show that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

See Figure 2. $A \subseteq B$ implies that every element of $A$ is also in $B$, $x \in A$ implies $x \in B$. Similarly $B \subseteq C$ implies that every element of $B$ is also in $C$, $y \in B$ implies $y \in C$. Thus $A \subseteq C$.

5. Use the Principle of Exclusion and Inclusion to show that $|A \cup B| + |A \cap B| = |A| + |B|$. (It may help your understanding if you first explore an example such as $A = \{1,2,3\}$ and $B = \{3,4\}$).

By the Principle of Inclusion Exclusion we have $|A| + |B| - |A \cap B| = |A \cup B|$. These quantities are just non-negative integers so if we add $|A \cap B|$ to the right and left side of the equation, we get the desired result.

6. What are the cardinalities of the following sets?

   (a) $A = \{\text{winter, spring, summer, fall}\}$. $|A| = 4$.
   
   (b) $B = \{x : x \in \mathbb{Z}, 0 < x < 7\}$. $|B| = 6$.
   
   (c) $P(B)$, that is, the power set of $B$. $|P(B)| = 2^6 = 64$.
   
   (d) $C = \{x : x \in \mathbb{N}, x \text{ is even}\}$. This set has infinitely many elements.

7. Suppose that we have a sample of 100 students at Queen’s who take at least one of the following language courses, French-101, Spanish-101, German-101. Also suppose that 65 take French-101, 45 take German-101, 42 take Spanish-101, 20 take French-101 and German-101, 25 take French-101 and Spanish-101, and 15 take German-101 and Spanish-101.

   (a) How many students take all three language courses?
Let F, S, and G denote the sets of students taking French Spanish and German respectively. The Principle of Inclusion and Exclusion tells us that
\[ |F \cup S \cup G| = |F| + |S| + |G| - |F \cap S| - |S \cap G| - |F \cap G| + |F \cap S \cap G| \]
The problem statement gives us values for each quantity in the equation except for \( |F \cap S \cap G| \). We can now simply fill in the numbers and solve for \( |F \cap S \cap G| \), as follows:

100 = 65 + 42 + 45 - 25 - 15 - 20 + |F \cap S \cap G|

So we conclude that \( |F \cap S \cap G| = 8 \).

(b) Draw a Venn diagram representing these 100 students and fill in the regions with the correct number.

(c) How many students take exactly 1 of these courses? Using the Venn diagram we can deduce that 28 + 10 + 18 = 56 students take exactly one of the language courses.

(d) How many students take exactly 2 of these courses? Using the Venn diagram we can deduce that 17 + 12 + 7 = 36 students take exactly two courses.

8. At an art class with 30 students, there are 14 women, and 16 men. Twenty-two of the students are right-handed. What is the minimum and maximum number of women that are right-handed?
With all of the men are right handed there must be 6 right handed women as the minimum. The maximum is when all of the women are right handed, that is 14.

9. Recall that the union operation is associative, that is $A \cup (B \cup C) = (A \cup B) \cup C$. Show that the relative complement set operation is not associative, that is, $A \setminus (B \setminus C) = (A \setminus B) \setminus C$, is incorrect for some sets $A$, $B$, $C$. (Note if relative complement is associative then the equation must be true for all sets $A$, $B$, $C$.)

Let $A = \{1, 2, 3\}$, $B = \{1, 2\}$ and $C = \{2, 3\}$. $A \setminus (B \setminus C) = \{2, 3\}$ and $(A \setminus B) \setminus C \neq \emptyset$.

10. Consider a set $S$ of $n$ elements, such that $\{a, b\} \subseteq S$.

(a) What is the cardinality of the power set of $S \setminus \{a\}$?

We know that $S$ has $n$ elements so $S \setminus \{a\}$ has $n - 1$ elements. The power set of $S \setminus \{a\}$ has $2^{n-1}$ elements.

(b) What is the cardinality of the power set of $S \setminus \{a, b\}$?

The power set of $S \setminus \{a, b\}$ has $2^{n-2}$ elements.

(c) How many subsets of $S$ are there that contain the element $a$?

Here is a way to construct the subsets of $S$ that contain the element $a$. For each subset $s \in P(S \setminus \{a\})$ construct the set $\{a\} \cup s$. This yields all subsets of $S$ that contain $a$. Since there are $2^{n-1}$ subsets of $S \setminus \{a\}$, there are $2^{n-1}$ subsets of $S$ that contain the element $a$.

We can obtain the same result by using a different argument. We know that there are $2^n$ subsets of $S$. The subsets of $S \setminus \{a\}$ are also subsets of $S$ that do not contain $a$. The total number of subsets of $S \setminus \{a\}$ is $2^{n-1}$. So the number of subsets of $S$ that contain $a$ is equal to $2^n - 2^{n-1} = 2^{n-1}$.

(d) How many subsets of $S$ are there that contain the element $a$ and exclude the element $b$?

Here is a way to construct the subsets of $S$ that contain $a$ and exclude $b$. For each subset $s \in P(S \setminus \{a, b\})$ construct the set $\{a\} \cup s$. This yields $2^{n-2}$ subsets of $S$. 