Probability Continued: Conditional Probability

Let A and B be two events on the same sample space (S, P) (from now on, when I say "Let A and B be two events", I will always mean "on the same sample space" unless I indicate otherwise).

We can speak of A **occurring** - we mean that the experiment underlying the sample space has been executed once, and the observed outcome of the experiment is an element of A. The point here is that for an event to occur, there must have been a sampling operation resulting in an outcome.

If the experiment is executed and we are told the outcome is some specific value such as k, then we can determine whether or not A occurred simply by checking to see if $k \in A$. But suppose we are only given partial information about the outcome such as "B occurred" ... can we determine if A occurred or not? If we can't be sure that A did or did not occur, can we assign a probability value to A based on the partial information we have?

Example: Suppose S = {1,2,3,4,5,6} and $P(x) = \frac{1}{6} \quad \forall\, x \in S$

Let A = {2, 4, 5} We can see immediately that P(A) = $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

Now let's consider some possibilities for B

Example 1: Suppose B = {4, 5}

If we know B occurred, the outcome was either 4 or 5. These are both in A so in this situation, we know A occurred. We could phrase this as "given that B occurred, the probability that A occurred is 1".

Example 2: Suppose B = {1, 3, 6}

If we know B occurred, the outcome was either 1, 3, or 6. None of these is in A so in this situation we know A did not occur. We could phrase this as "given that B occurred, the probability that A occurred is 0".

Example 3: Suppose B = {1, 2, 4}

If we know B occurred, the outcome was either 1, 2, or 4. Two of those are in A and the other is not. It seems pretty reasonable to say "given that B occurred, the probability that A occurred is $\frac{2}{3}$ " ... but can we put this on firmer ground than "seems pretty reasonable"?

Yes we can! The set of outcomes for which A and B both occur is simply given by $A \cap B$, which in this case is {2, 4}. But remember *we are given that B occurred*, so instead of the set of possible outcomes for this particular sample being S, it is just B.

So instead of computing P({2,4}) as $\frac{|\{2,4\}|}{|S|} = \frac{2}{6} = \frac{1}{3}$

we compute P({2,4}) as $\frac{|\{2,4\}|}{|B|} = \frac{2}{3}$

Remember where {2,4} came from ... it is $A \cap B$. So what we are saying here is "given that B occurred, the probability that A occurred is $\frac{|A \cap B|}{|B|}$

Now for some notation: instead of writing out "the probability of A, given that B has occurred" we simply write **P(A | B)** (The poor old vertical bar gets **another** meaning on top of all the meanings it already has.) So our first three examples can be summarized as

A = {2,4,5} B = {4, 5} P(A | B) = $\frac{|A \cap B|}{|B|} = \frac{2}{2} = 1$

A = {2,4,5} B = {1, 3, 6} P(A | B) = $\frac{|A \cap B|}{|B|} = \frac{0}{3} = 0$

A = {2,4,5} B = {1, 2, 4} P(A | B) = $\frac{|A \cap B|}{|B|} = \frac{2}{3}$

And a final example:

A = {2,4,5} B = {1,3,4,6} P(A | B) = $\frac{|A \cap B|}{|B|} = \frac{1}{4}$

Let's consider another sample space. Suppose we have a barn containing a cow, a moose, a horse, a llama, a hippo, a camel and a bear. Our sampling experiment is to open the barn door and observe which four animals come out first. We assume that all outcomes are equally probable (you can quite reasonably question the reality of this assumption, but we'll go with it for now).

How many outcomes are there? It should be clear that the answer is $\binom{7}{4} = 35$, so (under our assumption that each outcome has equal probability) each outcome has probability = $\frac{1}{35}$

Let event A = {all outcomes that include the horse and the moose} It's not hard to see that there are exactly 10 such outcomes, so |A| = 10 and P(A) = $\frac{10}{35}$

Let event B = {all outcomes that don't include the cow and don't include the llama} Again we can see that there are exactly 5 such outcomes, so |B| = 5 and P(B) = $\frac{5}{35}$

Now what is P(A | B)? To apply the formula we need to know $|A \cap B|$, which we can easily compute. There are precisely 3 outcomes that include the horse and the moose, and exclude the cow and llama: {horse, moose, hippo, camel} {horse, moose, hippo, bear} and {horse, moose, camel, bear}

So $P(A|B) = \frac{|A \cap B|}{|B|} = \frac{3}{5}$

Here is a way to make sense of this: If we don't know exactly which output occurred but we know it is in B, we are limited to just 5 possible outcomes. Out of those, there are 3 outcomes that are elements of A. So the probability that the (unknown) outcome is in A is $\frac{3}{5}$

In the previous examples, we were only able to use $|A \cap B|$ and |B| because all the elements of S have the same probability value. Now we must consider the situation where P(x) is not the same $\forall$ x . We will replace $|A \cap B|$ by $P(A \cap B)$, and |B| by P(B).

Example: S = {1,2,3,4,5,6} P(1) = 0.5 P(x) = 0.1 for $2 \leq x \leq 6$

Let A = {2,4,5}, as before. Now we see that P(A) = 0.3

Let B = {1,2,4}. Clearly P(B) = 0.7

What is P(A | B)? We will see it is $\frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.7} = \frac{2}{7}$ .... but why?

Here is the simplest way I have found to think about this.

In this example we can visualize the sample space as a probability pie cut into 6 slices. The slice labelled "1" is $\frac{1}{2}$ of the pie, and all the other slices are $\frac{1}{10}$ of the pie. For any event, the probability of the event is equal to the percentage of the pie contained in the event. When we look at the slices corresponding to B (i.e. 1, 2 and 4) we see they make up 70% of the pie. The slices that correspond to A ∩ B (i.e. 2 and 4) make up 20% of the original pie, but they make up $\frac{2}{7}$ of the area of B. So if we restrict ourselves to looking at B (which we **must do** since we are given that B occurred) the probability that the unknown outcome was also in A is given by $\frac{P(A \cap B)}{P(B)}$, which is just what we claimed above.

Does this make sense? The "raw" probability of A is 0.3, and we have just computed $P(A|B) = \frac{2}{7} = 0.286...$ so $P(A|B) < P(A)$. This is reasonable because if all we know is that B occurred, the probability that A **did not** occur is very high! Most of the probability associated with B is concentrated in "1" which is not an element of A. The probability of A ∩ B relative to B is less than the probability of A relative to S.

Another example:

A = {2,4,5} B = {1,3,4,6} P(A | B) = $\frac{P(A \cap B)}{P(B)} = \frac{P(4)}{P(B)} = \frac{\frac{1}{10}}{\frac{7}{10}} = \frac{1}{7}$

A last note: Since we now know that $P(A|B) = \frac{P(A \cap B)}{P(B)}$, we can write

$$P(A \cap B) = P(A|B) * P(B)$$

We can also compute $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$, which turns around to give

$$P(A \cap B) = P(B|A) * P(A)$$

Combining these two equations gives the result

$$P(A|B) * P(B) = P(B|A) * P(A)$$

We can use this! Suppose we know $P(A) = 0.3$, $P(B) = 0.6$ and $P(B|A) = 0.5$ ... we can compute

$$P(A|B) = \frac{0.5 * 0.3}{0.6} = 0.25$$

Exercise: Is it possible to have events A and B where $P(A) = 0.3$, $P(B) = 0.1$ and $P(B|A) = 0.5$?