# CISC 271   Class 2

## Graphs: The Adjacency Matrix

Text Correspondence, Goodaire & Parmentier: §9.1–9.2, §10.3

*Main Concepts:*

- *Graph $\mathcal{G}$: structure in discrete mathematics*
- *Vertex set $\mathcal{V}$: finite non-empty set*
- *Edge set $\mathcal{E}$: set of distinct pairs of vertices*
- *Adjacency matrix: square binary matrix of a graph*

**Sample Problem, Data Analysis:** How can we represent a graph as a matrix?

A *graph* is a term for the abstract structure that represents many common drawings and ideas. For example, when we look at a "map" of part of the London Underground subway system, company, we see subway stops as circles or ellipses that are connected by straight or curved lines; a sample of such a "map" is shown in Figure 2.1.

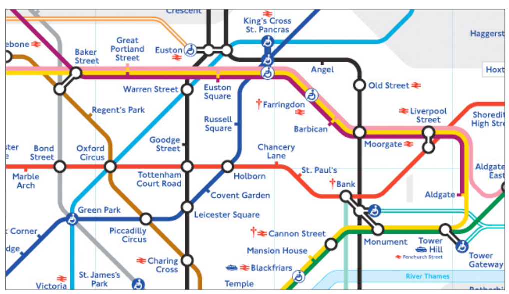


**Figure 2.1:** A "map" of part of the underground rail system of London, Great Britain. Subway stops are shown as circles, ellipses, or bars. Stops for which a train is directly available are shown as various straight and curved lines. (This image is sourced from `https://www.visitlondon.com` and is presented as "fair use".)

A graph is used to represent many familiar complex relationships. Examples in computer science include the network connectivity between computing nodes and the layout of transistors on a silicon chip. In this class, we are less interested in the applications of graphs or in the theory of computation for graphs. Our interest is in the representation of a graph as a matrix and some uses of linear algebra for a graph matrix.

In the intellectual field of discrete mathematics, a *graph* $\mathcal{G}$ is a structure that has two sets. We must be careful when defining and using graphs because there is no effective standardization of notation or meanings. When we read each text or other reference work, we need to read carefully to ensure that we understand the meaning of the author(s).

The first set of a graph $\mathcal{G}$, the *vertices*, is non-empty and finite. Each member of this set is a *vertex*. We will write the vertex set as $\mathcal{V}$ and each vertex as $v_j$.

The second set of a graph $\mathcal{G}$, the *edges*, is finite. Each member of the set is a pair of vertices. We will write the edge set as $\mathcal{E}$ and each edge as $e_k = \{v_i \ v_j\}$ or as $e_k = (v_i \ v_j)$, with the understanding that the second notation is not an ordered set.

We will generally write a graph as $\mathcal{G}(\mathcal{V}, \mathcal{E})$. We may use subscripts to distinguish graphs from each other.

In this course, we will make two assumptions about a graph that are important and that vary considerably between authors. The first assumption is that a graph is *undirected*. In the subway example, this implies that if it is possible to go from station $v_i$ to station $v_j$ without visiting any other station, then it is likewise possible to go from station $v_j$ to station $v_i$ without visiting any other station. If the graph represents city streets, then every street is "two-way" and there are no "one-way" streets in the graph.

The second important assumption that we will make is that there is no *loop* in a graph. By this, we intend that for each vertex $v_j$ there is no edge of the form $\{v_j \ v_j\}$.

For example, suppose that we construct a graph $\mathcal{G}_1(\mathcal{V}, \mathcal{E})$ that has five vertices and four edges:

$$\mathcal{G}_1 \stackrel{\text{def}}{=} \begin{cases} \mathcal{V}_1 & = & \{1\ 2\ 3\ 4\ 5\} \\ \mathcal{E}_1 & = & \{(1\ 2)\ \ (1\ 3)\ \ (2\ 3)\ \ (4\ 5)\} \end{cases} \tag{2.1}$$

We can produce a diagram of the graph $\mathcal{G}_1$ in a similar way that the map in Figure 2.1 is drawn. We can use a circle for each vertex and a line for each edge. The graph defined in Equation 2.1 can be diagrammed as shown in Figure 2.2.

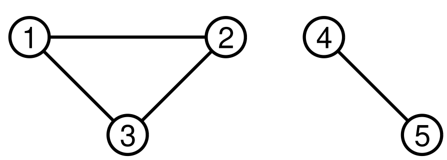


**Figure 2.2:** An illustration of the graph $\mathcal{G}_1$, having five vertices and four edges. The graph has two distinct sub-graphs that have no shared edge.

 

## 2.1 Relevant Definitions

In this course, we will use very limited graph theory. For our graphs, which are undirected and have no loops, some relevant definitions are:

**incident vertex and edge:** a vertex $v_i$ and an edge $e$ for which $e = (v_i \ v_j) = (v_j \ v_i)$

**adjacent vertices:** vertices $v_i$ and $v_j$ for which there exists an edge $e = (v_i \ v_j)$

**degree:** for a vertex $v_j$, the number of edges $e_i$ that are incident to $v_j$

**multiple edge:** distinct edges $e_p$ and $e_q$ that are equal, so that $e_p = e_q$

**loop:** an edge $e = (v_j \ v_j)$

**pseudograph:** a set $\{\mathcal{V} \ \mathcal{E}\}$ that has at least one multiple edge and/or at least one loop

**subgraph:** for a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, a graph $\mathcal{G}_S(\mathcal{V}_S, \mathcal{E}_S)$ is a *subgraph* of $\mathcal{G}$ is defined as: $\mathcal{V}_S \subseteq \mathcal{V}$ and $\mathcal{E}_S \subseteq \mathcal{E}$

**bipartite graph:** a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ such that: (a) $\mathcal{V}$ can be partitioned into two disjoint non-empty sets $\mathcal{V}_L$ and $\mathcal{V}_R$, and (b) each edge $e_k$ is incident with a vertex $v_i \in \mathcal{V}_L$ and is incident with a vertex $v_j \in \mathcal{V}_R$

From these definitions, we can make a few observations on the graph $\mathcal{G}_1$, such as:

- edge $(1 \ 2)$ is incident with vertex 1
- vertex 3 is not incident with $(1 \ 2)$
- the degree of vertex 3 is 2 and the degree of vertex 4 is 1
- vertex 2 is adjacent to vertex 3
- vertex 2 is not adjacent to vertex 4
- no edge is adjacent to vertex 4 and to any of vertices 1–3
- the graph $\mathcal{G}_1$ is not bipartite

The observation that graph $\mathcal{G}_1$ is not bipartite follows from the "triangle" formed by the first three vertices.

An example of a bipartite graph is the graph $\mathcal{G}_2$ that has six vertices and five edges:

$$\mathcal{G}_2 \stackrel{\text{def}}{=} \left\{ \begin{array}{lcl} \mathcal{V}_2 & = & \{1\ 2\ 3\ 4\ 5\ 6\} \\ \mathcal{E}_2 & = & \{(1\ 2)\ \ (2\ 3)\ \ (2\ 4)\ \ (4\ 5)\ \ (4\ 6)\} \end{array} \right. \tag{2.2}$$

The graph defined in Equation 2.2 can be drawn as shown in Figure 2.3. One way to partition the vertices of graph $\mathcal{G}_2$ is as $\mathcal{V}_L = \{2\ 5\ 6\}$ and $\mathcal{V}_R = \{1\ 3\ 4\}$.

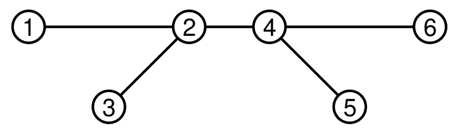


**Figure 2.3:** An illustration of the bipartite graph $\mathcal{G}_2$, having six vertices and five edges.

An observation on the graph $\mathcal{G}_2$ is that every nontrivial sub-graph is also a bipartite graph; this is a fact about all bipartite graphs, which we will not prove. The interested reader is encouraged to explore graph theory for more on these topics.

## 2.2 Adjacency Matrix of a Graph

Although the intellectual field of discrete mathematics uses a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ directly, we are more interested in using a matrix representation of a graph. There are two fundamental ways of representing a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ in linear algebra: as a square *adjacency* matrix that has a size that varies according to the number of vertices in $\mathcal{V}$; and as an *incidence* matrix that has a size that varies according to the number of vertices in $\mathcal{V}$ and the number of edges in $\mathcal{E}$. We will work with the adjacency matrix and recognize that the incidence matrix is also useful.

**Definition:** adjacency matrix $A(\mathcal{V}, \mathcal{E})$ of a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$

For any graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ that has $n$ vertices in $\mathcal{V}$ labeled $v_1, v_2, \ldots, v_n$ and that has $m$ edges labeled $(v_i\ v_j)$, the *adjacency matrix* $A(\mathcal{G})$ is the $n \times n$ matrix for which each entry is defined as

$$a_{ij} = \left\{ \begin{array}{ll} 1 & \text{if } (v_i\ v_j) \in \mathcal{E} \\ 0 & \text{if } (v_i\ v_j) \notin \mathcal{E} \end{array} \right. \tag{2.3}$$

 

According to Definition 2.3, any adjacency matrix $A$ of a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ must have these properties:

- $A$ is real, because either $a_{ij} = 0$ or $a_{ij} = 1$
- $A$ is symmetric, because if $(v_i \ v_j) \in \mathcal{E}$ then $a_{ij} = a_{ji} = 1$; this is true of all undirected graphs, which is the kind of graph that we are using
- $A$ has diagonal entries $a_{jj} = 0$, because the kind of graph that we are using has no loop
- The eigenvalues of $A$ are real, because $A$ is a real symmetric matrix
- The eigenvalues of $A$ add up to zero, because $\mathrm{tr}(A) = 0$

We can easily create the adjacency matrices for the two graphs that we have defined so far in this class. The matrix for graph $\mathcal{G}_1(\mathcal{V}_1, \mathcal{E}_1)$ is

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.4}$$

The adjacency matrix for graph $\mathcal{G}_2(\mathcal{V}_2, \mathcal{E}_2)$ is

$$A_2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \tag{2.5}$$

A simple calculation that we can perform is to multiply an adjacency matrix and the "ones" vector. The "ones" vector, which has $n$ entries and each entry is 1, is written as

$$\vec{1} \stackrel{\text{def}}{=} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

If we compute $[A_1]\vec{1}$, we find that the result is the vector

$$\vec{d}_1 = [A_1]\vec{1} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \tag{2.6}$$

Each entry of the vector $\vec{d}_1$ in Equation 2.6 is the *degree* of the vertex that has the corresponding row number. This makes sense because multiplication of a matrix and the "ones" vector takes the sum of the rows of the matrix; the sum of a row of an adjacency matrix is the number of edges that are incident with the vertex, which is the definition of the degree of the vertex.

We can likewise compute $[A_2]\vec{1}$ for graph $\mathcal{G}_2(\mathcal{V}_2, \mathcal{E}_2)$, to get

$$\vec{d}_2 = [A_2]\vec{1} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \tag{2.7}$$

A new graph can be derived from graph $\mathcal{G}_1(\mathcal{V}_1, \mathcal{E}_1)$ by adding one vertex and two edges. This graph $\mathcal{G}_3(\mathcal{V}_3, \mathcal{E}_3)$ can be defined as

$$\mathcal{G}_3 \stackrel{\text{def}}{=} \left\{ \begin{array}{lcl} \mathcal{V}_3 & = & \{1\ 2\ 3\ 4\ 5\ 6\} \\ \mathcal{E}_3 & = & \{(1\ 2)\ \ (1\ 3)\ \ (2\ 3)\ \ (4\ 5)\ \ (5\ 6)\} \end{array} \right. \tag{2.8}$$

The graph $\mathcal{G}_3(\mathcal{V}_3, \mathcal{E}_3)$ defined in Equation 2.8 can be diagrammed as shown in Figure 2.4.

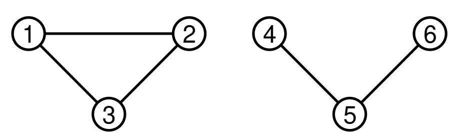


**Figure 2.4:** An illustration of the graph $\mathcal{G}_3$, having six vertices and five edges. The graph has two distinct sub-graphs that have no shared edge.

The graph $\mathcal{G}_1(\mathcal{V}_1, \mathcal{E}_1)$ and the $\mathcal{G}_3(\mathcal{V}_3, \mathcal{E}_3)$ have a common property: each graph has two distinct *components*. In graph theory, a component is a subgraph of a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with two special properties:

- For any two vertices of the component, there is a "walk" along incident edges between the vertices; and
- Each vertex of the component is not adjacent to any non-component vertex of $\mathcal{G}$

The adjacency matrix for graph $\mathcal{G}_3(\mathcal{V}_3, \mathcal{E}_3)$ is

$$A_3 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.9}$$

The graph property of having two distinct components has a correspondence in the adjacency matrix. In general, a graph that has distinct components can be partitioned with non-zero diagonal blocks and zero off-diagonal blocks.

We can modify graph $\mathcal{G}_3(\mathcal{V}_3, \mathcal{E}_3)$, creating a new graph $\mathcal{G}_4(\mathcal{V}_4, \mathcal{E}_4)$, by adding two edges. This new graph can be defined as

$$\mathcal{G}_4 \stackrel{\text{def}}{=} \begin{cases} \mathcal{V}_4 & = & \{1\ 2\ 3\ 4\ 5\ 6\} \\ \mathcal{E}_4 & = & \{(1\ 2)\ \ (1\ 3)\ \ (2\ 3)\ \ (2\ 4)\ \ (4\ 5)\ \ (4\ 6)\ \ (5\ 6)\} \end{cases} \tag{2.10}$$

The graph $\mathcal{G}_4(\mathcal{V}_4, \mathcal{E}_4)$ defined in Equation 2.10 can be diagrammed as shown in Figure 2.5. The

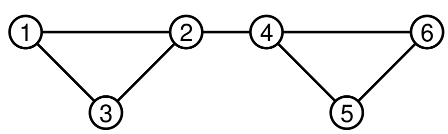


**Figure 2.5:** An illustration of the graph $\mathcal{G}_4$, having six vertices and seven edges. The graph has no distinct sub-graphs, but there are two "clusters" of vertices in the graph.

adjacency matrix for graph $\mathcal{G}_4(\mathcal{V}_4, \mathcal{E}_4)$ is

$$A_4 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \tag{2.11}$$

The graph $\mathcal{G}_4(\mathcal{V}_4, \mathcal{E}_4)$ is not bipartite, because there are two "triangles" that prevent the vertices from being appropriately partitioned. Graph $\mathcal{G}_4$ also cannot be partitioned into components, because there is always a "path" from any subgraph to a vertex that is not in the subgraph.

The graph $\mathcal{G}_4(\mathcal{V}_4, \mathcal{E}_4)$ is interesting to us because it suggests that there are "clusters" of vertices; in this instance the clusters appear, visually, to be $\{1 \quad 2 \quad 3\}$ and $\{4 \quad 5 \quad 6\}$. Although they are not components, the vertices of each "cluster" have a higher connectivity within the cluster than to outside the cluster.

We will use a different matrix, which is derived from the adjacency matrix, to explore clustering of vertices in graphs.

Extra Notes

## 2.3 Extra Notes: Incidence Matrix of a Graph

Another commonly used matrix of a graph is the *incidence matrix*. There are two ways to write this matrix – the ways are transposes of each other – so we must be careful when we read a source to determine which way the matrix is written. Here, we will use the `MATLAB` convention for the incidence matrix.

Suppose that a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ has $n$ vertices and $m$ edges. The incidence matrix is an $n \times m$ matrix that we will write as $C_{n \times m}(\mathcal{G})$, or in context simply as $C$. Each row of the incidence matrix $C$ corresponds to an edge $e_i \in \mathcal{E}$ of the graph $\mathcal{G}$ and each column the incidence matrix $C$ corresponds to a vertex $v_i \in \mathcal{V}$ of the graph $\mathcal{G}$.

**<u>Definition:</u>** incidence matrix $C(\mathcal{V}, \mathcal{E})$ of a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$

> For any graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ that has $n$ vertices in $\mathcal{V}$ labeled $v_1, v_2, \ldots, v_n$ and that has $m$ edges labeled $(v_i \ v_j)$, the *incidence matrix* $C(\mathcal{G})$ is the $n \times m$ matrix for which each entry is defined as
>
> $$c_{ij} = \begin{cases} 1 & \text{if } v_i \in e_j \\ 0 & \text{if } v_i \notin e_j \end{cases} \tag{2.12}$$

It is often easier to understand Definition 2.12 from examples. For the graph $\mathcal{G}_1$ that is specified in Equation 2.1, the incidence matrix is

$$C_1 \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.13}$$

From the first row of $C_1$ in Equation 2.13, we can say that vertex $v_1$ is incident with edge $e_1$, and edge $e_1$ also is incident with edge $e_2$. From the second column of $C_1$ in Equation 2.13, we can say that edge $e_2$ is incident with vertex $v_1$, and edge $e_2$ also is incident with vertex $v_3$.

For the bipartite graph $\mathcal{G}_2$ that is defined in Equation 2.2, the incidence matrix is

$$C_2 \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.14}$$

The columns of $C_2$ in Equation 2.14 correspond with the edges in graph $Gset_2$, and each non-zero entry of a column corresponds with a vertex to which the edge is incident.

In this course, we will not make practical use of the incidence matrix. There are many powerful applications of the incidence matrix, particularly for directed graphs, that the interested student is encouraged to explore.

End of Extra Notes