CISC 271 Class 7

Normal Matrices and Spectral Decomposition

Text Correspondence: §6.3

Main Concepts:

- *Normal: a matrix C such that $CC^T = C^TC$*
- *Orthogonal: a matrix Q such that $QQ^T = I$*
- *Symmetric: a matrix B with $b_{ij} = b_{ji}$*
- *Symmetric matrices have only real eigenvalues*
- *Spectral theorem: diagonalization uses matrix transpose*

Sample Problem, Data Analysis: When is a set of data vectors guaranteed to have an orthonormal basis?

Previously, we saw that the eigenvectors of a diagonalizable matrix $A \in \mathbb{R}^{n\times n}$ are a basis for the vector space $\mathbb{R}^n$. This means that any vector $\vec{u} \in \mathbb{R}^n$ can be represented as a linear sum of the eigenvectors $\vec{v}_j$, written as

$$
\begin{aligned}
\vec{u} &= \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots \alpha_n\vec{v}_n \\
&= \sum_{j=1}^{n} \alpha_j\vec{v}_j
\end{aligned} \tag{7.1}
$$

7.1 Orthogonal Matrices

The "best" kind of basis is an orthonormal basis, in which each vector is of unit length and is orthogonal to the other vectors. It is common to refer to such vectors as columns of a matrix Q, so each such vector is written as $\vec{q}_j$. The defining properties of an orthonormal basis are

$$
\begin{aligned}
i \neq j &\Rightarrow \vec{q}_i \cdot \vec{q}_j = 0 \\
i = j &\Rightarrow \vec{q}_i \cdot \vec{q}_j = 1
\end{aligned}
$$

By using the transpose to formulate the dot product, an orthonormal basis can also be defined as

$$\begin{aligned} i \neq j \quad &\Rightarrow \quad \vec{q}_i^{\,T} \vec{q}_j = 0 \\ i = j \quad &\Rightarrow \quad \vec{q}_i^{\,T} \vec{q}_j = 1 \end{aligned}$$

From this it follows that

$$Q^T Q = I \quad \text{and} \quad Q Q^T = I \tag{7.2}$$

An orthogonal matrix, defined in Equation 7.2, commutes with itself. This is a slightly unusual property; for example, the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

does *not* commute because

$$AA^T = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

What other matrices commute with their transposes?

7.2 Symmetric Matrices

One form of matrix that is frequently encountered in applications is a *symmetric* matrix. This is a matrix B for which

$$b_{ij} = b_{ji} \tag{7.3}$$

It is easy to see that, if B is symmetric, then

$$B^T B = B B^T = B^2$$

so the matrix commutes with its transpose.

What can we say about the eigenvalues and eigenvectors of a symmetric matrix B? As it happens, we can say a great deal. The first major result is

For a real symmetric matrix, every eigenvalue is a real number.

This can be proved by using complex conjugates, but complex analysis is beyond the scope of this course so the interested student should read the textbook and other sources for greater insight.

The second major result is

For a real symmetric matrix, eigenvectors that have distinct eigenvalues are orthogonal.

To show this, we will consider two eigenvalue/eigenvector pairs and use the transpose as a way of writing the dot product. We will be investigating whether

$$
\begin{aligned}
\vec{v}_1 \cdot \vec{v}_2 &\stackrel{?}{=} 0 \\
\text{or equivalently} \quad \vec{v}_1^T \vec{v}_2 &\stackrel{?}{=} 0
\end{aligned}
$$

We begin with the statement that the eigenvectors are distinct. For any pair, labeling one as λ_1 and the other as λ_2, we have

$$\lambda_1 \neq \lambda_2$$

Consider the dot product of $B\vec{v}_1$ with $\vec{v}_2$. Expanding the matrix-vector product, and factoring the scalar out of the dot product, gives

$$
\begin{aligned}
(B\vec{v}_1) \cdot \vec{v}_2 &= (\lambda_1 \vec{v}_1) \cdot \vec{v}_2 \\
&= \lambda_1 (\vec{v}_1 \cdot \vec{v}_2)
\end{aligned}
\tag{7.4}
$$

Alternatively, we can express the dot product as a transposed multiplication; doing this, then replacing $B^T = B$ because B is symmetric, then converting back to the dot product, gives

$$
\begin{aligned}
(B\vec{v}_1) \cdot \vec{v}_2 &= (B\vec{v}_1)^T \vec{v}_2 \\
&= (\vec{v}_1^T B^T) \vec{v}_2 \\
&= \vec{v}_1^T (B\vec{v}_2) \\
&= \vec{v}_1^T (\lambda_2 \vec{v}_2) \\
&= \vec{v}_1 \cdot (\lambda_2 \vec{v}_2) \\
&= \lambda_2 (\vec{v}_1 \cdot \vec{v}_2)
\end{aligned}
\tag{7.5}
$$

Because Equation 7.4 and Equation 7.5 have the same left-hand side, the right-hand terms are equal so

$$\lambda_1 (\vec{v}_1 \cdot \vec{v}_2) = \lambda_2 (\vec{v}_1 \cdot \vec{v}_2) \tag{7.6}$$

We began by assuming that $\lambda_1 \neq \lambda_2$ so, for Equation 7.6 to hold, we must have

$$\vec{v}_1 \cdot \vec{v}_2 = 0 \tag{7.7}$$

This proves that, for a symmetric matrix, the eigenvectors of distinct eigenvalues are orthogonal.

7.3 Normal Matrices

We now come to the main point of this class, which is the ***Spectral Theorem:***

> *Every normal matrix A has the factorization*
>
> $$A = Q\Lambda Q^T$$
>
> *where Q is an orthogonal matrix in which the columns are eigenvectors of A, and Λ is a diagonal matrix in which the entries are the eigenvalues of A.*

This has an important corollary, the ***Spectral Theorem for Symmetric Matrices:***

> *Every real symmetric matrix B has the factorization*
>
> $$B = Q\Lambda Q^T$$
>
> *where Q is an orthogonal matrix in which the columns are eigenvectors of B, and Λ is a real diagonal matrix in which the entries are the eigenvalues of B.*

The difference between these two theorems is that a real-valued symmetric matrix has real eigenvalues and eigenvectors, whereas an asymmetric normal matrix might have complex eigenvalues and eigenvectors.

We can now consider Equation 7.1 for a symmetric matrix (or other normal matrix). The eigenvectors are a basis, and more importantly their unit forms are an *orthonormal* basis. We can write Equation 7.1 concisely as

$$\vec{u} = Q\vec{\alpha} \tag{7.8}$$

Because Q is an orthogonal matrix, its inverse Q^{-1} is its transpose Q^T. We can therefore solve Equation 7.8 by pre-multiplying both sides by $Q^{-1} = Q^T$, so

$$Q^T\vec{u} = \vec{\alpha} \tag{7.9}$$

Consider expanding Equation 7.9 by rows. This gives

$$\begin{aligned} \vec{q}_1^T\vec{u} &= \alpha_1 \\ \vec{q}_2^T\vec{u} &= \alpha_2 \\ &\vdots \\ \vec{q}_n^T\vec{u} &= \alpha_n \end{aligned} \tag{7.10}$$

For an orthonormal basis, particularly for one that arises from a symmetric matrix, we can solve for the "weights" of the basis by using the dot-product form of each line in Equation 7.10 as

$$\alpha_k = \vec{q}_k \cdot \vec{u} \tag{7.11}$$

This property of orthonormal basis vectors is one that we will use in studying large data sets by principal components analysis.

7.4 Skew-Symmetric Matrices

A square matrix S is *skew symmetric*, or anti-symmetric, is defined as one for which

$$s_{ij} = -s_{ji} \tag{7.12}$$

Another way to write this matrix is

$$S^T = -S$$

We can confirm that a skew-symmetric matrix S is a normal matrix by expansion:

$$SS^T = S[-S] = [-S]S = S^T S \tag{7.13}$$

From either definition, most clearly from Equation 7.12, the diagonal entries are zero so the matrix S has the form

$$S = \begin{bmatrix} 0 & s_{12} & s_{13} & \cdots & & s_{1n} \\ -s_{12} & 0 & s_{23} & \cdots & & s_{2n} \\ -s_{13} & -s_{23} & 0 & \cdots & & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \\ & & & & 0 & s_{(n-1),n} \\ -s_{1n} & -s_{2n} & -s_{3n} & \cdots & -s_{(n-1),n} & 0 \end{bmatrix}$$

The ***Spectral Theorem for Skew-Symmetric Matrices:***

Every real skew-symmetric matrix S has the factorization

$$S = Q\Lambda Q^T$$

where Q is an orthogonal matrix in which the columns are eigenvectors of S, and Λ is a diagonal matrix in which the entries are the eigenvalues of S and are either zero or purely imaginary.

Skew-symmetric matrices may have nontrivial eigenvectors associated with the zero eigenvalues. Other properties of skew-symmetric matrices can be explored by interested students.

Extra Notes

Extra Notes: Other Normal Matrices

Is it possible for a matrix to be normal but none of orthogonal, symmetric, and skew-symmetric? The answer is positive and can be shown by example.

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

This matrix has none of the usual properties, but it is a normal matrix because

$$AA^T = A^T A = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

The eigenvalues are

$$\lambda_1 = 5 \quad \lambda_{2,3} = \pm\sqrt{3}\, i$$

and the eigenvectors (one real, two complex) are orthogonal. So this matrix has an orthonormal eigenvector basis but is not one of the matrix forms above.

This is a *circulant* matrix, which in general are normal matrices and which have an orthonormal eigenvector basis.

End of Extra Notes