CISC-102 WINTER 2020

HOMEWORK 3
SOLUTIONS

(1) Mathematical induction can be used to prove that the sum of the first n natural numbers is equal to $\frac{n(n+1)}{2}$. This can also be stated as:

We can prove that the proposition P(n),

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

is true for all $n \in \mathbb{N}$, by using mathematical induction.

I wrote out the proof, but somehow it got all scrambled as shown below. Rearrange the lines to get the correct proof.

1. **Induction step:** The goal is to show that P($k+1$) is true.

2. **Base:** for n =1, $1 = \frac{1(1+1)}{2}$

3. $= \frac{(k+1)(k+2)}{2}$

4. $\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$

5. $= \frac{k^2+k+2k+2}{2}$

6. **Induction hypothesis:** Assume that P(k), for Natural numbers $k \geq 1$ is true, that is:

7. $= \frac{k^2+3k+2}{2}$

8. $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

9. $= \frac{k(k+1)}{2} + (k+1)$

The correct order is:

2.
6.
8.
1.
4.
9.
5.
7.
3.

I will now spell it out for easier readability.

2. **Base:** for n =1, $1 = \frac{1(1+1)}{2}$

6. **Induction hypothesis:** Assume that P(k), for Natural numbers $k \geq 1$ is true, that is:

8. $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

1. **Induction step:** The goal is to show that P($k+1$) is true.

4. $\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$

 9. $= \frac{k(k+1)}{2} + (k+1)$

 5. $= \frac{k^2+k+2k+2}{2}$

 7. $= \frac{k^2+3k+2}{2}$

 3. $= \frac{(k+1)(k+2)}{2}$

(2) Prove using mathematical induction that the proposition P(n),

$$\sum_{i=2}^{n} i = \frac{(n-1)(n+2)}{2}$$

is true for all $n \in \mathbb{N}, n \geq 2$

Base: for $n = 2$, $2 = \frac{1(2+2)}{2}$

Induction hypothesis: Assume that P(k) is true, that is:

$$\sum_{i=2}^{k} i = \frac{(k-1)(k+2)}{2}.$$

for $k \geq 2$.

Induction step: The goal is to show that P($k+1$) is true, that is:

$$\sum_{i=2}^{k+1} i = \frac{(k)(k+3)}{2}.$$

Consider the sum

$$\begin{aligned}
\sum_{i=2}^{k+1} i &= \sum_{i=2}^{k} i + (k+1) \text{(arithmetic)} \\
&= \frac{(k-1)(k+2)}{2} + (k+1) \text{(Use the induction hypothesis)} \\
&= \frac{k^2 + k - 2 + 2k + 2}{2} \text{(get common denominator and add)} \\
&= \frac{k^2 + 3k}{2} \text{(arithmetic)} \\
&= \frac{k(k+3)}{2} \text{(factor to arrive at goal)}
\end{aligned}$$

We have shown that P(k) true implies that P($k+1$) is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}, n \geq 2$. □

(3) Prove using mathematical induction that the proposition P(n),

$$\sum_{i=3}^{n} i = \frac{(n-2)(n+3)}{2}$$

is true for all $n \in \mathbb{N}, n \geq 3$

Base: for $n = 3$, $3 = \frac{(3-2)(3+3)}{2}$

Induction hypothesis: Assume that P(k) is true, that is:

$$\sum_{i=3}^{k} i = \frac{(k-2)(k+3)}{2}.$$

for $k \geq 3$.

Induction step: The goal is to show that P($k+1$) is true, that is:

$$\sum_{i=3}^{k+1} i = \frac{(k-1)(k+4)}{2}.$$

Consider the sum

$$\begin{aligned}\sum_{i=3}^{k+1} i &= \sum_{i=3}^{k} i + (k+1) \text{(arithmetic)} \\ &= \frac{(k-2)(k+3)}{2} + (k+1) \text{(Use the induction hypothesis)} \\ &= \frac{k^2+k-6+2k+2}{2} \text{(get common denominator and add)} \\ &= \frac{k^2+3k-4}{2} \text{(arithmetic)} \\ &= \frac{(k-1)(k+4)}{2} \text{(factor to arrive at goal)}\end{aligned}$$

We have shown that P(k) true implies that P($k+1$) is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}, n \geq 3$. □

(4) Prove using mathematical induction that the proposition P(n)

$$n! \leq n^n$$

is true for all $n \in \mathbb{N}$.

Base: for n =1, $1! = 1 = 1^1$

Induction hypothesis: Assume that P(k) is true, that is:

$$k! \leq k^k$$

for $k \geq 1$.

Induction step: The goal is to show that P$(k+1)$ is true, that is:

$$(k+1)! \leq (k+1)^{k+1}.$$

We have:

$$\begin{aligned}(k+1)! =& k!(k+1) \text{(Definition of factorial)}\\ \leq& k^k(k+1) \text{(Use the induction hypothesis)}\\ \leq& (k+1)^k(k+1) \text{(because } k \leq k+1\text{)}\\ =& (k+1)^{k+1} \text{(multiply)}\end{aligned}$$

We have shown that P(k) true implies that P$(k+1)$ is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}$. □

(5) Given a set of n points on a two dimensional plane, such that no three points are on the same line, it is always possible to connect every pair of points with a line segment. The figure illustrates this showing 5 points, that are pairwise connected with 10 line segments. Prove using mathematical induction that the total number of line segments is $\frac{n(n-1)}{2}$ for any number of points $n \in \mathbb{N}, n \geq 2$.

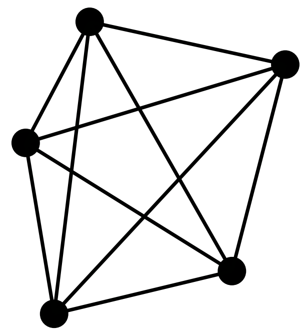

FIGURE 1. Five points, pairwise connected with 10 line segments.

Base: Given two points there is exactly one segment that connects them.

Induction Hypothesis: Assume that k points can be connected by $\frac{k(k-1)}{2}$ line segments for some fixed natural numer $k, k \geq 2$.

Induction Step: Consider $k+1$ points. We can partition the points into two subsets with k points in one and a single point in the other. The induction hypothesis implies that there are $\frac{k(k-1)}{2}$ line segments connecting the k points. The $k+1^{st}$ point can now be connected to these k points with k line segments. Therefore we have $\frac{k(k-1)}{2}+k = \frac{(k+1)k}{2}$ line segments connecting all $k+1$ points.

We have shown that P(k) true implies that P($k+1$) is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}, n \geq 2$. □

(6) Let T_n denote the number of two element subsets of a set with n elements. Now observe that $T_{n+1} = T_n + n$. Can you explain why this is true.?

If we introduce a new $n+1^{st}$ element to a set of n there are n additional two element subsets to the count for a set of size n.

(7) Use the result from the previous question and mathematical induction to prove that the number of two element subsets of a set of size n is $\frac{n(n-1)}{2}$.

Let P(n) denote the proposition that the number of two element subsets of a set of size n is $T(n) = \frac{n(n-1)}{2}$.

Proof: Using mathematical induction.

Base: There is one two element subset of a two element set. That is, $T(2) = \frac{2(2-1)}{2}$.

Induction Hypothesis: Assume that $T(k) = \frac{k(k-1)}{2}$, for a fixed value $k \geq 2$.

Induction Step: We argued that $T(k+1) = T(k) + k$, thus we have:

$$\begin{aligned} T(k+1) =& T(k) + k \\ =& \frac{k(k-1)}{2} + k \text{(apply induction hypothesis)} \\ =& \frac{k^2 - k + 2k}{2} \\ =& \frac{(k+1)k}{2} \end{aligned}$$

We have shown that P(k) true implies that P$(k+1)$ is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}, n \geq 2$. □

(8) Consider the following proof that $n + 1 = n$, for all natural numbers n.

Induction Hypothesis: Assume that $k + 1 = k$ for a fixed natural number k.

Induction step:

$$\begin{aligned} k + 2 =& k + 1 + 1 \\ =& k + 1 \text{(apply induction hypothesis)} \\ =& k + 1 \end{aligned}$$

We have shown that P(k) true implies that P$(k+1)$ is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}$. □

This can't possibly be right! What's wrong?

The Base case is missing.