Queen's University

CISC-203 Practice Midterm 2019

INSTRUCTIONS

- You have 40 minutes. Attempt all four questions.
- You may bring in one 8.5×11 sheet of paper containing notes, and use it during the midterm.
- **Answer each question in the space provided** (on the question paper). There is an extra page at the end of the exam if more space is needed. **Please write legibly.**
- *Note:* In questions dealing with counting, combinatorics or probability it is not expected that you should compute large numerical values: it is fine to give the final answer in a form like "$\frac{56!}{29!}$" or "$\binom{40}{21}$" as long as you clearly explain how you arrived at the answer.
- **Please note:** You are asked to write your answers using a *non-erasable pen.* Answers written in pencil or erasable ink will be marked, but they will not be considered for remarking after the midterms are returned.

STUDENT NUMBER: One digit in each square, please!

Student number (written in words):

MARKS

Problem 1	/X
Problem 2	/Y
Problem 3	/Z
Problem 4	/W
Total	/X + Y + Z + W

Code: askjkhkhdhfkjhqwewe1yu2tuytuyue1tduayt4duytu12yqeqwe

1. Consider relations $R \subseteq A \times B$ and $S \subseteq B \times C$. Prove that

$$(R \odot S)^{-1} = S^{-1} \odot R^{-1}.$$

Here $R \odot S$ denotes the composition of relations R and S.

Solution. First we note that both $(R \odot S)^{-1}$ and $S^{-1} \odot R^{-1}$ are relations from set C to set A (that is, subsets of $C \times A$).

Consider any elements $c \in C$ and $a \in A$. Now

$$(c, a) \in (R \odot S)^{-1} \text{ iff } (a, c) \in (R \odot S) \text{ iff } (\exists b \in B) : (a, b) \in R \text{ and } (b, c) \in S$$
$$\text{iff } (\exists b \in B) : (b, a) \in R^{-1} \text{ and } (c, b) \in S^{-1} \text{ iff } (c, a) \in S^{-1} \odot R^{-1}.$$

2. A three-digit natural number is a number of the form $d_1d_2d_3$, where d_1, d_2, and d_3 are elements from the set $\{0, 1, \ldots, 9\}$ and $d_1 \neq 0$. In other words, a three-digit natural number is any number between 100 and 999.

   (i) How many three-digit natural numbers can be formed if each digit must be distinct?

   Solution. For the first digit d_1 we have 9 choices (any of digits 1, 2, ..., 9). When the first digit is fixed, there are 9 choices for the 2nd digit (any digit that is not d_1). For the third digit there is then 8 choices (any digit that is not d_1 or d_2).

   By the product rule the total number of choices is $9 \cdot 9 \cdot 8 = 648$.

   (ii) How many three-digit natural numbers can be formed if the number is even and digits may be repeated? (A number is even if the digit d_3 is an element from the subset $\{0, 2, 4, 6, 8\}$.)

   Solution. We can count the number of three-digit numbers that are even with possibly-repeated digits by determining the options we have for each digit. For the digit d_1, we have 9 options since the digit cannot be 0. For the digit d_2, we have 10 options since there are no restrictions on the digit. For the digit d_3, we have 5 options, since the digit must come from the aforementioned subset.

   By the product rule there are $9 \times 10 \times 5 = 450$ three-digit natural numbers that are even with possibly-repeated digits.

   (iii) How many three-digit natural numbers can be formed if the number is even, digits may be repeated, and the number contains the digit 5?

   Solution. From part (b), we know that there are 450 three-digit natural numbers that are even with possibly-repeated digits. From this value, we subtract the number of three-digit natural numbers that are even with possibly-repeated digits where none of the digits are 5. There are a total of $8 \times 9 \times 5 = 360$ such numbers, since neither d_1 nor d_2 may be 5 and d_3 cannot be 5 by our definition of an even number.

   Altogether, there are $450 - 360 = 90$ three-digit natural numbers that are even, with possibly-repeated digits, and containing the digit 5.

3. How many functions from the set $\{1, 2, \ldots, n\}$ to $\{0, 1\}$ $(n \geq 1)$ there exist

   (i) that are one-to-one?

   (ii) that map both 1 and n to 0?

   (iii) that map exactly one of the integers $1, \ldots n-1$ to 1?

   Justify your answers!
   Note: In some of the cases one may need to consider separately a few small value of n.

   Solution.

   (i) 2 functions if $n = 1$ (two possible values for 1)
   2 functions if $n = 2$: the identity map and the function that swaps the elements
   0 functions if $n \geq 3$: since the range has only two elements we cannot define a one-to-one function

   (ii) 1 function if $n = 1$: 1 must be mapped to 0
   2^{n-2} functions when $n \geq 2$: the values of 1 and n are fixed. The values of 2, ...n-1 can be chosen to be either 0 or 1. The product rule gives 2^{n-2} choices.

   (iii) $2(n-1)$. Justification: there are $n-1$ ways to choose the integer $x \in \{1, \ldots, n-1\}$ that is mapped to 1. Additionally we can map n to either 0 or 1. The product rule give $2(n-1)$ as the total number.

4. A standard deck of playing cards contains 52 cards. Each card has a rank from the set $\{2, 3, 4, 5, 6, 7, 8, 9, 10, \text{J}, \text{Q}, \text{K}, \text{A}\}$ and a suit from the set $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$.

   Assume that the ranks J, Q, and K have numerical values of 10 and assume that the rank A has a numerical value of 11.

   For the following questions, let X be a random variable corresponding to the numerical value of the first card drawn from a shuffled deck and let Y be a random variable corresponding to the numerical value of the second card drawn from the same shuffled deck.

   Below $P(e)$ stands for probability of event e and $E(X)$ is the expected value of random variable X.

   (i) What is $P(X \text{ is even})$?

   Solution. The playing cards with an even numerical value are 2, 4, 6, 8, 10, J, Q, and K. There are four copies of each of these cards in the deck; that is, one for each suit. Therefore, there are 32 cards in the deck with an even numerical value, and $P(X \text{ is even}) = 32/52 = 8/13$.

   (ii) What is $E(X)$?

   Solution. Since the deck is shuffled, each numerical value appears with probability $1/13$. Therefore, the expectation of X is

$$E(X) = \frac{2+3+4+5+6+7+8+9+10+10+10+10+11}{13} = \frac{95}{13}.$$

   (iii) Are X and Y independent? Explain why or why not.

   Solution. No, X and Y are not independent. This is because the action of drawing the first card affects the outcome of the second card drawn: the second card cannot be the same as the first card (and probabilities of other occurrences also change).

   (iv) Using your answer from part (b), determine $E(X+Y)$.

   Solution. The expectation of Y is equal to the expectation of X by the same reasoning given in the solution to part (b). By linearity of expectation, we have

$$E(X+Y) = E(X) + E(Y) = \frac{95}{13} + \frac{95}{13} = \frac{190}{13}.$$

(Extra page)