MIDTERM SOLUTIONS

Note: The questions have notational variations. Your paper may not be exactly as shown.

INSTRUCTIONS

- You have 40 minutes. Attempt all four questions.
- You may bring in one 8.5×11 sheet of paper containing notes, and use it during the midterm.
- **Answer each question in the space provided** (on the question paper). There is an extra page at the end of the exam if more space is needed. **Please write legibly.**
- *Note:* In questions dealing with counting, combinatorics or probability it is not expected that you should compute large numerical values: it is fine to give the final answer in a form like "$\frac{56!}{29!}$" or "$\binom{40}{21}$" as long as you clearly explain how you arrived at the answer.
- **Please note:** You are asked to write your answers using a *non-erasable pen.* Answers written in pencil or erasable ink will be marked, but they will not be considered for remarking after the midterms are returned.

Student number (written in words):

MARKS

Problem 1	/6
Problem 2	/6
Problem 3	/6
Problem 4	/6
Total	/24

Code: askjkhkhdhfkjhqwewe1yu2tuytuyue1tduayt4duytu12yqeqwe

1. A bit is an element of $\{0, 1\}$. How many bit strings of length 10 contain

   (i) exactly four 1s?

   (ii) at most four 1s?

   (iii) at least four 1s?

   (iv) an equal number of 0s and 1s?

   Justify your answers.

   Solution.

   (i) Selecting the positions of the four 1-bits completely determines the string. The number of ways to choose 4 positions out of 10 is

   $$\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = 210.$$

   (ii) The number of ways to choose at most 4 positions out of 10 is

   $$\binom{10}{4} + \binom{10}{3} + \binom{10}{2} + \binom{10}{1} + \binom{10}{0} = 210 + 120 + 45 + 10 + 1 = 386.$$

   (iii) The number of ways to choose at least 4 positions out of 10 is

   $$\binom{10}{4}+\binom{10}{5}+\binom{10}{6}+\binom{10}{7}+\binom{10}{8}+\binom{10}{9}+\binom{10}{10} = 210+252+210+120+45+10+1 = 848.$$

   (The calculation is simplified by recalling that $\binom{n}{k} = \binom{n}{n-k}$.)

   (iv) The numbers of 0s and 1s are equal when we have exactly 5 1s. As calculated above the number of ways to select 5 positions out of 10 is

   $$\binom{10}{5} = 252.$$

2. Consider a function $f : A \to B$. Prove:

   f has a left inverse if and only if f is one-to-one.

   Recall that $h : B \to A$ is a left inverse of function f if $h \circ f = 1_A$.

   Note: In order to prove an "iff" statement you need to prove the implication in both directions.

   Solution.

   (implication from left to right): Suppose that $h : B \to A$ is left inverse of f. Consider $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. Now

$$a_1 = 1_A(a_1) = (h \circ f)(a_1) = h(f(a_1)) = h(f(a_2)) = 1_A(a_2) = a_2.$$

   Thus, $f(a_1) = f(a_2)$ implies $a_1 = a_2$ and f is one-to-one.

   (implication from right to left): Suppose f is one-to-one and choose $a_0 \in A$ to be one fixed element of A. We define a function $h : B \to A$ by setting for $b \in B$:

$$h(b) = \begin{cases} a \text{ if } f(a) = b; \\ a_0 \text{ if } b \text{ is not in the image of } f. \end{cases}$$

   Since f is one-to-one, for any $b \in B$, there is at most one $a \in A$ such that $f(a) = b$. This means that, for all $a \in A$, h must map $f(a)$ back to a. Thus, for all $a \in A$,

$$(h \circ f)(a) = h(f(a)) = a$$

   and h is left inverse of f.

3. Answer the following questions and <u>justify</u> your answers!

   Let $n \geq 5$. How many functions

$$f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, \ldots, n\}$$

   there exist

   (i) that are onto?

   Solution. If $n = 5$, an onto function is a bijection. The number of bijections (permutations) of a 5 element set is 5!.
   If $n > 5$, the function f cannot be onto.

   (ii) that are one-to-one?

   Solution. Since $n \geq 5$ a one-to-one function can always be defined. The number of one-to-one functions from a 5-element set to an n element set is the number of 5-permutations of an n-elements set

$$P(n, k) = \frac{n!}{(n-k)!}.$$

   (iii) that map both 4 and 5 to 1?

   Solution. Since the images of 4 and 5 are determined, only the values of 1, 2, 3 can be selected freely. Each value has n choices and using the product rule the total number of choices is n^3.

   (iv) that map exactly one of the integers 1, 2, 3, 4, 5 to 1?

   Solution. The value of the integer to be mapped to 1 can be selected in 5 ways.

   The image of each of the remaining four integers has $n - 1$ choices.

   Using the product rule the total number of choices is $5 \cdot (n-1)^4$.

4. A standard deck of playing cards contains 52 cards. Each card has a rank 2, 3, 4, 5, 6, 7, 8, 9 or 10, or "jack", "queen", "king" or "ace". Each card is one of the suits "clubs", "diamond", "heart" or "spade". For example, the "king of spades" is the card that has rank "king" and suit "spade".

   Below by a *five-card poker hand* we mean a randomly selected (using a uniform distribution) set of 5 cards from the deck of 52 cards.

   (i) What is the probability that a five-card poker hand does not contain the king of spades?

   Solution. The total number of poker hands is $\binom{52}{5}$.

   The number of poker hands not containing the king of spades is $\binom{51}{5}$.

   The probability of not containing the king of spades is

$$\frac{\binom{51}{5}}{\binom{52}{5}} = \frac{51! \cdot 5! \cdot 47!}{52! \cdot 5! \cdot 46!} = \frac{47}{52}.$$

   (ii) What is the probability that a five-card poker hand contains the queen of spades?

   Solution. As in (a) we know that the probability of not containing the queen of spades is $\frac{47}{52}$.

   The probability of containing the queen of spades is "one minus the probability of not containing the queen of spades", that is, $1 - \frac{47}{52} = \frac{5}{52}$.

   (iii) What is the probability that a five-card poker hand contains at least one ace? (There are in total four aces.)

   Solution. The total number of poker hands is $\binom{52}{5}$.

   The number of poker hands not containing an ace is $\binom{48}{5}$.

   The probability of having at least one ace is "one minus the probability of not having an ace", that is,

$$1 - \frac{\binom{48}{5}}{\binom{52}{5}}.$$

   (iv) What is the probability that a five-card poker hand contains five cards of the same suit? (Each of four suits has 13 cards.)

   Solution. The total number of poker hands is $\binom{52}{5}$.

   The number of poker hands where all cards are spades is $\binom{13}{5}$. Similarly, the number of hands where all cards are hearts (or diamonds or clubs) is $\binom{13}{5}$. Since the different possibilities are disjoint the total number of hands where all five cards have the same suit is $4 \cdot \binom{13}{5}$.

   Answer: $\frac{4 \cdot \binom{13}{5}}{\binom{52}{5}}$.

(Extra page)