1. Let R_1 be a relation on the set of Integers such that $(a, b) \in R_1$ if $b = a + 2$. For each of the following questions answer true or false.

   (a) (1) R_1 is reflexive. F
   (b) (1) R_1 is transitive. F
   (c) (1) R_1 is an onto function. T

2. Let R_2 be a relation on the set of Natural numbers such that $(a, b) \in R_2$ if $b = a + 2$. For each of the following questions answer true or false.

   (a) (1) R_2 is symmetric. F
   (b) (1) R_2 is antisymmetric. T
   (c) (1) R_2 is an onto function. F

3. Let $S = \{1, 2, 3, ..., n\}$ with $P(S)$ the powerset of S. Consider a relation $R_4 \subseteq P(S) \times P(S)$ such that $(X, Y) \in R_4$ if $X \subset Y$. For each of the following questions answer true or false.

   (a) (1) R_4 is reflexive. F
   (b) (1) R_4 is symmetric. F
   (c) (1) R_4 is antisymmetric. T
   (d) (1) R_4 is transitive. T

4. For each pair of integers a and b use the division algorithm theorem to find integers q and r such that $a = bq + r$.

   (a) (1) $a = 34, b = 11$

   $34 = (3)11 + 1$

   $q = 3 \quad r = 1$

   (b) (1) $a = -34, b = 11$

   $-34 = (-4)11 + 10$

   $q = -4 \quad r = 10$

5. An example of a relation R on the set $\{1, 2, 3\}$ is R=$\{(1,2), (1,3)\}$. Given an example of a relation R on the set $\{1, 2, 3\}$, such that:

   (a) (2) R is both symmetric and antisymmetric.

   $\emptyset$

   (b) (2) R is neither symmetric nor antisymmetric.

   $\{(1,2), (2,1), (1,3)\}$

   (c) (2) R is reflexive, symmetric, antisymmetric and transitive.

   $\{(1,1)\ (2,2)\ (3,3)\}$

   (d) (2) R is not reflexive, not symmetric, not antisymmetric, and is transitive.

   $\{(1,2), (2,1), (1,1)\ (1,3), (2,3), (2,2)\}$

6. (4) Use Euclid's algorithm to find gcd(540,102). Show all of the steps.

$540 = (5)\,102 + 30$

$102 = (3)\,30 + 12$

$30 = (2)\,12 + 6$

$12 = (2)\,6 + 0$

$\gcd(540, 102)$
$= \gcd(102, 30)$
$= \gcd(30, 12)$
$= \gcd(12, 6)$
$= \gcd(6, 0)$
$= 6$

7. (3) Let a, b be positive integers. Use the division algorithm theorem to prove that if $a|b$ and $b|a$ then $a = b$.

if $a|b$ then $b = pa$ $p \in \mathbb{Z}$
If $b|a$ then $a = qb$ $q \in \mathbb{Z}$
Therefore $b = pqb$ and $p = q = 1$
so $b = a$

8. (3) Let $a, b, c, d \in \mathbb{Z}$. Prove, using the division algorithm theorem, that if $a|b$ and $c|d$, then $a|bd$.

if $a|b$ then $b = pa$ $p \in \mathbb{Z}$
$d \in \mathbb{Z}$

$bd = pad$ and $a | apd$
so $a | bd$.

9. (5) Consider a recursive function defined as:

$$A(1) = 1, A(2) = 2, A(n) = 2A(n-1) - A(n-2) \text{ for } n \geq 3.$$

Use the second form of mathematical induction to prove that $A(n) = n$ for all $n \in \mathbb{N}$.

Base $A(1) = 1,\ A(2) = 2$

Ind. Hyp. $A(j) = j$ for $j \in \mathbb{N}$ $1 \leq j \leq k$

Ind. Step. $A(k+1) = 2A(k) - A(k-1)$

$= 2k - (k-1)$

$= k+1$ $\boxed{\mathbb{Z}}$

10. (5) Consider the recursive function

$$F(1) = 1, F(2) = 2, F(n) = F(n-1) + F(n-2) \text{ for } n \geq 3.$$

Use the second form of mathematical induction to prove that $F(n) < 2^n$, for all $n \in \mathbb{N}$.

Base: $F(1) = 1 < 2^1$, $F(2) = 2 < 2^2$

Ind Hyp: $F(j) < 2^j$ $j \in \mathbb{N}$, $1 \leq j \leq k$

Ind. Step. $F(k+1) = F(k) + f(k-1)$

$< 2^k + 2^{k-1}$

$< 2^k + 2^k$

$= 2^{k+1}$ $\boxed{\mathbb{Z}}$