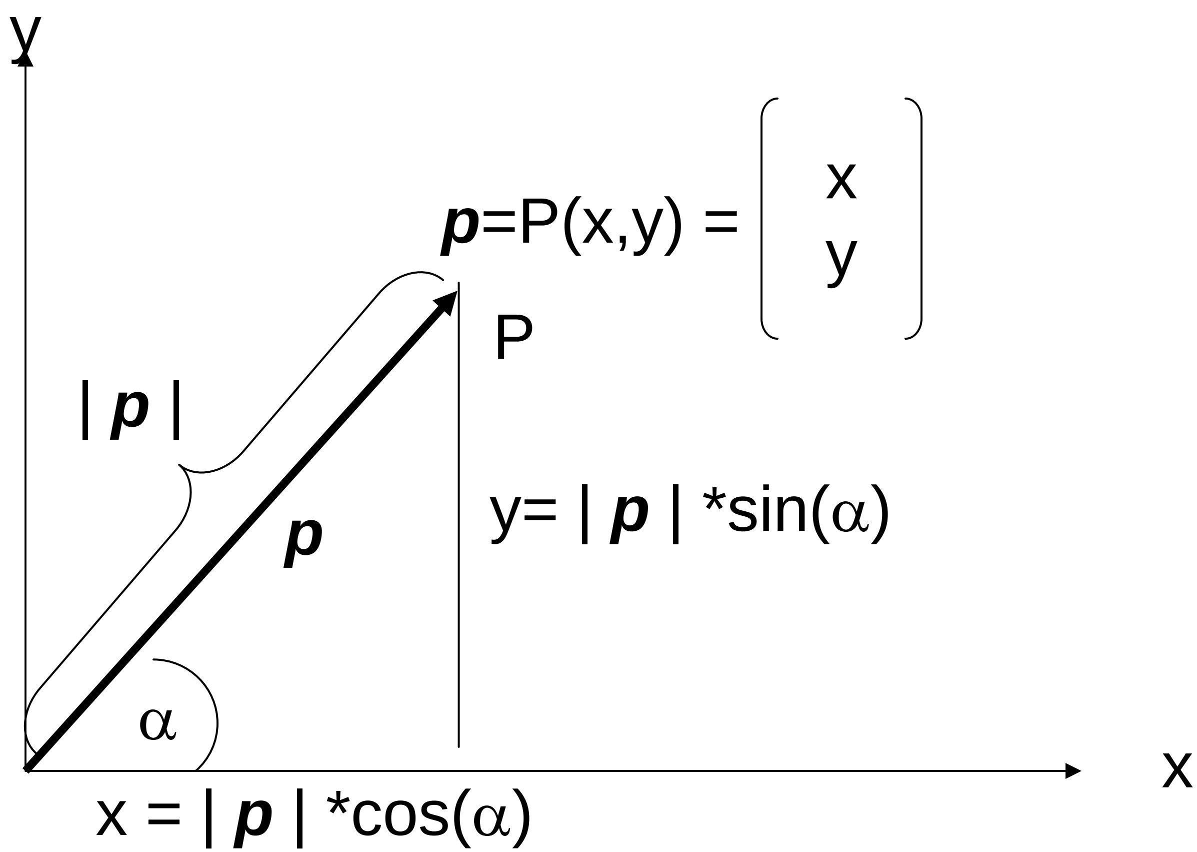
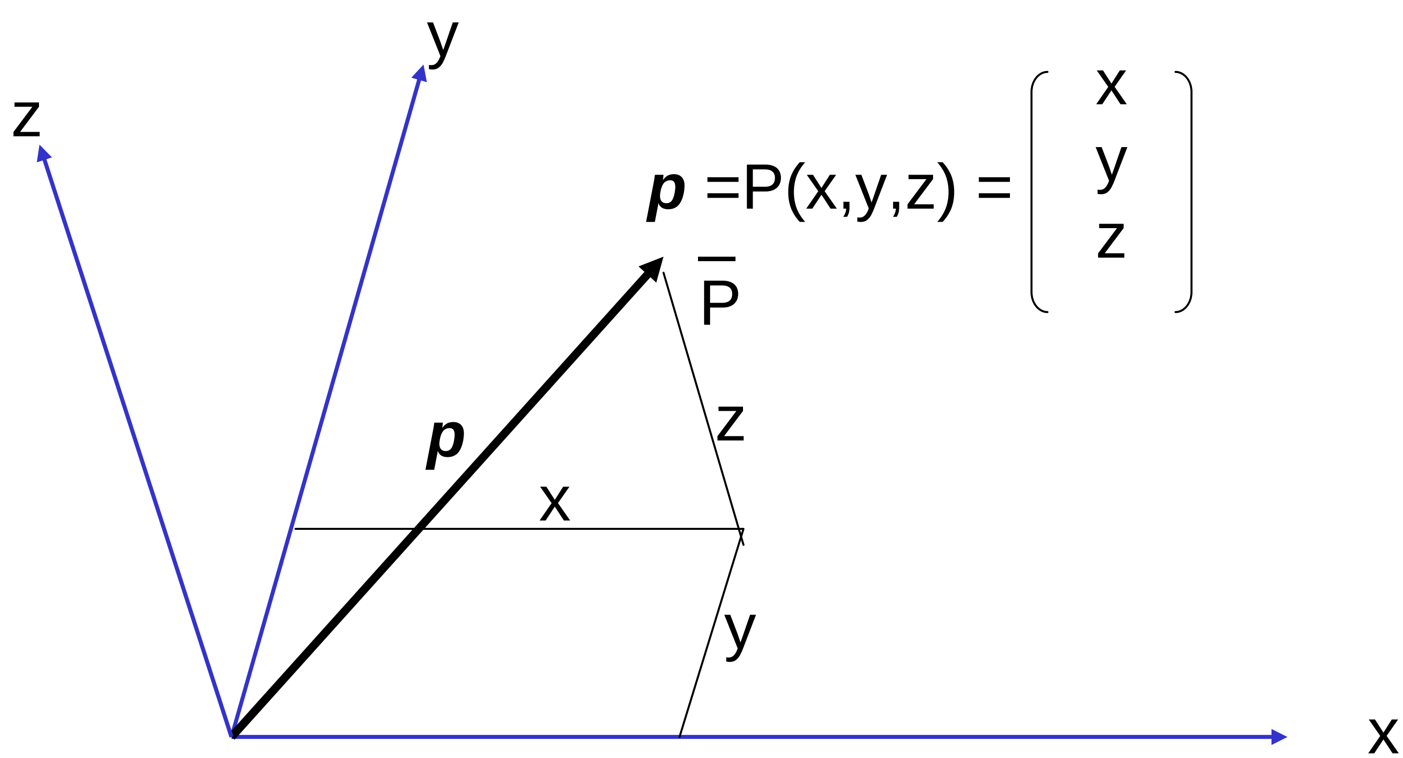
Basic Math – Vectors & Lines

Vector in 2D



length ($\boldsymbol{p}$) = | $\boldsymbol{p}$ | = sqrt(x^2 + y^2)

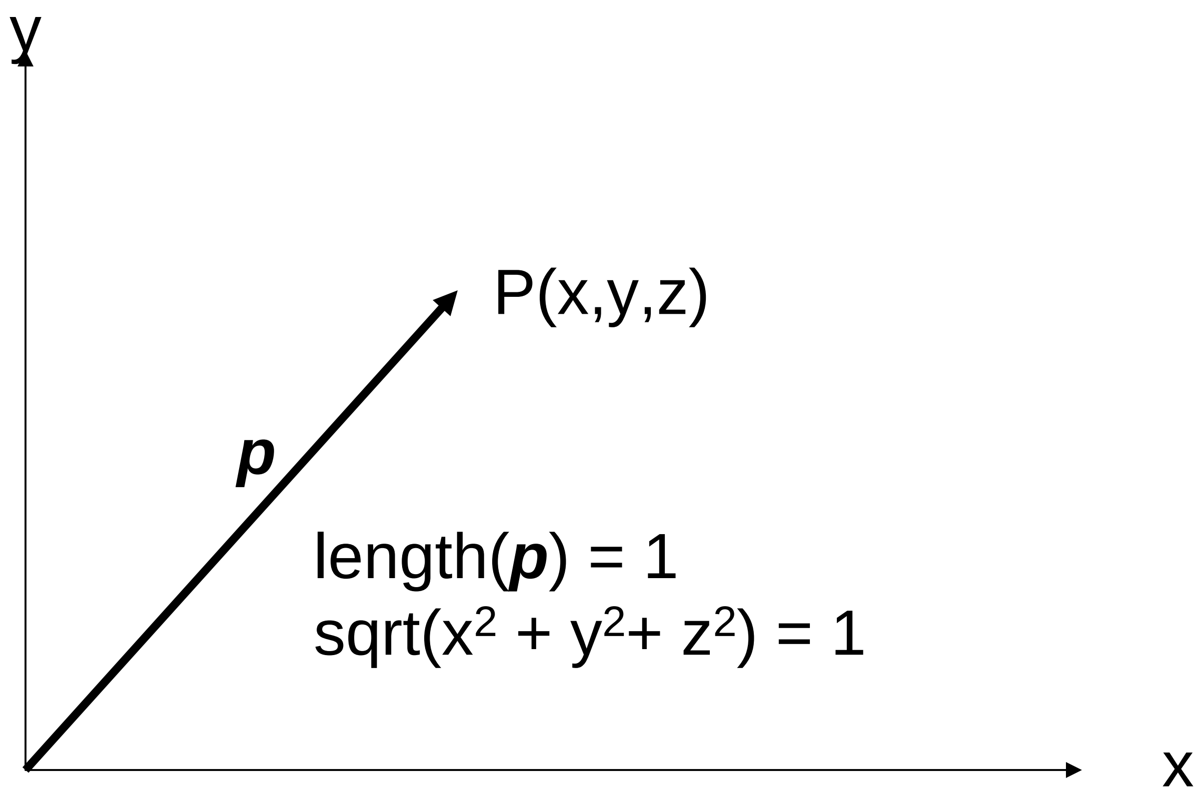
Vector in 3D



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

length ($\boldsymbol{p}$) = | $\boldsymbol{p}$ | = sqrt($x^2 + y^2 + z^2$)

Unit vector

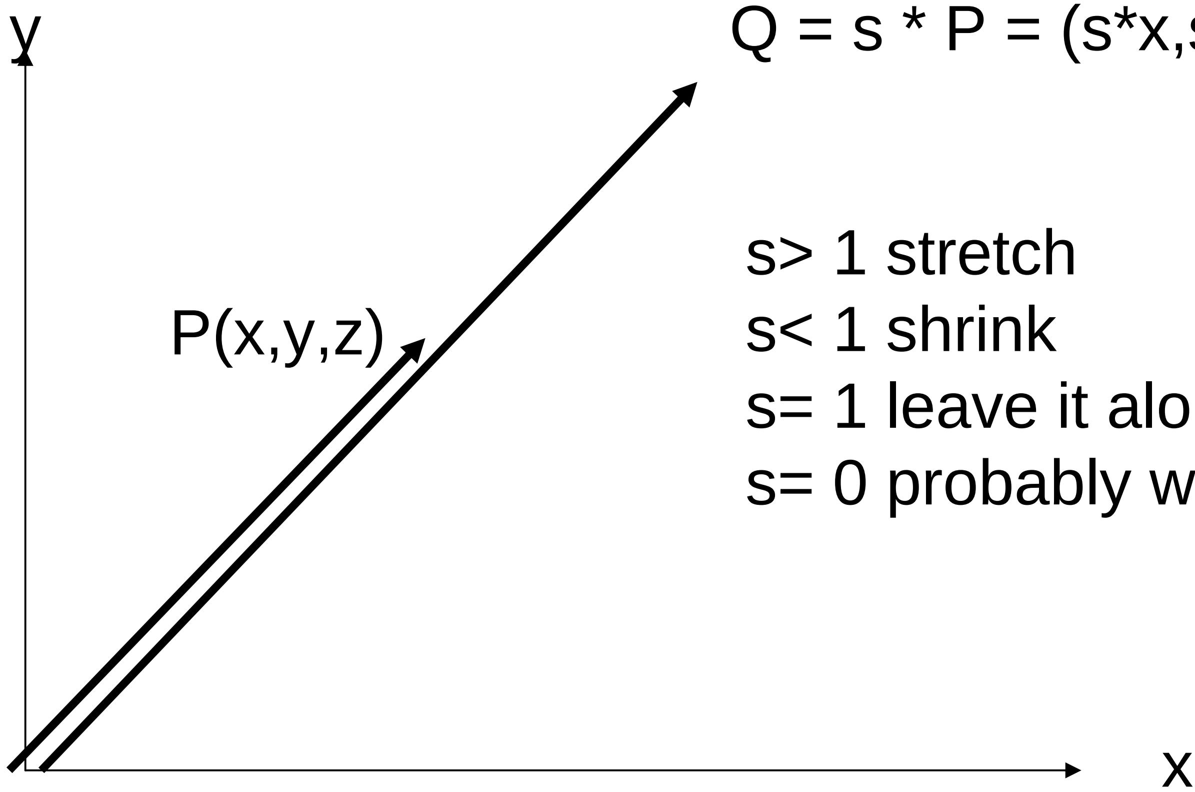


length(***p***) = 1

$\mathrm{sqrt}(x^2 + y^2 + z^2) = 1$

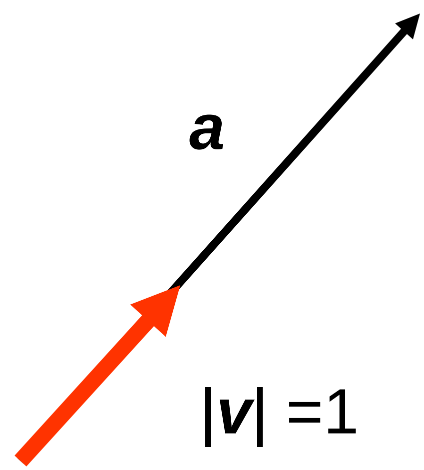
Scaling up/down a vector

$Q = s * P = (s*x, s*y, s*z)$



s> 1 stretch
s< 1 shrink
s= 1 leave it alone
s= 0 probably wrong…

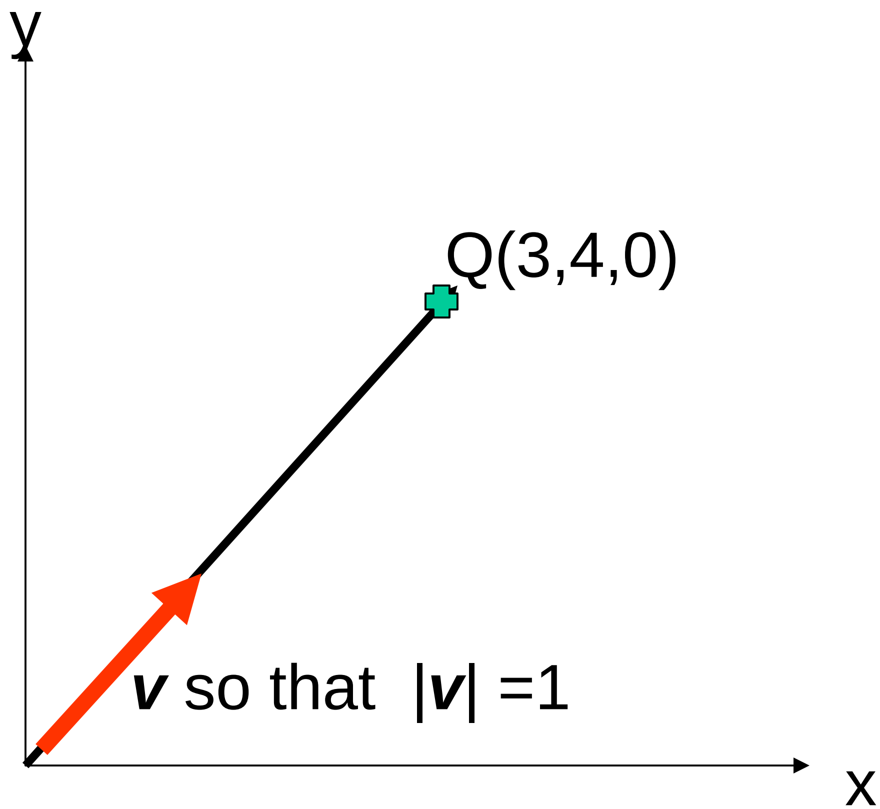
Create unit vector from a vector
a.k.a. “normalize” a vector



Scale down the vector by its own length:

$\boldsymbol{v} = \boldsymbol{a} \,/\, |\boldsymbol{a}|$ or $\boldsymbol{v} = \boldsymbol{a}\,/\mathrm{length}(\boldsymbol{a})$

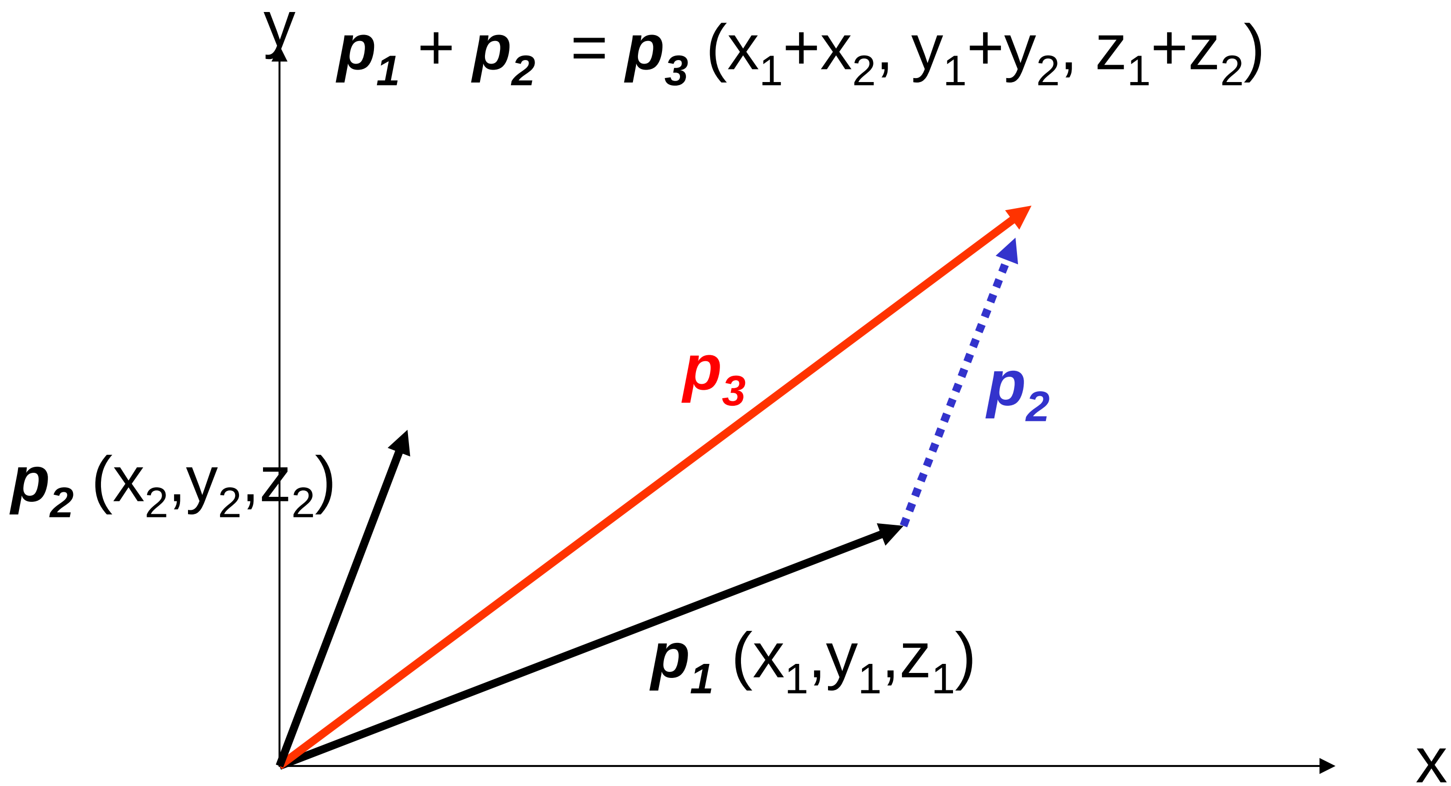
Create unit vector from a vector (Example)



length(Q) =sqrt(3*3 + 4*4+0*0) = sqrt(25) = 5

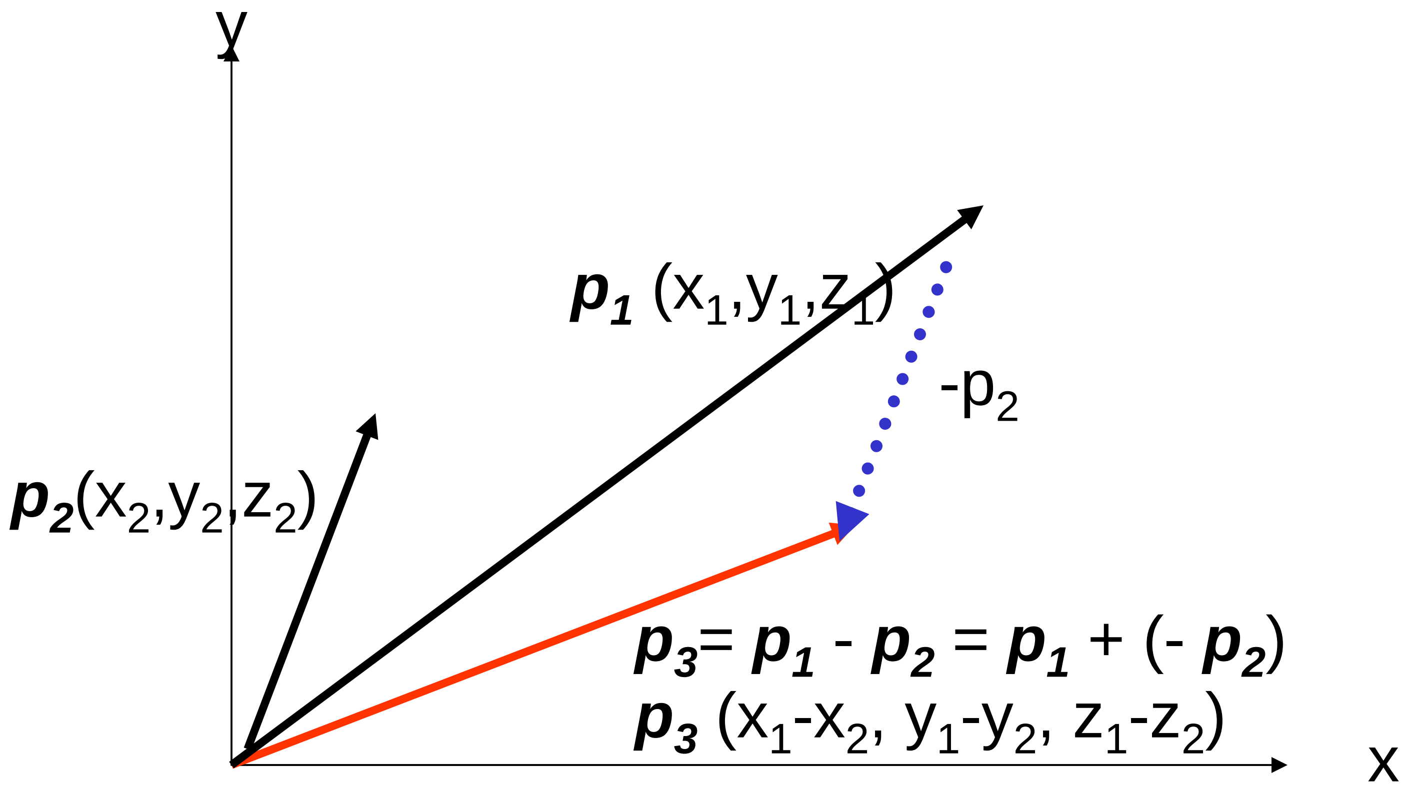
v = (3/5, 4/5, 0)

Sum of vectors



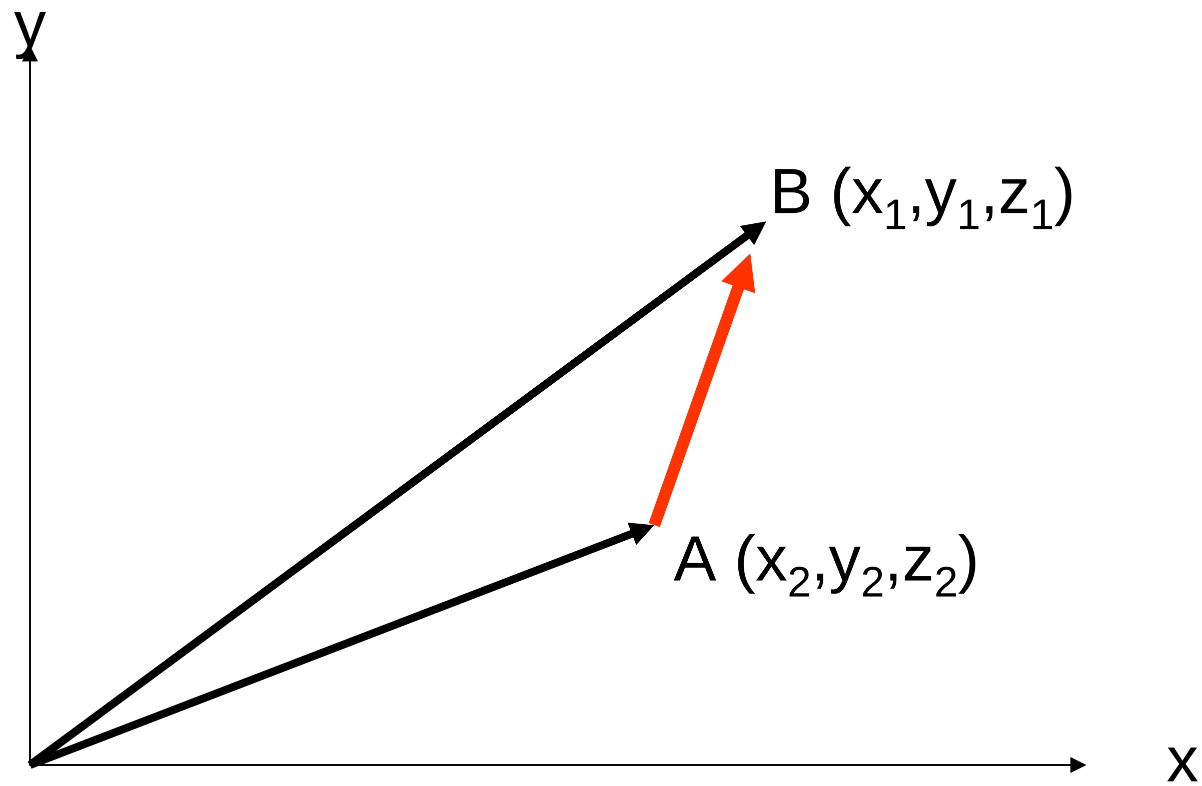
Catenate the vectors !

Subtraction of vectors



Reverse $\boldsymbol{p_2}$ and catenate to $\boldsymbol{p_1}$!

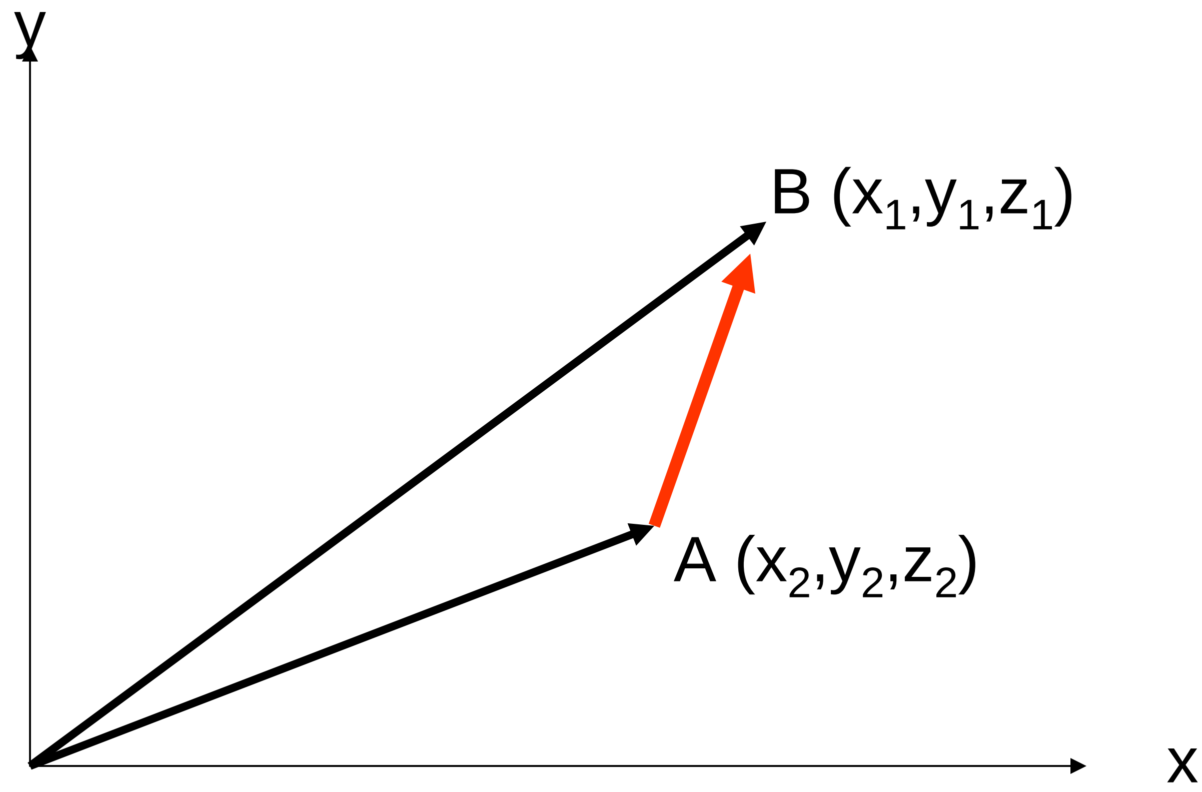
Vector from A to B



AB = B – A = $(x_1-x_2,\ y_1-y_2,\ z_1-z_2)$

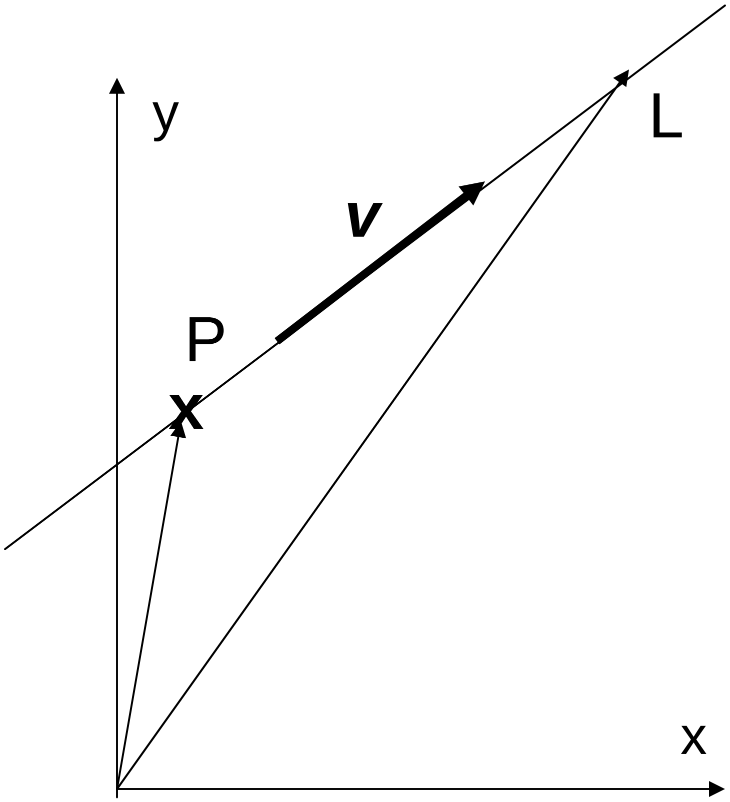
Subtract A from B

Distance between A to B



Distance = length(AB) = length(B – A) =
sqrt($(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2$)

Vector equation of a line



$L = P + t^*\boldsymbol{v}$

or broken to x,y,z components

$L_x = P_x + t^*v_x$

$L_y = P_y + t^*v_y$

$L_z = P_z + t^*v_z$

Infinite line:

t=(-inf,inf)

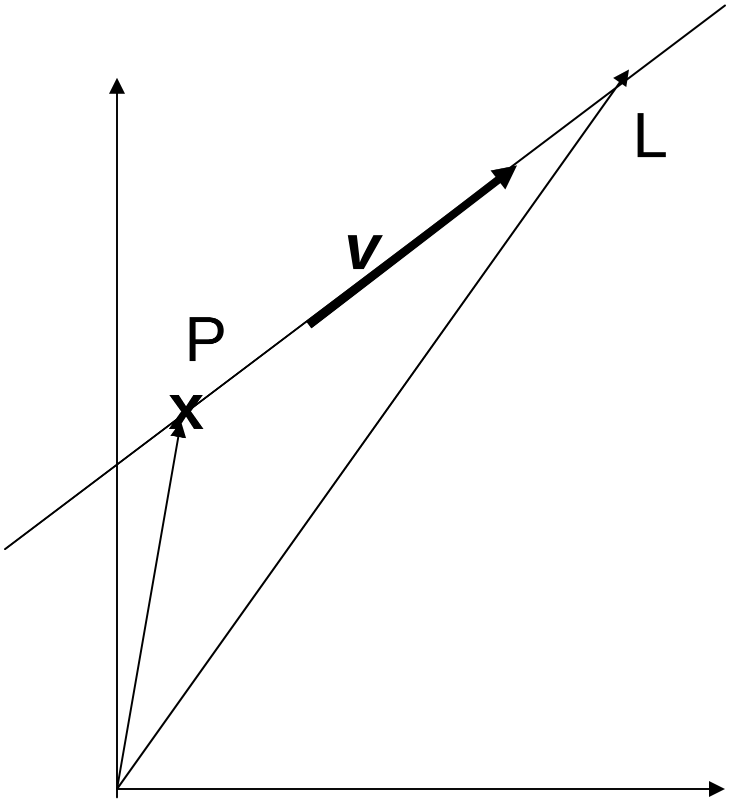
Ray:

t=(0,inf)

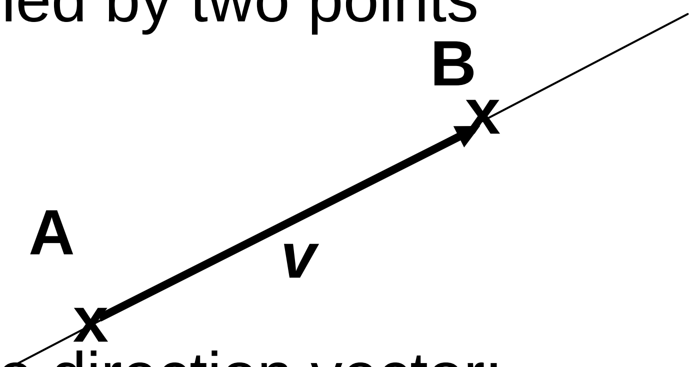
Segment:

$t=(t_{min}, t_{max})$

Create the direction vector of a line



Line defined by two points



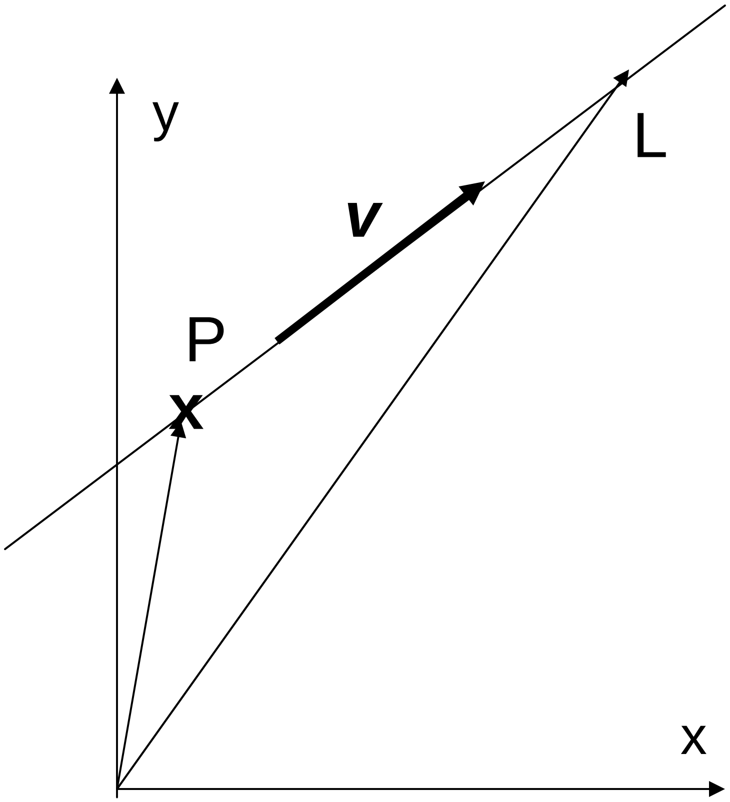
Obtain the direction vector:

AB = B – A   subtract

Optional:

$\boldsymbol{v}$= AB / |AB| Normalize – if necessary

Vector equation of a line



$L = P + t^*\boldsymbol{v}$

then

$L - P = t^*\boldsymbol{v}$

then

$|L - P| = t|\boldsymbol{v}|$

Now if $|\boldsymbol{v}| = 1$ (***v*** was normalized)

then

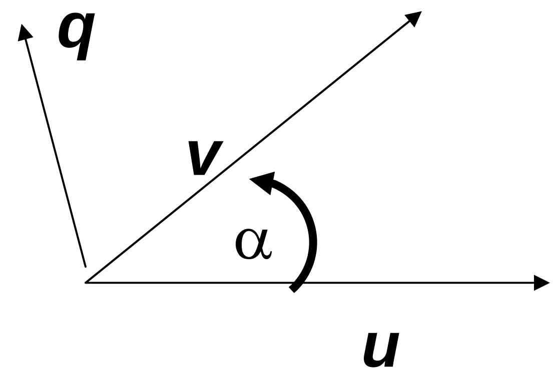
$|L - P| = t$

Cross product

Cross product (or vector product) of u and v is denoted as $\boldsymbol{q} = \boldsymbol{u} \times \boldsymbol{v}$, where ***u,v,q*** are 3D vectors denoted as $\boldsymbol{u}(x_1,y_1,z_1)$, $\boldsymbol{v}(x_2,y_2,z_2)$, $\boldsymbol{q}(x_3,y_3,z_3)$,

$|\boldsymbol{q}| = |\boldsymbol{u}| * |\underline{\mathbf{v}}| * \sin(\alpha)$

q is perpendicular to both ***u*** and ***v***, and



$x_3=y_1*z_2-y_2*z_1$

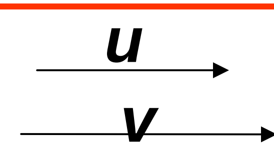
$y_3=x_2*z_1-x_1*z_2$

$z_3=x_1*y_2-x_2*y_1$

NOT COMMUTATIVE! ORDER MATTERS !

Cross product = 0 if and only if $\sin(\alpha)=0$

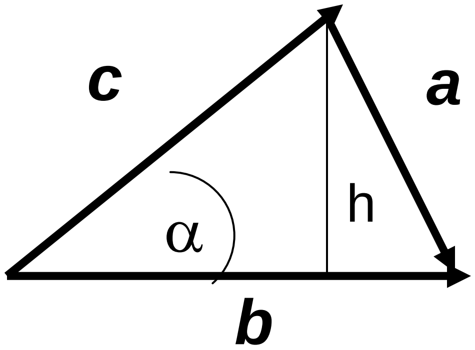
i.e. ***u*** and ***v*** are parallel



Area of a triangle

A= ½ |***b***|* h --- area of triangle
h=|***c***|*sin(α)

A= ½ |***b***|* |***c***| *sin(α)
A= ½ | ***b*** x ***c*** |



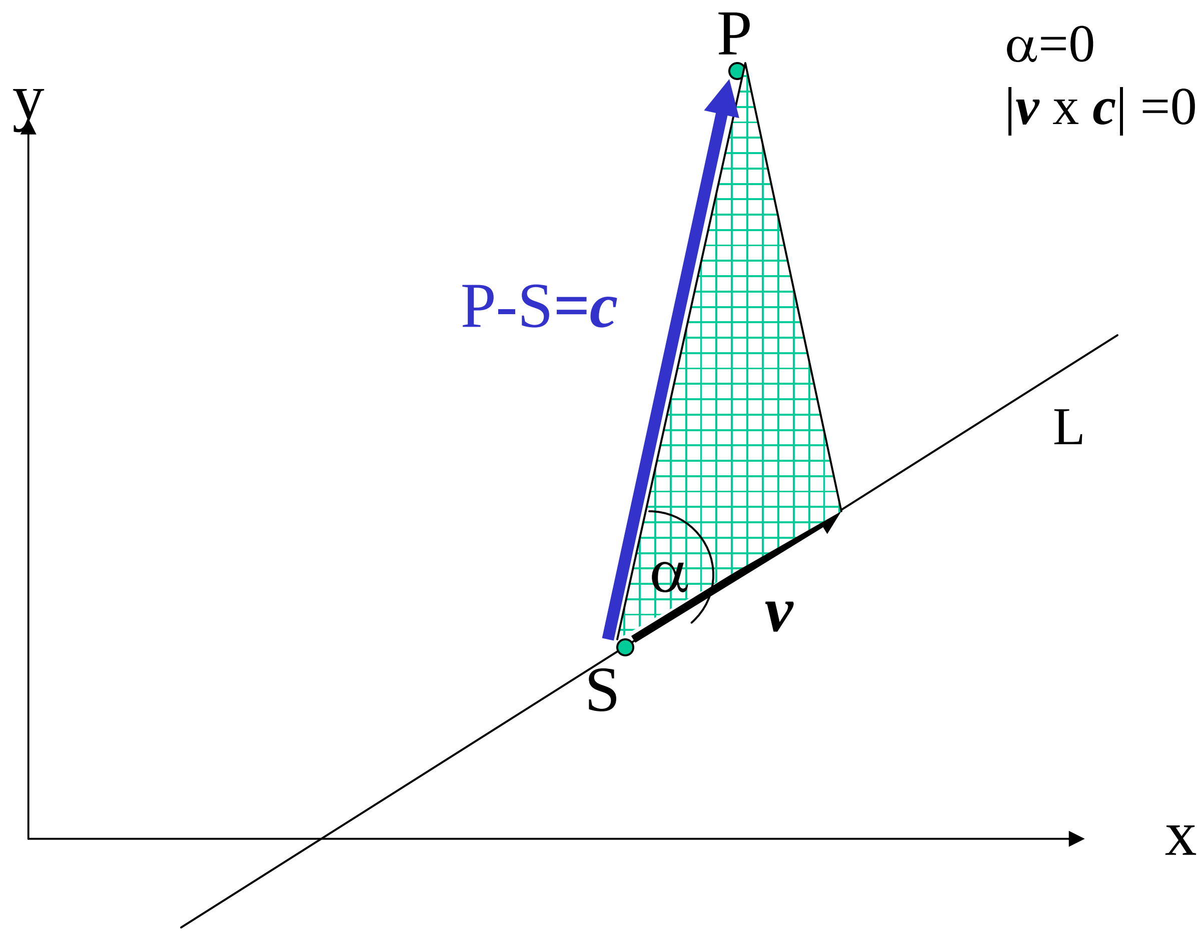
For non-zero b and c, the area is 0 if and only if sin(α) =0
i.e. c and b are parallel, so α=0

Is the point P on the line?

P is on the line i.o.i

$\alpha = 0$

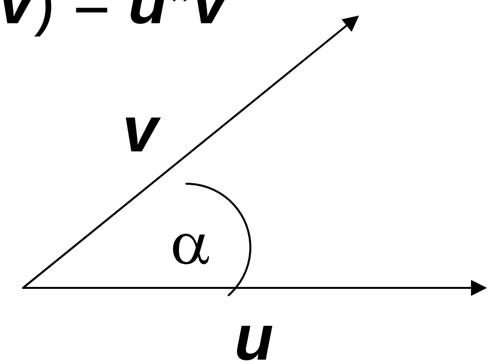
$|\boldsymbol{v} \times \boldsymbol{c}| = 0$



Dot product

Dot product (or scalar product) = dot(***u***,***v***) = ***u********v***

$\boldsymbol{u}(x_1,y_1,z_1)$ $\boldsymbol{v}(x_2,y_2,z_2)$



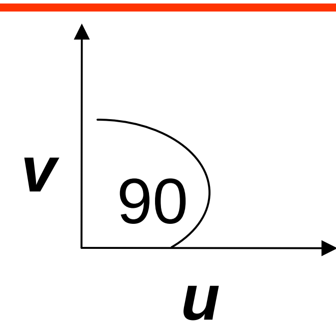
dot product = $\boldsymbol{u} * \boldsymbol{v} = |\boldsymbol{u}| * |\boldsymbol{v}| * \cos(\alpha) = (x_1x_2 + y_1y_2 + z_1z_2)$

Result is a scalar number, not a vector

It is commutative, so order does not matter

Dot product = 0 if and only if cos(α)=0,

i.e. ***u*** and ***v*** are perpendicular



Dot product and the length of a vector

Length = square root of the dot product with itself

$\boldsymbol{v}$=(x,y,z)

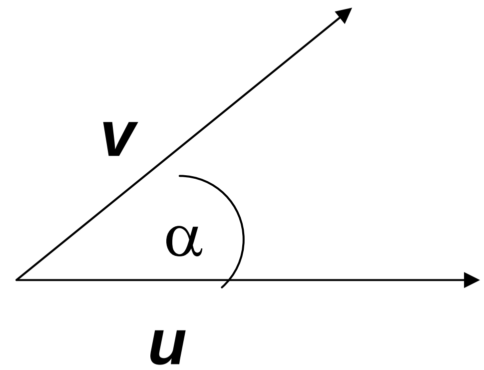
length($\boldsymbol{v}$) = sqrt($x^2 + y^2 + z^2$) = sqrt(dot($\boldsymbol{v}$,$\boldsymbol{v}$))

Some More Dot Product Facts

ab	$= ba$	commutative
$(ab)c$	$\neq a(bc)$	not associative
$(a + b)c$	$= ac + bc$	distributive with addition

Angle between vectors

dot($\boldsymbol{u}$,$\boldsymbol{v}$)=length($\boldsymbol{u}$) * length($\boldsymbol{v}$) * cos(α)
dot($\boldsymbol{u}$,$\boldsymbol{v}$)= $|\boldsymbol{u}|$ * $|\boldsymbol{v}|$ * cos(α)

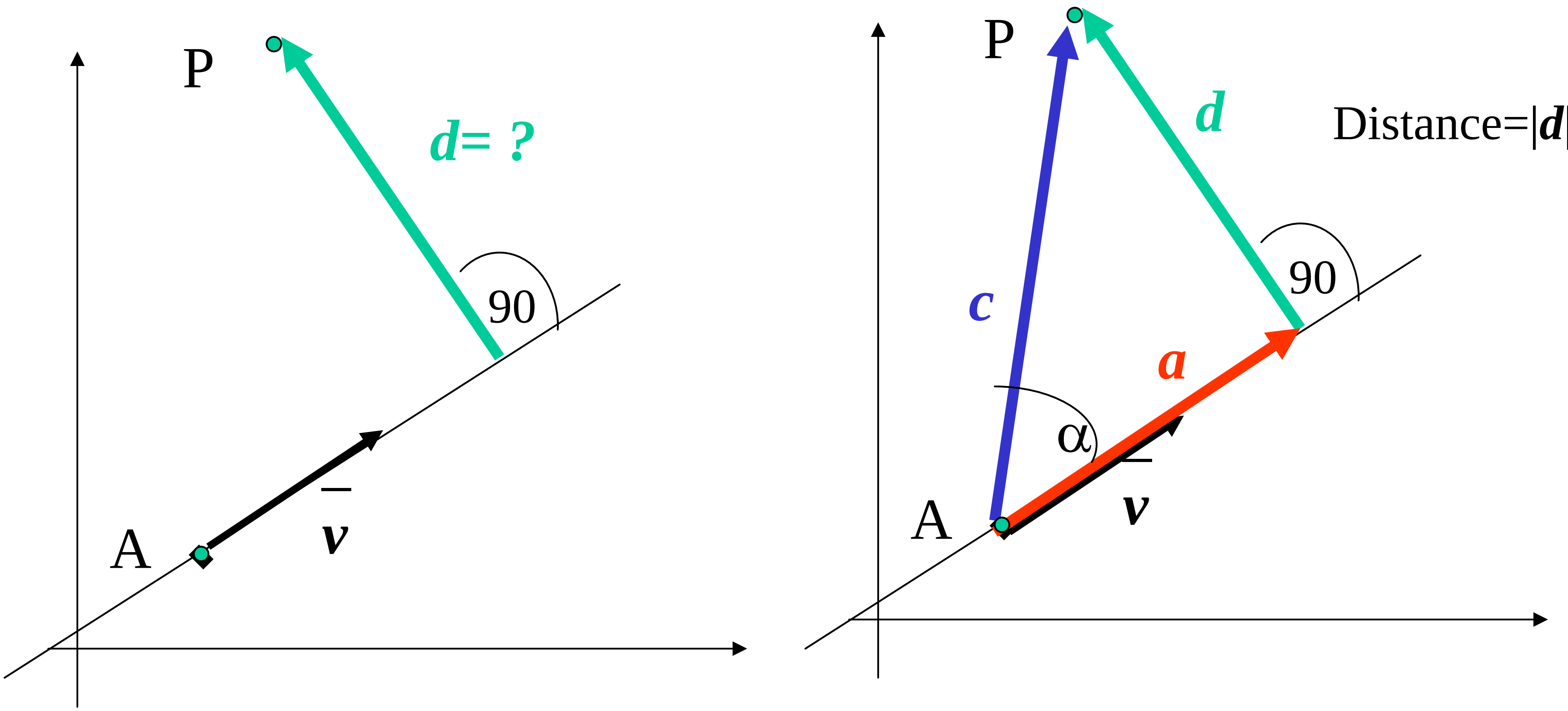


If $\boldsymbol{u}$ and $\boldsymbol{v}$ are unit vectors:

$|\boldsymbol{u}|$ =1 and $|\boldsymbol{v}|$ =1, so

dot($\boldsymbol{u}$,$\boldsymbol{v}$) =cos(α)

Distance of a point from a line



$\boldsymbol{v}$ is a known unit vector, so length($\boldsymbol{v}$) =1

$\boldsymbol{c}$=P-A -- this is a known vector

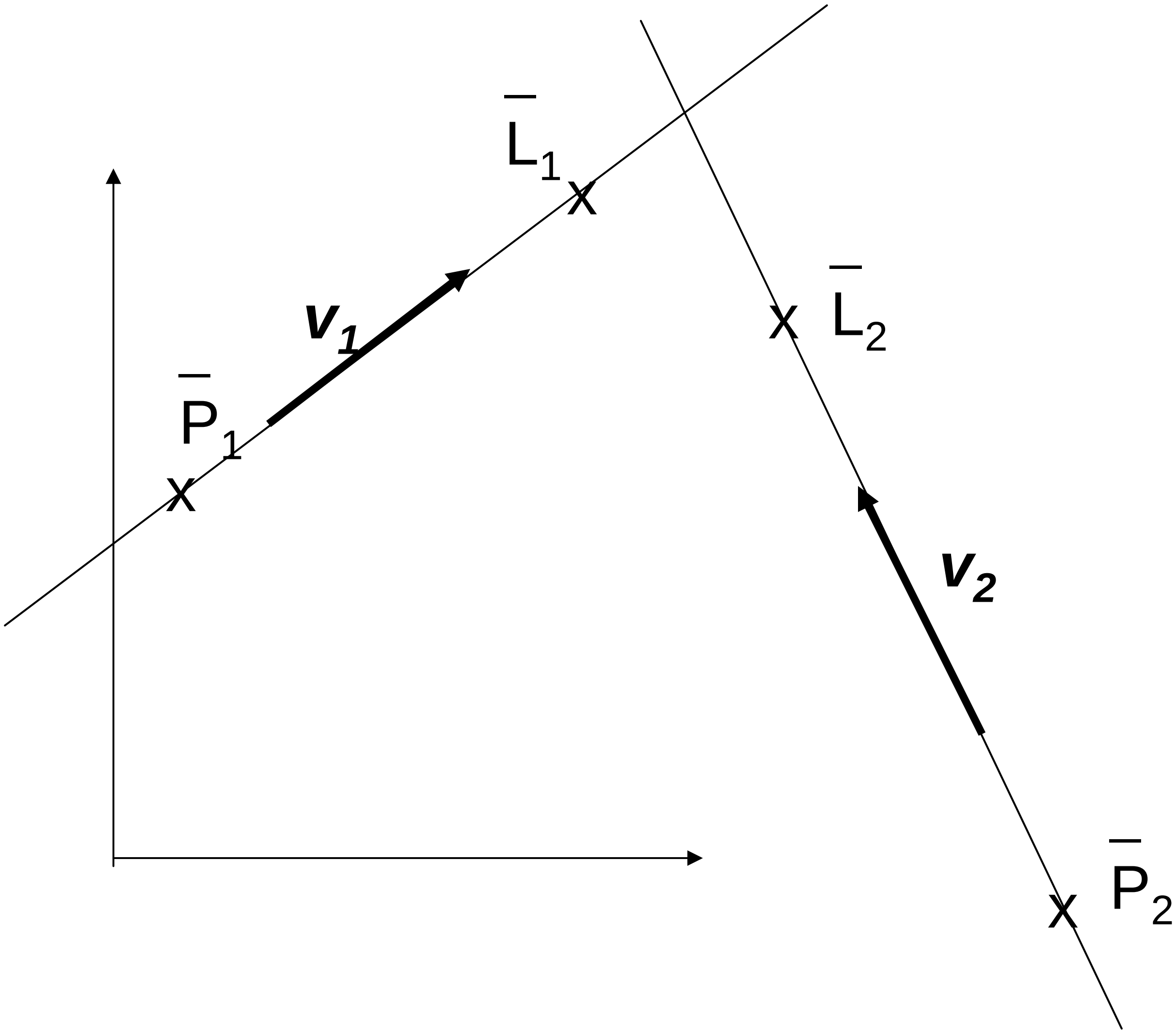
dot($\boldsymbol{v},\boldsymbol{c}$) = $\boldsymbol{vc}$ = $|\boldsymbol{v}| * |\boldsymbol{c}| * \cos(\alpha) = |\boldsymbol{c}| * \cos(\alpha) = |\boldsymbol{a}|$

$\boldsymbol{a} = \boldsymbol{v} * |\boldsymbol{a}| = \boldsymbol{v}(\boldsymbol{vc})$

$\boldsymbol{d} = \boldsymbol{c} - \boldsymbol{a} = \boldsymbol{c} - \boldsymbol{v}(\boldsymbol{vc})$

dist = $|\mathbf{d}| = |\boldsymbol{c} - \boldsymbol{v}(\boldsymbol{vc})|$ or $\mathbf{d}^2 = \boldsymbol{c}^2 - (\boldsymbol{vc})^2$

Intersection of 2 lines?



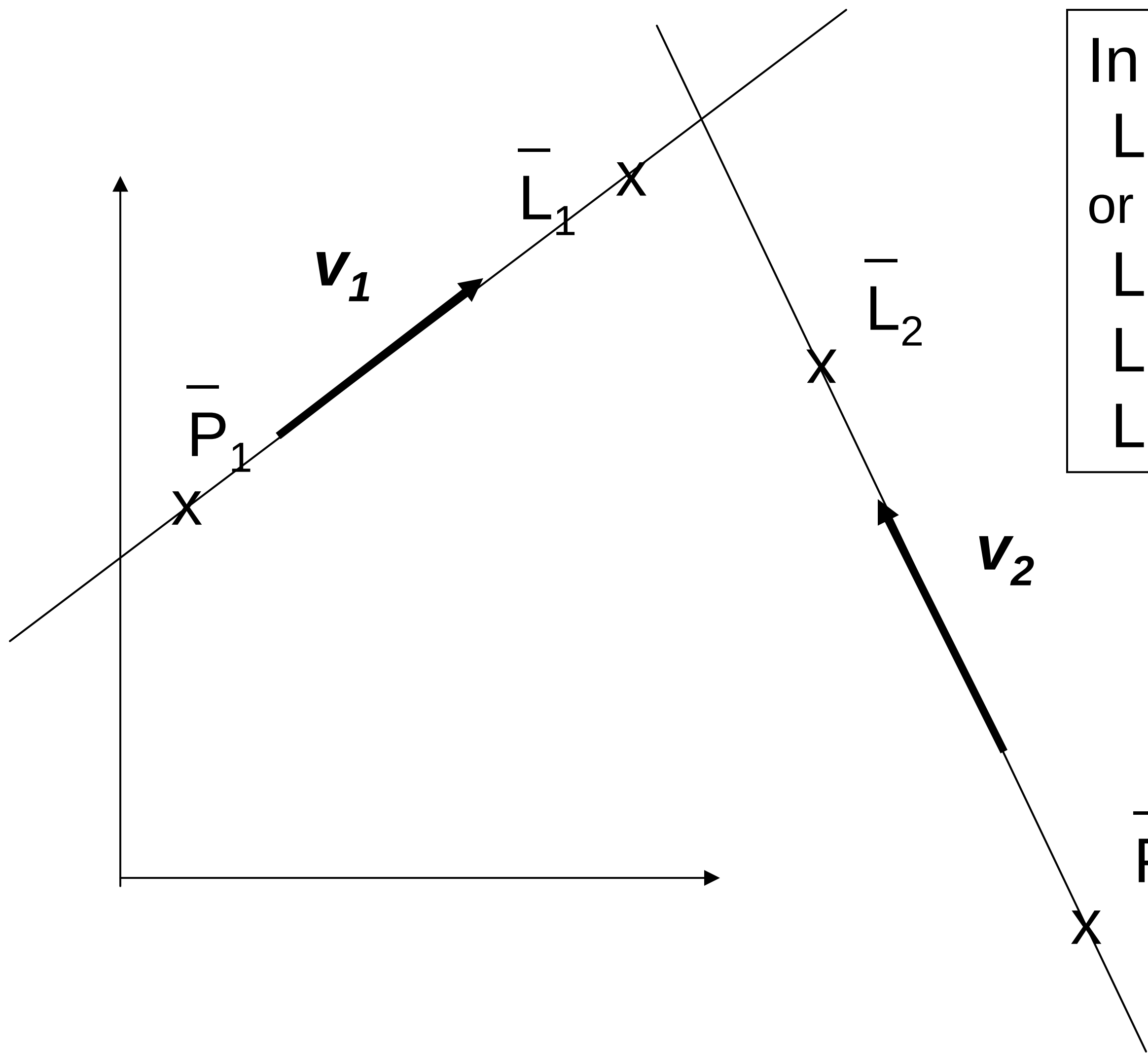
$L_1 = P_1 + t_1 * \boldsymbol{v_1}$
$L_2 = P_2 + t_2 * \boldsymbol{v_2}$

$L_{1x} = P_{1x} + t_1 * v_{1x}$
$L_{1y} = P_{1y} + t_1 * v_{1y}$
$L_{1z} = P_{1z} + t_1 * v_{1z}$

$L_{2x} = P_{2x} + t_2 * v_{2x}$
$L_{2y} = P_{2y} + t_2 * v_{2y}$
$L_{2z} = P_{2z} + t_2 * v_{2z}$

where
t=(-inf,inf)
u=(-inf,inf)

Intersection of 2 lines



In the intersection point :

$L_1=L_2$

or by component

$L_{1x} = L_{2x}$

$L_{1y} = L_{2y}$

$L_{1z} = L_{2z}$

Intersection of 2 lines – IMPOSSIBLE

In the intersection point :

$L_{1x} – L_{2x} = 0 = P_{1x} - P_{2x} + \mathbf{t_1} * v_{1x} - \mathbf{t_2} * v_{2x}$
$L_{1y} – L_{2y} = 0 = P_{1y} - P_{2y} + \mathbf{t_1} * v_{1y} - \mathbf{t_2} * v_{2y}$
$L_{1z} – L_{2z} = 0 = P_{1z} - P_{2z} + \mathbf{t_1} * v_{1z} - \mathbf{t_2} * v_{2z}$

where

$t=(-\inf,\inf)$

$u=(-\inf,\inf)$

Trouble: 3 eqs, 2 unknowns → no guaranteed solution.

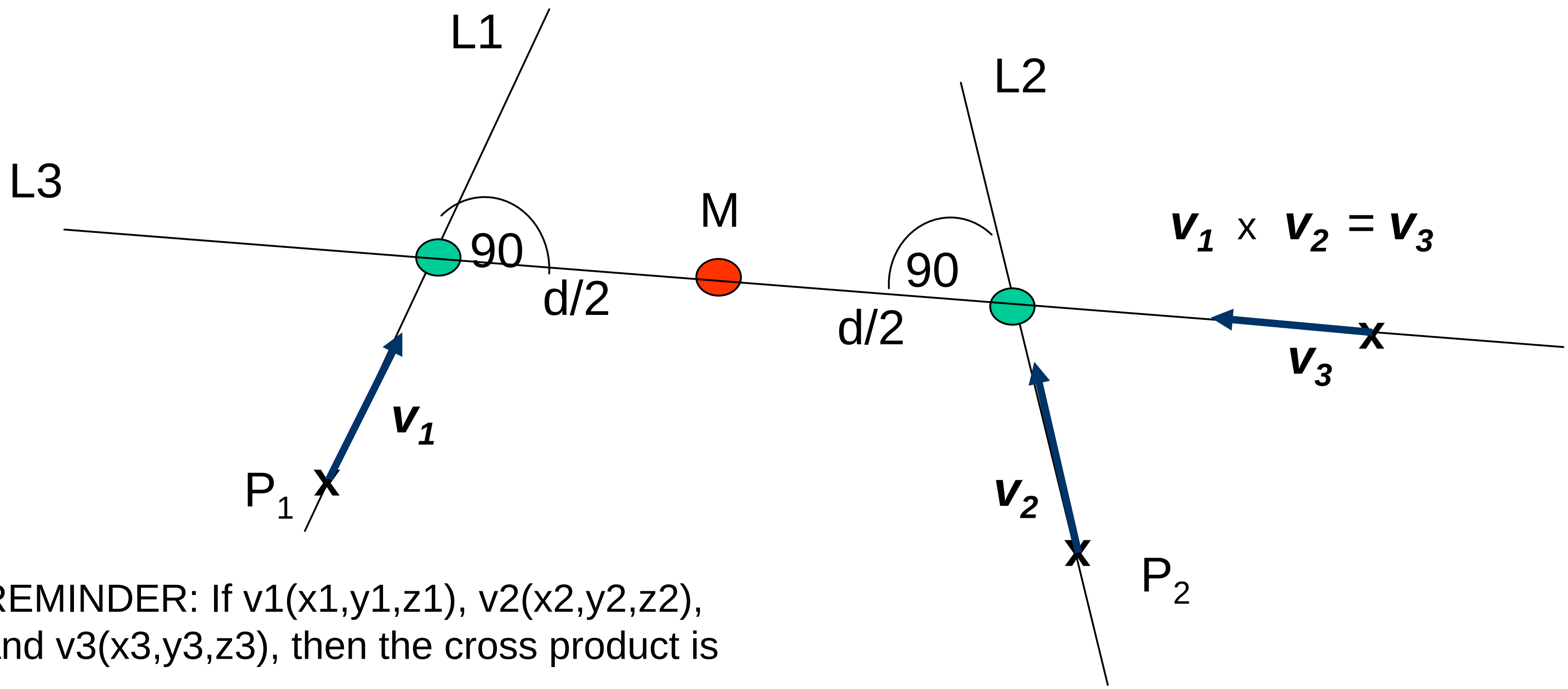
The lines might just intersect, but they do not have to.

When they intersect, one of the three eqs cancels out. In other words: a linear combination of any two gives the third one. (We have to be extremely lucky for this to happen.) Generally, two lines avoid each other.

We approximate: find the shortest distance between the two lines and find the point in midway.

Approximate Intersection of 2 lines

Intuition: Find the line that is perpendicular to both lines



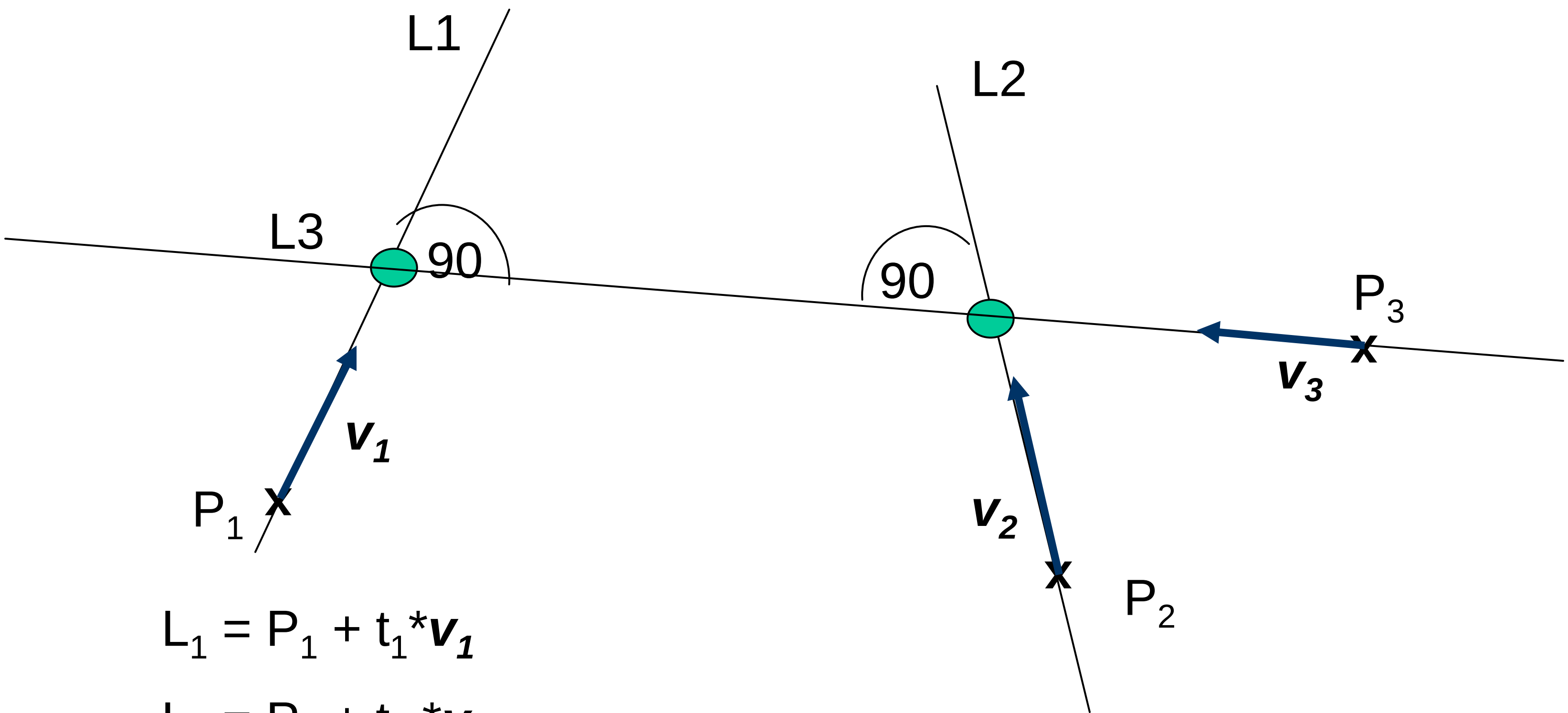
REMINDER: If v1(x1,y1,z1), v2(x2,y2,z2), and v3(x3,y3,z3), then the cross product is

x3=y1*z2-y2*z1

y3=x2*z1-x1*z2

z3=x1*y2-x2*y1

Write up the equation of each line

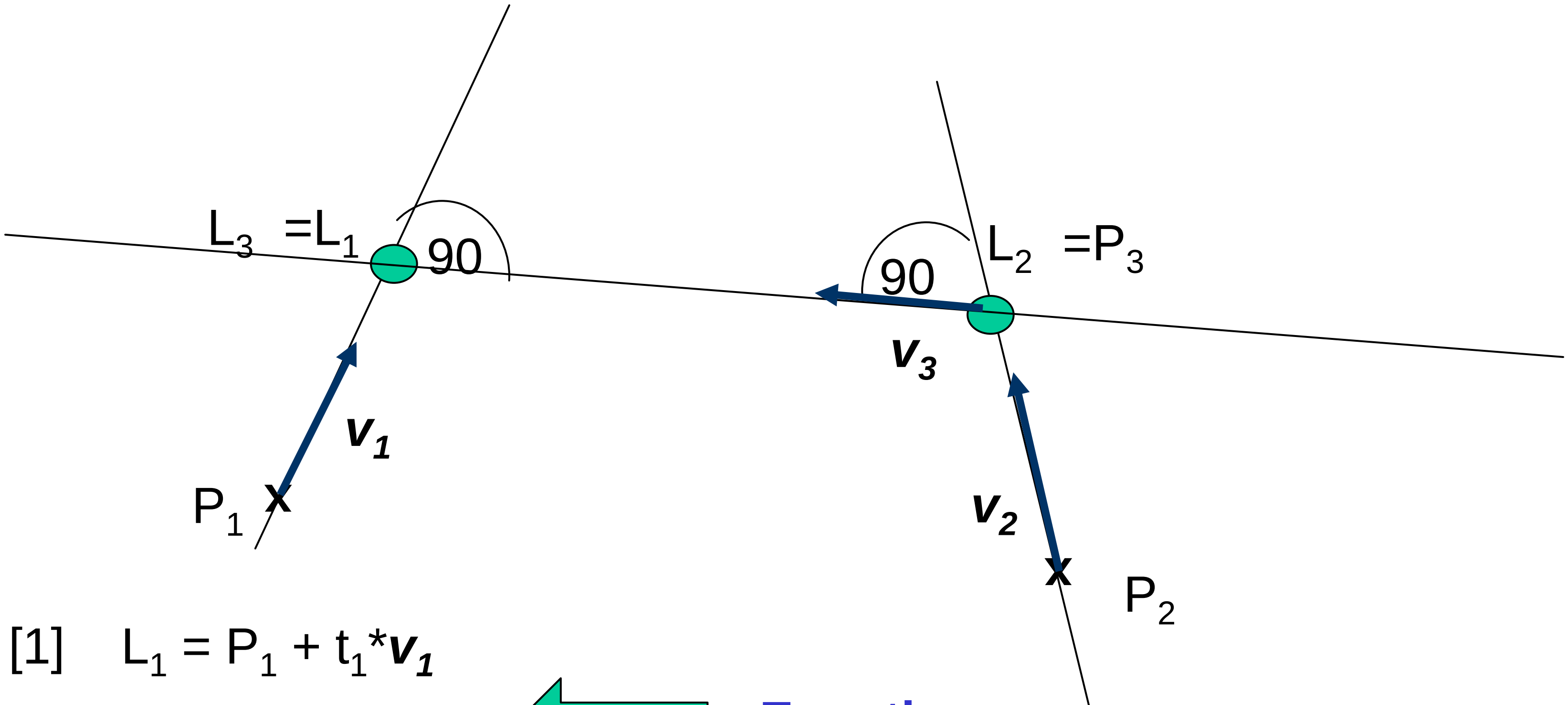


$L_1 = P_1 + t_1 * \boldsymbol{v_1}$

$L_2 = P_2 + t_2 * \boldsymbol{v_2}$

$L_3 = P_3 + t_3 * \boldsymbol{v_3}$

Derive conditions, make an equation system



[1] $L_1 = P_1 + t_1 * \boldsymbol{v_1}$

[2] $L_2 = P_2 + t_2 * \boldsymbol{v_2}$

[3] $L_3 = P_3 + t_3 * \boldsymbol{v_3}$, where

$P_3 = L_2$

$L_3 = L_1$

$\boldsymbol{v_1} \times \boldsymbol{v_2} = \boldsymbol{v_3}$

Equations

Conditions

Solve the vector equation system

[1] $L_1 = P_1 + t_1 * \boldsymbol{v_1}$

[2] $L_2 = P_2 + t_2 * \boldsymbol{v_2}$

[3] $L_3 = P_3 + t_3 * \boldsymbol{v_3}$

where

$P_3 = L_2$ -- replace P_3 in [3]

$L_3 = L_1$ -- replace L_3 in [3]

[1] $L_1 = P_1 + t_1 * \boldsymbol{v_1}$ *-- plug [1] in [3]*

[2] $L_2 = P_2 + t_2 * \boldsymbol{v_2}$ *-- plug [1] in [3]*

[3] $L_1 = L_2 + t_3 * \boldsymbol{v_3}$

$P_1 + t_1 * \boldsymbol{v_1} = P_2 + t_2 * \boldsymbol{v_2} + t_3 * \boldsymbol{v_3}$ *– arrange*

$P_1 - P_2 = -t_1 * \boldsymbol{v_1} + t_2 * \boldsymbol{v_2} + t_3 * \boldsymbol{v_3}$ *– break it up*

$P_{1x} - P_{2x} = -t_1 * v_{1x} + t_2 * v_{2x} + t_3 * v_{3x}$

$P_{1y} - P_{2y} = -t_1 * v_{1y} + t_2 * v_{2y} + t_3 * v_{3y}$

$P_{1z} - P_{2z} = -t_1 * v_{1z} + t_2 * v_{2z} + t_3 * v_{3z}$

- 3 linear equations
- 3 unknowns
- SOLVABLE

Solve the linear equations

Use any of the three methods:

1. Gaussian elimination
2. Substitution
3. Matrix inversion

$$\begin{pmatrix} P_{1x} - P_{2x} \\ P_{1y} - P_{2y} \\ P_{1z} - P_{2z} \end{pmatrix} = \begin{pmatrix} -v_{1x} & v_{2x} & v_{3x} \\ -v_{1y} & v_{2y} & v_{3y} \\ -v_{1z} & v_{2z} & v_{3z} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} -v_{1x} & v_{2x} & v_{3x} \\ -v_{1y} & v_{2y} & v_{3y} \\ -v_{1z} & v_{2z} & v_{3z} \end{pmatrix}^{-1} \begin{pmatrix} P_{1x} - P_{2x} \\ P_{1y} - P_{2y} \\ P_{1z} - P_{2z} \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}}$$

Plug t_1 and t_2 back into L_1 and L_2 line equations

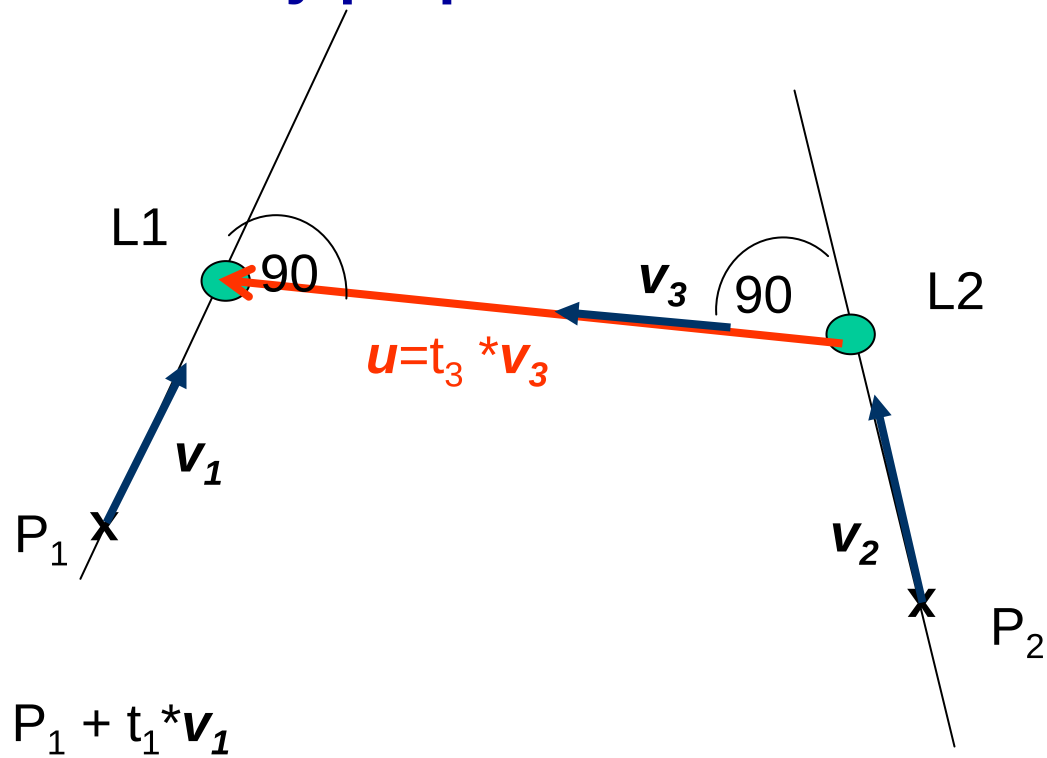
$$L_1 = P_1 + t_1 * \boldsymbol{v_1}$$

$$L_2 = P_2 + t_2 * \boldsymbol{v_2}$$

$$M = (L_1 + L_2)/2$$

$$d = \text{length}(L_1 - L_2)$$

Find the mutually perpendicular vector between L1 & L2



$L_1 = P_1 + t_1 * \boldsymbol{v_1}$

$L_2 = P_2 + t_2 * \boldsymbol{v_2}$

$\boldsymbol{v_3} = \boldsymbol{v_1} \times \boldsymbol{v_2} / |\boldsymbol{v_1} \times \boldsymbol{v_2}|$

$\boldsymbol{v_3}$ is normalized!!!

We look for the vector ***u*** that is perpendicular to both L1 and L2. This vector is expressed as $\boldsymbol{u}=t_3 * \boldsymbol{v_3}$. The length of it is t_3 and its direction is determined by the $\boldsymbol{v_3}$ unit direction vector. $\boldsymbol{v_3}$ is the cross product of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$, being perpendicular to both.

Find the mutually perpendicular vector between L1 & L2

$$L_1 - L_2 = t_3 * \boldsymbol{v_3}$$

$$\Downarrow$$

$$P_1 - P_2 + t_1 * \boldsymbol{v_1} - t_2 * \boldsymbol{v_2} = t_3 * \boldsymbol{v_3}$$

$$\Downarrow$$

NICE TRICK !!!

$$P_1 - P_2 = -t_1 * \boldsymbol{v_1} + t_2 * \boldsymbol{v_2} + t_3 * \boldsymbol{v_3}$$

– dot produce by $\boldsymbol{v_3}$

$$\Downarrow$$

$$(P_1 - P_2)\, \boldsymbol{v_3} = -t_1 * \underbrace{\boldsymbol{v_1 v_3}}_{0} + t_2 * \underbrace{\boldsymbol{v_2 v3}}_{0} + t_3 * \underbrace{\boldsymbol{v_3 v_3}}_{1}$$

$$\Downarrow$$

$$\mathbf{(P_1 - P_2)\, \boldsymbol{v_3} = t_3}$$ ☺

Use the same "dot product trick" for t_1 and t_2

$$P_1 - P_2 = -t_1 * \boldsymbol{v_1} + t_2 * \boldsymbol{v_2} + t_3 * \boldsymbol{v_3}$$

(1) $$(P_1 - P_2)\,\boldsymbol{v_1} = -t_1 * \underbrace{\boldsymbol{v_1}\,\boldsymbol{v_1}}_{1} + t_2 * \boldsymbol{v_1}\,\boldsymbol{v_2} + t_3 * \underbrace{\boldsymbol{v_1}\,\boldsymbol{v_3}}_{0}$$ *if dot produce by* $\boldsymbol{v_1}$

(2) $$(P_1 - P_2)\,\boldsymbol{v_2} = -t_1 * \boldsymbol{v_2}\,\boldsymbol{v_1} + t_2 * \underbrace{\boldsymbol{v_2}\,\boldsymbol{v_2}}_{1} + t_3 * \underbrace{\boldsymbol{v_2}\,\boldsymbol{v_3}}_{0}$$ *if dot produce by* $\boldsymbol{v_2}$

We recognize three dot products of vectors:

$(P_1 - P_2)\, v_1$ and $(P_1 - P_2)\, v_1$ and $v_1 v_2$

REMEMBER: dot product is a scalar number

Let a_1, a_2 and d denote those scalars..

(1) $$\underbrace{(P_1 - P_2)\,\boldsymbol{v_1}}_{a_1} = -t_1 * \underbrace{\boldsymbol{v_1}\,\boldsymbol{v_1}}_{1} + t_2 * \underbrace{\boldsymbol{v_1}\,\boldsymbol{v_2}}_{d} + t_3 * \underbrace{\boldsymbol{v_1}\,\boldsymbol{v_3}}_{0}$$

(2) $$\underbrace{(P_1 - P_2)\,\boldsymbol{v_2}}_{a_2} = -t_1 * \underbrace{\boldsymbol{v_1}\,\boldsymbol{v_2}}_{d} + t_2 * \underbrace{\boldsymbol{v_2}\,\boldsymbol{v_2}}_{1} + t_3 * \underbrace{\boldsymbol{v_2}\,\boldsymbol{v_3}}_{0}$$

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(1) $a_1 = -t_1 + t_2 d$

(2) $a_2 = -t_1 d + t_2$

This solves easily for t1 and t2.

$t_1 = (a_2 d - a_1)\,(1-d^2)$

$t_2 = -(a_1 d - a_2)\,(1-d^2)$

$\boldsymbol{t_3 = (P_1 - P_2)\, v_3}$

☺

Plug t_1 and t_2 back into L_1 and L_2 line equations

$L_1 = P_1 + t_1 * \boldsymbol{v_1}$

$L_2 = P_2 + t_2 * \boldsymbol{v_2}$

The mid point is $M = (L_1 + L_2)/2$

The distance between L_1 and L_2 is t_3