CISC-102 WINTER 2020

HOMEWORK 8
SOLUTIONS

(1) How many ways are there to select a 5 card poker hand from a standard deck of 52 cards, such that none of the cards are clubs?

We make the selection using 52-13 = 39 club free cards. Thus we have

$$\binom{39}{5}$$

ways to make the selection.

(2) How many ways are there to select a 5 card poker hand from a standard deck of 52 cards, such that at least one of the cards is a club?

We subtract selections with no clubs from all possible selections, yielding the following expression:

$$\binom{52}{5} - \binom{39}{5}$$

(3) You are planning a dinner party and want to choose 5 people to attend from a list of 11 close personal friends.

(a) In how many ways can you select the 5 people to invite.

$$\binom{11}{5}$$

(b) Suppose two of your friends are a couple and will not attend unless the other is invited. How many different ways can you invite 5 people under these constraints?

$$\binom{9}{3} + \binom{9}{5}$$

The first binomial coefficient in the sum counts selections with the couple, and the second counts selections without the couple.

(c) Suppose two of your friends are enemies, and will not attend unless the other is not invited. How many different ways can you invite 5 people under these constraints?

$$\binom{9}{5} + 2\binom{9}{4}$$

The first binomial coefficient in the sum counts selections without the enemies. The second counts selections with either enemy A, or enemy B.

(4) What is the number of ways to colour n different objects, one colour per object with 2 colours? What is the number of ways to colour n different objects with 2 colours, so that each colour is used at least once.

There are 2^n ways of colouring the n different objects using 2 colours. There exactly two ways where only one colour is used. So there are $2^n - 2$ ways to colour n different objects with 2 colours, so that each colour is used at least once.

(5) What is the number of ways to colour n identical objects with 3 colours? What is the number of ways to colour n identical objects with 3 colours so that each colour is used at least once?

The number of ways to colour n objects with 3 colours can be viewed as counting the number of binary strings with n 0's and 2 1's. This yields the expression:

$$\frac{(n+2)!}{n!2!} = \binom{n+2}{2} = \binom{n+2}{n}$$

To ensure that each colour is used at least once we pre-assign one object per colour leaving $n-3$ objects to be coloured with no further restrictions. We map this problem to counting binary strings with $n-3$ 0's and 2 1's. This yields the expression:

$$\frac{(n-3+2)!}{(n-3)!2!} = \binom{n-1}{2} = \binom{n-1}{n-3}$$

(6) Consider the equation

$$x_1 + x_2 + x_3 + x_4 = 7$$

A non-negative integer solution to this equation assigns non-negative integers (integers $x, x \geq 0$) to the variables x_1, x_2, x_3, x_4 so that the sum is 7. For example one possible solution is $x_1 = 1, x_2 = 3, x_3 = 1, x_4 = 2$. And another distinct solution is $x_1 = 2, x_2 = 3, x_3 = 1, x_4 = 1$ How many distinct non-negative integer solutions are there to this equation?

We count binary strings with 3 1's and 7 0's. The first example solution is encoded as 0100010100 and the second solution as 0010001010.

The solution is:

$$\frac{10!}{3!7!} = \binom{10}{3} = \binom{10}{7}$$

(7) From 100 used cars siting on a lot, 20 are to be selected for a test designed to check safety requirements. These 20 cars will be returned to the lot, and again 20 will be selected for testing for emission standards.

(a) In how many ways can the cars be selected for safety requirement testing?
$\left(\begin{array}{c} 100 \\ 20 \end{array} \right)$

(b) In how many ways can the cars be selected for emission standards testing?
$\left(\begin{array}{c} 100 \\ 20 \end{array} \right)$

(c) In how many different ways can the cars be selected for both tests?
$\left(\begin{array}{c} 100 \\ 20 \end{array} \right)\left(\begin{array}{c} 100 \\ 20 \end{array} \right)$

(d) In how many ways can the cars be selected for both tests if exactly 5 cars must be tested for safety and emission?
$\left(\begin{array}{c} 100 \\ 5 \end{array} \right)\left(\begin{array}{c} 95 \\ 15 \end{array} \right)\left(\begin{array}{c} 80 \\ 15 \end{array} \right)$

(8) Consider the equation

$$\sum_{i=0}^{2} \binom{3}{i}\binom{2}{2-i} = \binom{5}{2}. \tag{1}$$

(a) Use algebraic manipulation to prove that the left hand and right hand sides of equation (1) are in fact equal.
First we work out the left hand side of equation (1).

$$\begin{aligned}\sum_{i=0}^{2} \binom{3}{i}\binom{2}{2-i} &= \binom{3}{0}\binom{2}{2} + \binom{3}{1}\binom{2}{1} + \binom{3}{2}\binom{2}{0} \\ &= 1 + 6 + 3 \\ &= 10\end{aligned}$$

And the right hand side of equation (1).

$$\begin{aligned}\binom{5}{2} &= \frac{5!}{3!2!} \\ &= \frac{5 \times 4}{2} \\ &= 10\end{aligned}$$

(b) Use a counting argument to prove that the left hand and right hand sides of equation (1) are in fact equal.
On the right we count the number of ways of selecting 2 balls from a bag of 5 different balls without regard to ordering.
On the left we have two bags one with 3 balls and the other with 2 balls, which we call the 3bag and 2bag respectively. We now sum the products of selecting, without ordering, 0 from the 3bag times 2 from the 2bag, 1 from the 3bag and 1 from the 2bag, and 2 from the 3 bag and 0 from the 2 bag.

(9) Now consider a generalization of the previous equation.

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}. \tag{2}$$

Use a counting argument to prove that the left hand and right hand sides of equation (2) are in fact equal.

On the right we count the number of ways of selecting k balls without ordering from a bag of $m+n$ balls. On the left we count selections from two bags one with m balls and the other with n balls. We sum products of selecting $k-i$ and i balls from the two bags.

(10) In the notes you will find Pascal's triangle worked out for rows 0 to 8. The numbers in row 8 are 1 8 28 56 70 56 28 8 1. Work out the values of rows 9 and 10 of Pascal's triangle with the help of the equation:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}. \tag{3}$$

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row 8:     1   8   28   56   70   56   28    8    1
row 9:   1   9   36    84   126 126   84    36   9   1
row10:1   10 45   120   210 252   210   120 45  10   1
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(11) Show that $\binom{n}{0} = \binom{n-1}{0}$, and that $\binom{n-1}{n-1} = \binom{n}{n}$ by an algebraic argument as well as a counting argument.

Recall that $0! = 1$. So we have:

$$\binom{n}{0} = \frac{n!}{n!0!} = 1,$$

and

$$\binom{n-1}{0} = \frac{(n-1)!}{(n-1)!0!} = 1.$$

The counting argument is that choosing nothing from any number of items is always 1, and in particular for n items and $n-1$ items.

$$\binom{n}{n} = \frac{n!}{n!0!} = 1,$$

and

$$\binom{n-1}{n-1} = \frac{(n-1)!}{(n-1)!0!} = 1.$$

The counting argument is that there is only one way to choose all items, and in particular for n items and $n-1$ items.

(12) Prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0$$

Note that this equation can also be written as follows:

$$\sum_{i=0}^{n} \binom{n}{i} (-1^i) = 0$$

HINT: This can be viewed as a special case of the binomial theorem.

Observe that by the binomial theorem we have:

$$0 = (1-1)^n = \sum_{i=0}^{n} \binom{n}{i} (-1^i) 1^{n-i}$$

And this proves the result.