1. **Solution.** (i) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$: Suppose $x \in \overline{A \cap B}$. This implies $x \notin (A \cap B)$, that is, $x \notin A$ or $x \notin B$. The latter is the same as $x \in \overline{A}$ or $x \in \overline{B}$, that is $x \in \overline{A} \cup \overline{B}$.

   (ii) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$: Suppose $x \in \overline{A} \cup \overline{B}$. This implies $x \notin A$ or $x \notin B$. Since x cannot be an element of both A and B, this means $x \notin A \cap B$, that is, $x \in \overline{A \cap B}$.

2. **Solution.**

   (a) First note that both $R \odot (S_1 \cap S_2)$ and $(R \odot S_1) \cap (R \odot S_2)$ are relations from set A to set C.

   The inclusion is proved by the following chain of implications:
   $(a, c) \in R \odot (S_1 \cap S_2)$ implies
   $(\exists b \in B) : (a, b) \in R$ and $(b, c) \in S_1 \cap S_2$ implies
   $(\exists b \in B) : (a, b) \in R$ and $(b, c) \in S_1$ and $(b, c) \in S_2$ implies
   $(\exists b \in B)([(a, b) \in R$ and $(b, c) \in S_1]$ and $[(a, b) \in R$ and $(b, c) \in S_2])$ implies
   $(\exists b \in B)[(a, b) \in R$ and $(b, c) \in S_1]$ and $(\exists b \in B)[(a, b) \in R$ and $(b, c) \in S_2]$ implies
   $(a, c) \in R \odot S_1$ and $(a, c) \in R \odot S_2$ implies
   $(a, c) \in (R \odot S_1) \cap (R \odot S_2)$.

   (b) Below is one possible example – there exist infinitely many others.
   Let $A = \{a\}$, $B = \{b_1, b_2\}$, $C = \{c\}$ and choose

   $$R = \{(a, b_1), (a, b_2)\}, \quad S_1 = \{(b_1, c)\}, \quad S_2 = \{(b_2, c)\}.$$

   Now $R \odot (S_1 \cap S_2) = \emptyset$ because $S_1 \cap S_2 = \emptyset$. On the other hand, $(a, c) \in (R \odot S_1) \cap (R \odot S_2)$.

3. **Solution.** (a) TRUE, (b) FALSE, (c) TRUE
   (d) TRUE, (e) TRUE, (f) FALSE

4. **Solution.**

   (a) The total number of strings of length 4 is 10^4. There are 10 strings that have 4 identical digits. The number of strings of length 4 that do not have 4 identical digits is $10000 - 10 = 9990$.

   (b) When choosing the first digit we have 10 choices. For a fixed first digit, we have 9 choices for the second digit (to avoid repetition). Similarly, for the 3rd digit we have 8 choices and for the 4th digit 7 choices. The total number of strings that do no repeat a digit is $10 \cdot 9 \cdot 8 \cdot 7 = 5040$.

(c) There are 5 even digits that can be chosen as the last digit. Each of the first 3 digits has 10 choices and using the product rule the total number of choices is $10 \cdot 10 \cdot 10 \cdot 5 = 5000$.

(d) There are 4 ways to choose the positions of the digits 8 (1, 2, 3 or 1, 2, 4, or 1, 3, 4, or 2, 3, 4). The remaining digit can be chosen in 9 ways. Using the product rule the total number of choices if $4 \cdot 9 = 36$.

5. **Solution.** We partition elements of A into $|B|$ subsets as follows:

   two elements a_1 and a_2 belong to the same subset iff $f(a_1) = f(a_2)$.

   The pigeonhole principle (Theorem 1.2 in the second set of notes,) guarantees that some subset has $\lceil \frac{|A|}{|B|} \rceil$ elements.

6. **Solution.** (i) Suppose that f has a right inverse $g : B \to A$. For any $b \in B$ we have

$$b = 1_B(b) = (f \circ g)(b).$$

   Since $(f \circ g)(b) = f(g(b))$, this means that b is the f image of $g(b)$ and f is onto.

   (ii) Conversely suppose that f is onto. For each $b \in B$, choose $a_b \in A$ such that $f(a_b) = b$. Define a function $g : B \to A$ by setting $g(b) = a_b$ for all $b \in B$.

   We verify that g is the right inverse of f. For any $b \in B$ we have

$$(f \circ g)(b) = f(a_b) = b = 1_B(b),$$

   that is, $f \circ g = 1_B$.