1. Consider permutations (with no repetition) of the sequence of nine letters ABCDEFGH. Answer the following questions and justify your answers.

(i) How many permutations contain the string FAD ?

Since FAD must be one block, the possible permutations rearrange the block FAD and the individual letters B, C, E, G, H. The number of permutations of 6 elements is $6! = 720$

(ii) How many permutations contain the string BADEF ?

Now BADEF must be one block and the possible permutations rearrange this block with letters C, G, H. The number of permutations is $4! = 24$

(iii) How many permutations contain both the strings ABC and FGH ?

Both strings ABC and FGH must appear as a block. The possible permutations rearrange these blocks with the letters D and E. The number of permutations is $4! = 24$

(iv) How many permutations contain both the strings CE and DE ?

If CE and DE were to appear in the sequence, the letter E would be immediately preceded both by C and D which is impossible.
The number of permutations is 0.

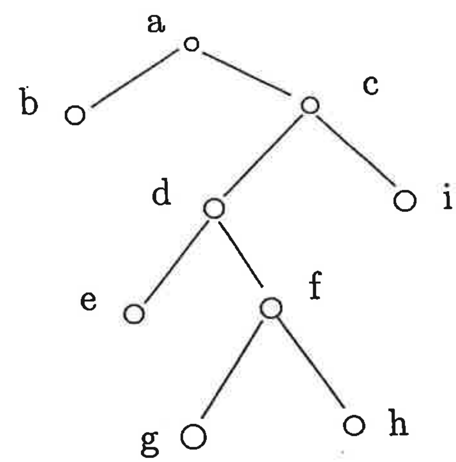


Figure 1: Tree T_1

2. Consider the rooted ordered binary tree T_1 given in Figure 1

(i) In which order does the pre-order traversal visit the vertices of T_1?

a, b, c, d, e, f, g, h, i

(ii) In which order does the post-order traversal visit the vertices of T_1?

b, e, g, h, f, d, i, c, a

(iii) In which order does the in-order traversal visit the vertices of T_1?

b, a, e, d, g, f, h, c, i

(iv) In which order does the level-order traversal visit the vertices of T_1?

a, b, c, d, i, e, f, g, h

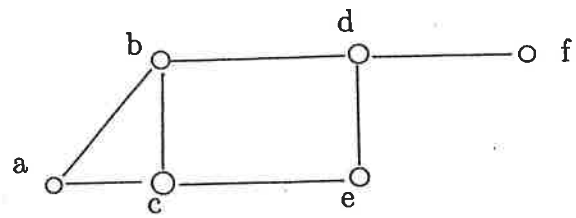


Figure 2: Graph G

3. Consider the graph G given in Figure 2.

   (i) Determine whether G has a **Hamiltonian cycle** and construct a Hamiltonian cycle if one exists. If G does not have a Hamiltonian cycle give a rigorous argument to show why no Hamiltonian cycle exists.

   G has no Hamiltonian cycle. Justification: if G had a Ham. cycle it would have one that begins and ends in f. This means that the edge $\{d, f\}$, and hence vertex d, would be visited more than once.

   (ii) Determine whether G has a **Hamiltonian path** and construct a Hamiltonian path if one exists. If G does not have a Hamiltonian path give a rigorous argument to show why no Hamiltonian path exists.

   Hamiltonian path for G:

   (f, d, e, c, b, a)

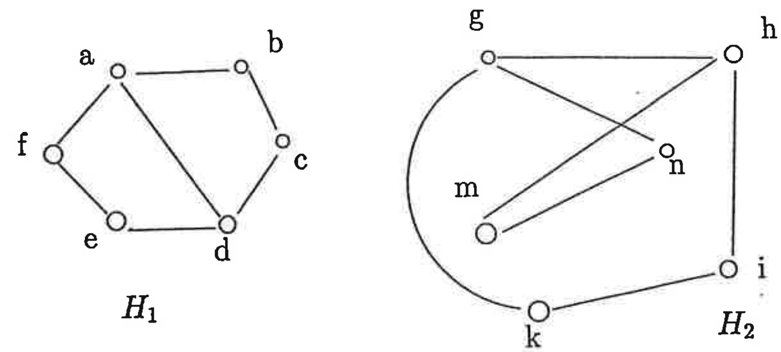


Figure 3: Graphs H_1 and H_2

4. (i) Give adjacency matrices for the graphs H_1 and H_2 given in Figure 3. In the matrices remember to clearly indicate what is the ordering of rows and columns.

$$A_{H_1} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \qquad A_{H_2} = \begin{array}{c} \\ g \\ h \\ i \\ k \\ m \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} g & h & i & k & m & n \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(ii) Are the graphs H_1 and H_2 given in Figure 3 isomorphic? Either give an explicit isomorphism between H_1 and H_2 or give a rigorous argument showing that an isomorphism does not exist.

Isomorphism φ:

$\varphi(a) = g$ $\qquad$ $\varphi(d) = h$

$\varphi(b) = n$ $\qquad$ $\varphi(e) = i$

$\varphi(c) = m$ $\qquad$ $\varphi(f) = k$

5. Consider a function $f : A \to B$. Prove:

   f has a left inverse if and only if f is one-to-one.

   Recall that $h : B \to A$ is a left inverse of function f if $h \circ f = 1_A$.

   Note: In order to prove an "iff" statement you need to prove the implication in both directions.

($\Rightarrow$) Suppose $h : B \to A$ is left inverse of f.

Consider $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. Now

$$a_1 = (h \circ f)(a_1) = h(f(a_1)) = h(f(a_2)) = (h \circ f)(a_2) = a_2$$

Thus $f(a_1) = f(a_2)$ implies $a_1 = a_2$ and f is one-to-one.

($\Leftarrow$) Suppose f is one-to-one and choose $a_0 \in A$ to be one fixed element of A. We define a function $h : B \to A$ by setting for $b \in B$:

$$h(b) = \begin{cases} a & \text{if } f(a) = b \\ a_0 & \text{if } b \text{ is not in the image of } f. \end{cases}$$

Since f is one-to-one, for any $b \in B$, there is at most one $a \in A$ such that $f(a) = b$. This means that, for all $a \in A$, h must map $f(a)$ back to a. Thus for all $a \in A$:

$$(h \circ f)(a) = h(f(a)) = a$$

and h is left inverse of f.

6. A standard deck of playing cards contains 52 cards. Each card has a rank 2, 3, 4, 5, 6,7, 8, 9 or 10, or "jack", "queen", "king" or "ace". Each card is one of the suits "clubs", "diamond", "heart" or "spade", and each suit has 13 cards (or the ranks listed above).

   Below by a five-card poker hand we mean a randomly selected (using a uniform distribution) set of 5 cards from the deck of 52 cards.

   Answer the following questions and justify your answers.

   (i) What is the probability that a five-card poker hand contains four cards of the same rank?

   Total number of poker hands is $\binom{52}{5}$.

   The number of hands with 4 cards of same rank: choose a rank, choose the 5^{th} card from remaining 48: $13 \cdot 48$

   Probability: $13 \cdot 48 / \binom{52}{5}$

   (ii) What is the probability that a five-card poker hand contains five cards of the same suit?

   The number of hands with 5 cards of same suit: choose a suit, choose 5 cards out of 13: $4 \cdot \binom{13}{5}$

   Probability: $4 \cdot \binom{13}{5} / \binom{52}{5}$

   (iii) What is the probability that a five-card poker hand contains three cards of one rank and two cards of a another rank? This is called a "full house". An example of a full house would be three kings and two cards of rank 7.

   Number of hands containing a full house: choose a rank R_1 that has "three of a kind", choose a rank R_2, $R_2 \neq R_1$, that has a pair, choose 3 cards of rank R_1 and two cards of rank R_2:

   $13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2} = 13 \cdot 12 \cdot 4 \cdot 6$

   Probability: $\dfrac{13 \cdot 12 \cdot 4 \cdot 6}{\binom{52}{5}}$

7. Tim Horton's at Bioscience has 4 different varieties of cookies.

   In how many ways can 12 cookies be chosen? When choosing a number of cookies only the variety of the cookie matters. Also the order in which the selections are made does not matter. Justify your answer.

The same variety can be chosen more than once.
The number of choices is the number of 12-combinations of a set of 4 elements with repetition, that is,

$$\frac{(12+4-1)!}{12!\,(4-1)!} = \frac{15!}{12!\,3!} = \frac{15\cdot 14\cdot 13}{6} = 455$$

8. Roll two fair dice and consider the following events:

   Event A: the numbers on the two dice sum up to 8

   Event B: the numbers on the two dice are both even

   Calculate the conditional probability $P(A \mid B)$ and show your work.

Sample space has 36 elements.
The event A consists of elements $(2,6), (3,5), (4,4), (5,3), (6,2)$ and the probability of A is $P(A) = \frac{5}{36}$
The event B consists of elements $(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)$, and the probability is $P(B) = \frac{1}{4}$
The event $A \cap B$ consists of $(2,6), (4,4), (6,2)$ and

$$P(A \cap B) = \frac{3}{36} = \frac{1}{12}$$

Now $P(A \mid B) = \frac{1}{12} \Big/ \frac{1}{4} = \frac{1}{3}$

9. Using the characteristic root method find a solution for the recurrence

$$a_n = 3a_{n-1} + 4a_{n-2}$$

with initial values $a_0 = 3$ and $a_1 = 6$. That is, give a closed form expression for a_n as a function of n. This includes finding the values of the associated constants. Please remember to show your work.

The characteristic equation

$$r^2 - 3r - 4 = 0$$

has roots 4 and -1. Thus solutions are of the form

$$a_n = \alpha_1 \cdot 4^n + \alpha_2 (-1)^n$$

From the initial values we get:

$$a_0 = \alpha_1 + \alpha_2 = 3$$

$$a_1 = 4\alpha_1 - \alpha_2 = 6$$

By adding the two equations we get $5\alpha_1 = 9$, or $\alpha_1 = \frac{9}{5}$

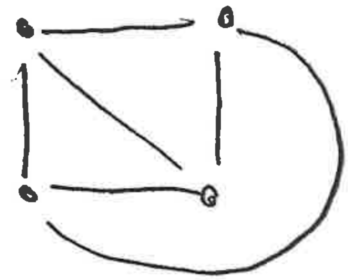
Substituting to first equation get $\alpha_2 = \frac{6}{5}$

Answer: $a_n = \frac{9}{5} \cdot 4^n + \frac{6}{5}(-1)^n$

10. Recall that K_n is the complete graph with n vertices.

(i) Show that K_4 is planar.

Planar embidding of K_4:



(ii) What is the chromatic number of K_7? Justify your answer.

$\chi(K_7) = 7$. Seven colors are sufficient because can color each vertex with a differnt color. Seven colors are necessary because in K_7 any two vertices are connected by an edge.

(iii) Show that K_5 is nonplanar. You can assume known Euler's formula for planar graphs.

Euler's formula: $v - e + f = 2$

v = vertices
e = edges
f = faces

If K_5 is planar, Euler's formula gives $f = 2 - 5 + 10 = 7$. Let b be the number of boundary edges surrounding each face. Since each face is surrounded by at least 3 edges we get $b \geq 3f$. Since each edge is a boundary edge for two faces we have $b = 2e$, and thus $2e \geq 3f$. This is a contradiction because $e = 10$ and $f = 7$.