CISC-102
Fall 2020
Week 1

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Goodwin G-532
Office Hours: Tuesday 12:30-2:30

Homework

- Homework every week. Keep up to date or you risk falling behind.
- Homework will be solved in class on due date.
- Homework is not handed in, and not graded.
- All quizzes as well as the final exam are based on homework questions.

Assessment

Grades will be made up of,

Three in class midterm quizzes, each worth 20%, total: 60%

Final exam: 40%

NOTE: A minimum of 50% must be obtained on the final exam to pass the course.

The quizzes will be scheduled as follows:
Quiz 1: Wednesday, January 29.
Quiz 2: Wednesday, February 26.
Quiz 3: Wednesday, March 25.

Please make every effort to be present for the midterm quizzes. However, writing any of the quizzes is up to you, all quizzes are optional. At the end of the term I will tally four grades for everyone in the class as follows.

1. 3 quizzes 20% each and 40% Final.
2. Best 2 quiz grades 20% each and 60% Final.
3. Best single quiz grade 20% and 80% Final
4. 100% Final.

You will then get the maximum of the grades 1, 2, 3, or 4, with the exception that if you get 49% or less on the final exam, then that will be your grade.

Attending class is not compulsory, and I will not take attendance.

However…

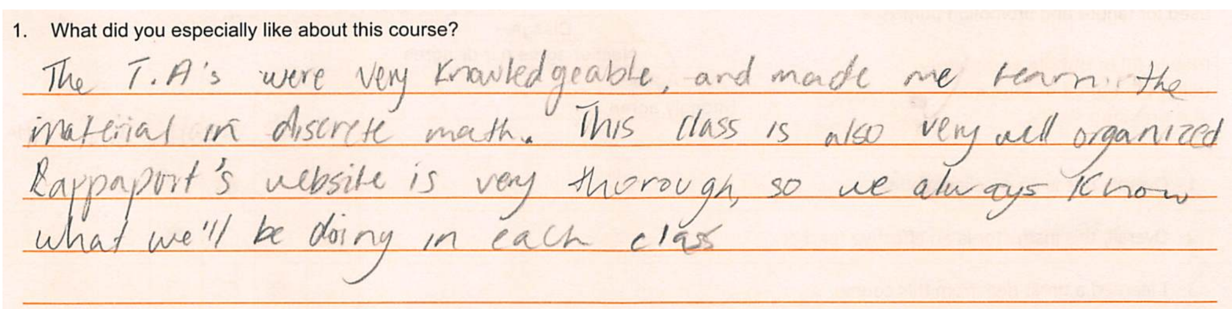
2. Do you have any specific suggestions for improvements to this course?

Somehow get the students that don't come to class to come! because they are seriously missing out.

Less proofs? lol.

Teaching Assistants and Office hours.

There are 4 Teaching Assistants assigned to this course. Each teaching assistant will provide two hours a week where any student in the class can partake to ask questions about the course. I will also provide two office hours a week for out of class questions.

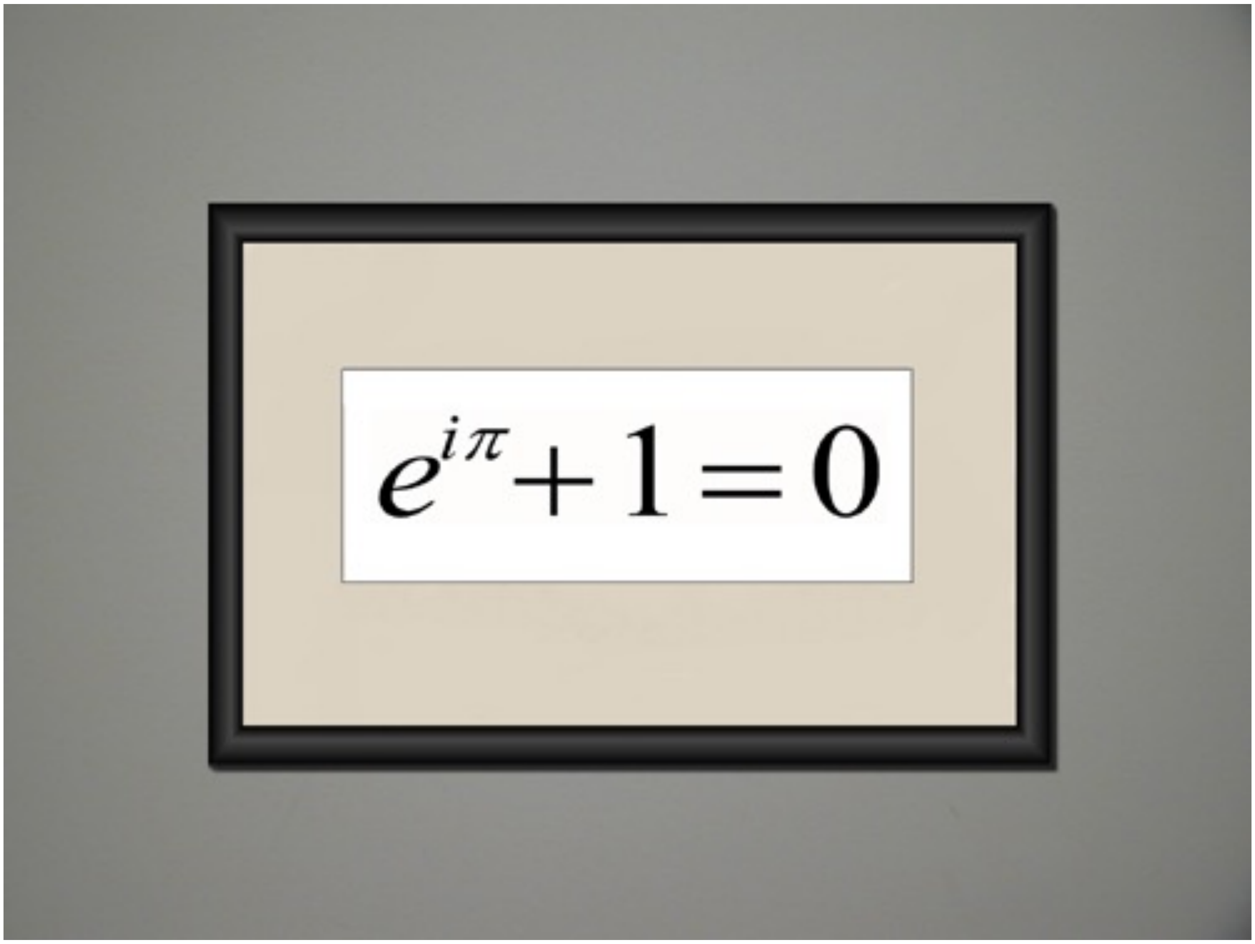


1. What did you especially like about this course?

The T.A's were very knowledgeable, and made me learn the material in discrete math. This class is also very well organized Rappaport's website is very thorough, so we always know what we'll be doing in each class

Motivation

- Discrete math is used in cryptography allowing us the convenience of online shopping.
- Learning discrete mathematics is the direct pre-requisite to mastering algorithm design and analysis skills.
- You should view this course as a language course. You will be learning the language of mathematics and computing!
- Math can be fun.
- Math is beautiful!



Beauty

- The picture on the previous page is a work of art titled "Beauty". (Prints can be purchased on-line.)

- The equation

$$e^{i\pi} + 1 = 0$$

  consists of the most important concepts in mathematics:
  - numbers
    0,1 (integers)
    π, e (irrational real numbers)
    i (a complex number)
  - operations
    $+ \times$ and exponentiation (exp. function)
  - and the relation =

UGLY?

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

This expression is known as the *binomial identity.*

Does the binomial identity seems like a mess you would like to avoid?

By the end of the term you will be able to read this and other similar "complicated" looking mathematical expressions.

Motivation

- Math is a human invention just like music, painting, sculpture, poetry, hockey, basketball, soccer, fishing …

And how do you become proficient at music, painting, hockey … ?

Practice, practice, practice.

- 10,000 “rule” holds that **10,000 hours** of "deliberate practice" are needed to become world-class in any field. (Working 40 hours per week during for a 4 year university degree gets you about half way there.)
- The homework that you do for this course can be viewed as “deliberate practice”.

The Perfect Introductory Problem: Counting hand shakes

Alice is having a birthday party at her house, and has invited Bob, Carl, Diane, Eve, Frank, and George.

They all shake hands with each other.

Q: How many handshakes?

George says, “ I know the answer and I can prove it to you. There are 7 of us, so I shake hands with 6 other people. That’s also true for everyone else. So the total number of hand shakes is $6 \times 7 = 42$.”

Frank says, “ I have another way of working this out. Suppose there’s only two of us, just George and I. That’s 1 handshake.

George: “No that’s two handshakes! I shake your hand and you shake my hand. That makes 2 handshakes.

Who’s right?

Sets

The hand shake problem is stated imprecisely, and I slacking a clear definition of what one hand shake is. We could say that the act of two people touching hands constitutes one hand shake, but that too leaves open questions about what part of each hand touches.

We convert the hand shake problem into an a math problem using proper mathematical notation.

The basic building block will be the *set*, where a set is just a collection of distinct objects .

Examples

A = {1,3,5,7,9},
B = {x | x is an integer, $0 \le x < 10$}
C = {x : x is an odd integer, $0 < x < 10$}

Warning: the following notation is given informally without definitions. Definitions to follow sometime soon

$A \subseteq C$ (A is a subset of C)
$C \subseteq A$ (C is a subset of A)
$A = C$ (A and C are equal, that is the elements of A and C are the same.)

NOTE:

If $A \subseteq C$ and $C \subseteq A$ then $A = C$.
If $A = C$ then $A \subseteq C$ and $C \subseteq A$.

$A \subseteq B$ (A is a subset of B)
$B \nsubseteq A$ (B is not a subset of A)

$A \subset B$ (A is a proper subset of B)
$B \not\subset A$ (B is not a proper subset of A)

$1 \in A$ (1 is an element of A)
$\{ 1 \} \subseteq A$
$\{ 1 \} \subset A$

Sets can have infinitely many elements

$\mathbb{N}$ = the set of *natural numbers*: 1, 2, 3, . . .
$\mathbb{Z}$ = the set of all integers: …, –2, –1, 0, 1, 2, …
$\mathbb{Q}$ = the set of rational numbers
$\mathbb{R}$ = the set of real numbers
$\mathbb{C}$ = the set of complex numbers

Observe that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

U **:** All sets under investigation in any application of set theory are assumed to belong to some fixed large set called the *universal set.*

$\varnothing$ **:** A set with no elements is called the *empty set* or *null set* .

For any set *A*, we have: $\varnothing \subseteq A \subseteq \boldsymbol{U}$

What is the smallest natural number, 0 or 1?

Schaum's notes defines the Natural numbers starting with 1.

The standard ISO 80000-2 defines the Natural numbers starting with 0.

I use Schaum's definition.

The handshake problem

Let S = {a,b,c,d,e,f,g} denote the set of party goers, and a handshake can be represented as a two element subset of S. (For example {a,b} denotes the handshake between Alice and Bob.)

Q. How many two element subsets are there of the set S?

Generalizing the handshake problem

Suppose that S is a set consisting of *n* elements.

Q. How many two element subsets are there of the set S?

The hand shake problem seems frivolous but it is actually a representation of an important mathematical concept. For example if we wanted to know which handshake was the “best” we would have to compare n(n-1)/2 of them. Let n = 35, 000, 000 (the population of Canada) we would have to compare 612,499,982,500,000 or roughly 612 trillion hand shakes. (Too much!)

If we test one handshake per second it would take roughly 31,688 Years, 269 Days, 1 Hour. (Too long!)

Some additional counting problems involving subsets.

Suppose that S is a set consisting of n elements.

Q1. How many one element subsets are there of the set S? (Easy)

Q2. How many zero element subsets are there of the set S? (Easy)

Q3. How many n element subsets are there of the set S? (Easy)

Q4. Suppose $n \geq 3$. How many *three* element subsets are there of the set S? (Harder, to be solved later.)

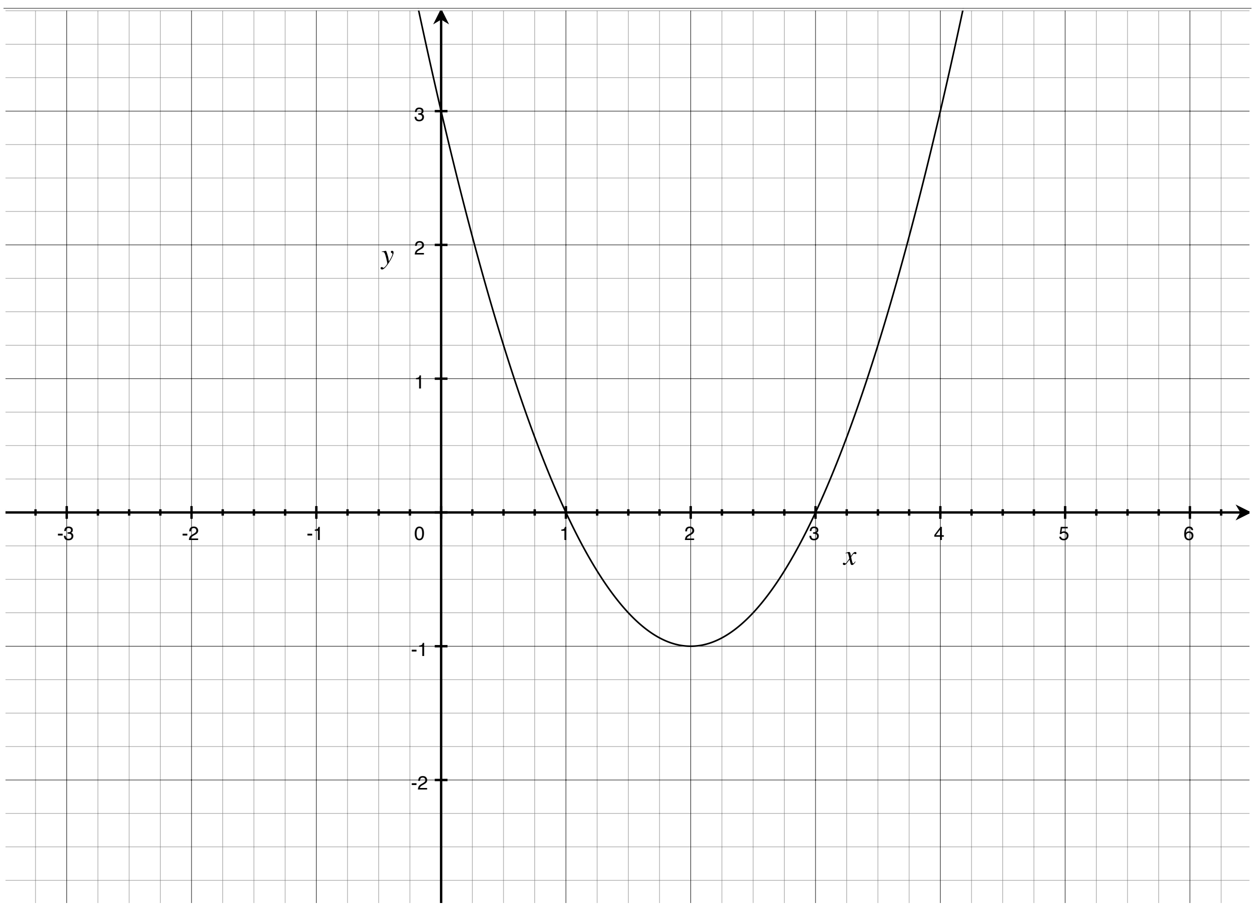
Q5. Suppose $0 \leq k \leq n$ what is a formula for the number of k element subsets of the set S? (More general and harder to be solved later.)

Problems from Schaum's Notes (SN)

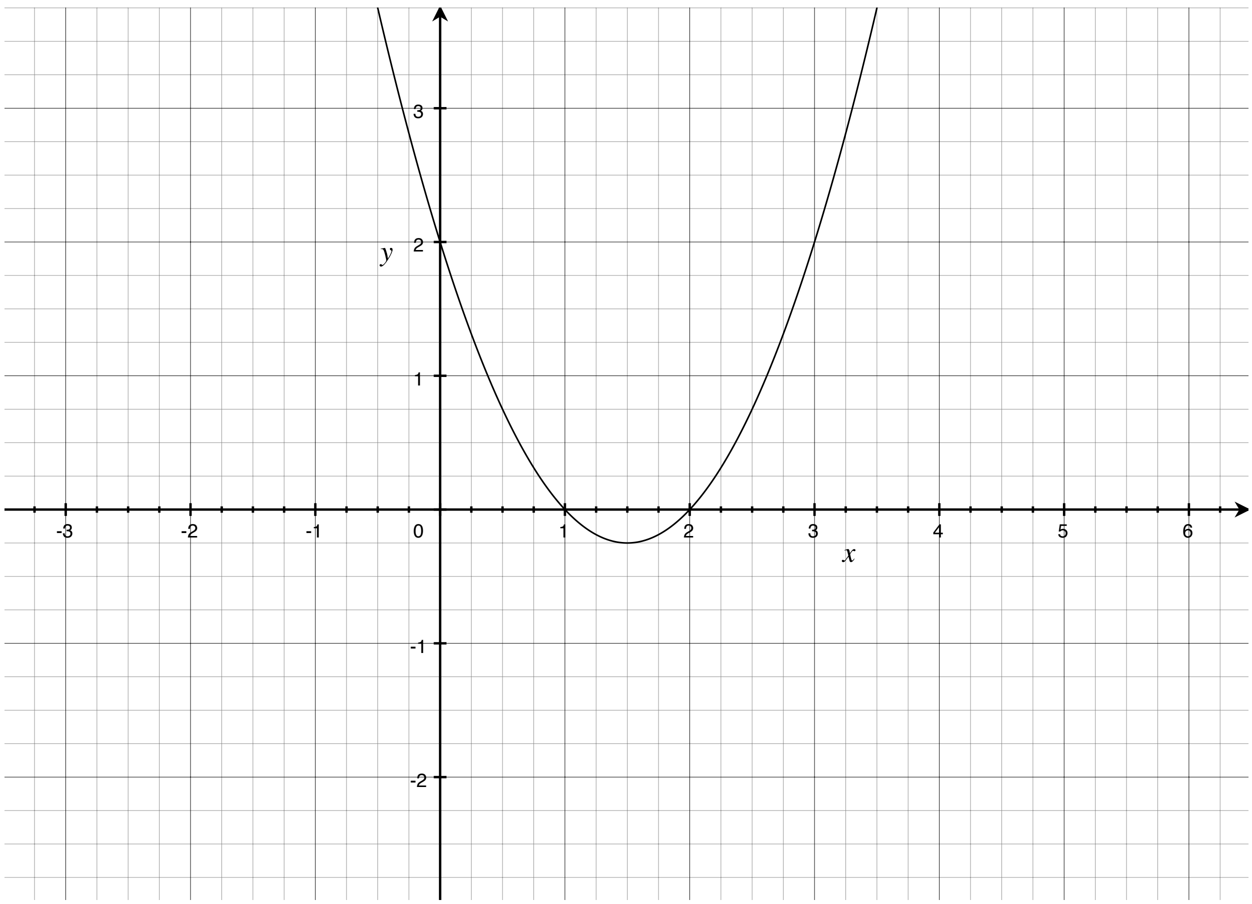
1.26 Which of the following sets are equal?
A = $\{x \mid x^2 - 4x + 3 = 0\}$,
B = $\{x \mid x^2 - 3x + 2 = 0\}$,
C = $\{x \mid x \in \mathbb{N}, x < 3\}$,
D = $\{x \mid x \in \mathbb{N}$, x is odd, $x < 5\}$,
E = {1, 2},
F = {1,2,1},
G = {3, 1},
H = {1,1,3}.

NOTE: To determine the elements of sets A and B, you need to be able to *factor quadratic equations*. This is a topic that you may or may not be familiar with. For this course it is assumed that you are able to do this factoring or pick up this skill on your own. All examples that you will see in this course will have integer solutions. Here's a link to a web page with some good tips for factoring quadratic equations: https://www.mathsisfun.com/algebra/factoring-quadratics.html

Graph of $x^2 - 4x + 3$.
The function crosses the x-axis at two points $x = 1$, and $x = 3$.
Note: $x^2 - 4x + 3 = (x - 1)(x - 3)$.



Graph of $x^2 - 3x + 2$.
The function crosses the x-axis at two points $x = 1$, and $x = 2$.
Note: $x^2 - 4x + 3 = (x - 1)(x - 2)$.



Sets

The following definitions are from Schaum's Notes. Sometimes I will also give an alternate definition to avoid unnecessary confusion.

Definition:
A *set* may be viewed as any well-defined collection of objects, called the *elements* or members of the set.

This sentence defines in a mathematical sense the term *set* and the term *element.*

Key things to remember about sets.
- Always use curly braces { }.
- The elements are *well-defined,* that is, each element can be distinguished from another.
- A set is an *un-ordered* collection of elements.

Notation

A = {1, 2, 3} is a set of 3 elements.

1 $\in$ A (1 is an element of the set A.)

B = {1, 3, 2}

So A = {1, 2, 3} = B = {1, 3, 2}.

Subset

Let A and B be two sets, where every element of A is also an element of B.

For example:
A = {red, black}, B = {red, black, green}.

Let a denote an arbitrary object.
Observe that: if $a \in$ A then $a \in$ B.
We can say that A is contained in B, or A is a subset of B.

Definition: Let X and Y be two sets such that $a \in$ X implies $a \in$ Y. We then can say that X is a *subset* of Y, and notate it as $X \subseteq Y$.

Alternate Definition: If every element in the set X is also an element of the set Y then X is a subset of Y, notated as $X \subseteq Y$.

Suppose X and Y are two sets such that:
$X \subseteq Y$ and $Y \subseteq X$.

Therefore, every element of X is an element of Y and every element of Y is an element of X. So in fact the sets are equal.

Definition: Let X and Y be two sets.
If $X = Y$ then $X \subseteq Y$ and $Y \subseteq X$.
And if $X \subseteq Y$ and $Y \subseteq X$ then $X=Y$.

These two sentences can be expressed in a single sentence as:

$X = Y$ **if and only if** $X \subseteq Y$ and $Y \subseteq X$.

Definition: Let X and Y be two sets.
If $X \subseteq Y$ and $X \neq Y$ then we say that X is a *proper subset* of Y, and notate it as:
$X \subset Y$.

Alternate Definition: X is a *proper subset* of Y if every element of X is also an element of Y and there exists at least one element of Y that is not an element of X.

Subsets and Proper subsets

$x \leq y$ (x less than y **or** equal to y)
$x < y$ (x less than y)

$X \subseteq Y$ (X proper subset of Y **or** equal to Y)
$X \subset Y$ (X proper subset of Y)

Find the definitions and examples in Schaum's Notes for the symbols.

$\nsubseteq$ (not a subset) $\not\subset$ (not a proper subset)
$\supseteq$ (superset) $\nsupseteq$ (not a superset)
$\supset$ (proper superset) $\not\supset$ (not a proper superset)

Disjoint sets

Let A and B be two sets. If A and B have no elements in common then we say that they are *disjoint*.

Using subset notation we can say that if A and B are disjoint then $A \not\subseteq B$ and $B \not\subseteq A$.

However, if $A \not\subseteq B$ and $B \not\subseteq A$ then A and B may not be disjoint. (Can you think of an example where $A \not\subseteq B$ and $B \not\subseteq A$ but still A and B have elements in common, that is, the sets are not disjoint.)

U **:** All sets under investigation in any application of set theory are assumed to belong to some fixed large set called the *universal set*.

$\varnothing$ **:** A set with no elements is called the *empty set* or *null set* .

> The empty set is a subset of every set, and the universal set is a superset of every set.

Using symbols the blue sentence can be expressed as follows:

For any set A, we have: $\varnothing \subseteq \boldsymbol{A} \subseteq \boldsymbol{U}$

Examples

Consider the set A = { 1, 2, 3}.

A is a set consisting of 3 elements.

$\{1\} \subseteq A$, ($\{1\}$ is a subset of A)

$\{1\} \subset A$, ($\{1\}$ is a proper subset of A)

$1 \in A$ (1 is an element of A)

$\{1,2,3\} \subseteq A$

$\{1,2,3\} \not\subset A$

$\{1,2\} \subset A$

$\emptyset \subseteq A$ and $\emptyset \subset A$
$A \subseteq \mathbb{N}$ and $A \subset \mathbb{N}$

Examples:

People in a room.
Coins in your pocket.

Note: If you have two (or more) quarters in your pocket then you need to be able to distinguish one from the other if you want to consider the coins as a set. If you have no coins in your pocket then the set of coins in your pocket is the empty set.

Seating Arrangements

There is a large table at the party and Alice wants to experience every possible seating arrangement. How many ways can 7 people sit at a table?

This “seating arrangement question” is equivalent to asking for the number of different ways to order 7 people.

Number of ways to order 1 person? 1.

Number of ways to order 2 people?
(1,2) (2,1). 2×1

Number of ways to order 3 people?
(3,1,2)(3,2,1)(1,3,2)(2,3,1)(1,2,3)(2,13). $3 \times 2 \times 1$

Number of ways to order 4 people?
Guess: $4 \times 3 \times 2 \times 1 = 4!$

Permutations

There are $n!$ ways to order n distinct objects

Recall $n!$ (n factorial) is given by the expression:

$n! = n \times (n-1) \times (n-2) \times \ldots \times 1$

A notational shorthand that makes this product explicit without the need for ellipses (…) is:

$$\prod_{i=1}^{n} i$$

Selection with ordering

How many ways are there to pick 7 people out of a class of 70 and seat them into 7 numbered chairs? (Selection with ordering.)

1st pick has 70 choices.
2nd pick has 69 choices.
3rd pick has 68 choices.
4th pick has 67 choices.
5th pick has 66 choices.
6th pick has 65 choices.
7th pick has 64 choices.

So the number of ways to pick 7 people out of a class of 70 is:

$$70 \times 69 \times 68 \times 67 \times 66 \times 65 \times 64 = 70!/63!$$

Lottery Tickets

Lotto 6-49, choose 6 numbers from 1 to 49.

How many ways are there to choose to these numbers?

A very simplified version of this game is Lotto 1-49, where players choose 1 number from 49. There are 49 choices.

Note that the probability (the odds) of winning Lotto 1- 49 is 1/49. (one choice divided by the total number of choices)

Consider Lotto 2 - 49, where you have to pick 2 numbers from 49.

A tempting but wrong guess would be 49×48 choices.

Suppose choice 1 is 42, and choice 2 is 18. That is equivalent to choice 1 is 18 and choice 2 is 42, so 49×48 double counts all possibilities.

The actual answer is $49 \times 48 / 2!$.

(Note: this is the same as asking for the number of two element subsets of a set of 49 elements.)

Combinations (Selection without ordering)

For lotto 6-49, players choose 6 numbers from 1 to 49.

How many ways are there to choose to these numbers?

Solution:
49×48×47×46×45×44/6! =13,983,816.

This can also be written as:

$$\binom{49}{6} = \frac{49!}{43!6!}$$

and pronounced 49 choose 6.

What is the probability that any single choice is the winning number?

1/13,983,816

The current price of a Lotto 6-49 card is $3.

If the only prize awarded is the jackpot then the "fair" prize for choosing the winning numbers should be $41,951,448.

The minimum jackpot is $5,000,000.
There have been occurrences when the Lotto 6-49 jackpot exceeded the fair prize.

The largest jackpot (as of Aug. 2017) was $64,000,000 on October 17, 2015.

And here is an article about the U.S. lottery "Powerball". (Pick 5 from 69, and then a sixth from a different pool of 26.)

http://money.cnn.com/2017/08/19/news/powerball-drawing/index.html

Can you determine the odds of winning the jackpot?

Set Operators

Operators on sets are

union $\cup$ and intersection $\cap$.

Definitions:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Logical Operators

$p \wedge q$ pronounced *p* and *q*

Both *p* and *q* have to be true for the compound proposition *p* and *q* to be true.

$p \vee q$ pronounced *p* or *q*

At least one of *p* or *q* must be true for the compound proposition *p* or *q* to be true.

We can rewrite our definition for set union and set intersection using logical operators as follows:

$$A \cup B = \{x : x \in A \vee x \in B\}$$

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

For example:

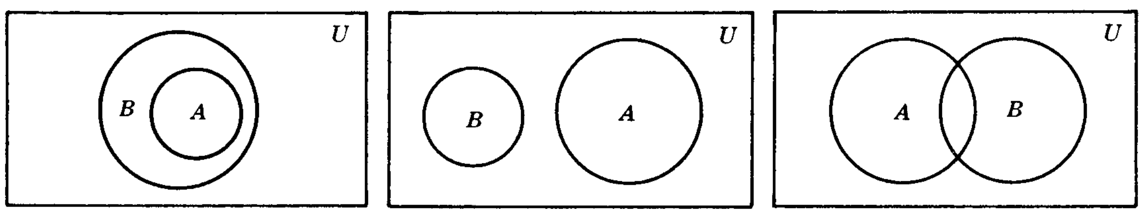
Suppose A is the set of guitars and B is the set of red musical instruments.

- An element x is in the set of A union B if it is a guitar or if it is a red musical instrument.
- An element of x is in the set of A intersection B if x is red and x is a guitar.

Venn Diagrams

Useful for providing intuitive insight.

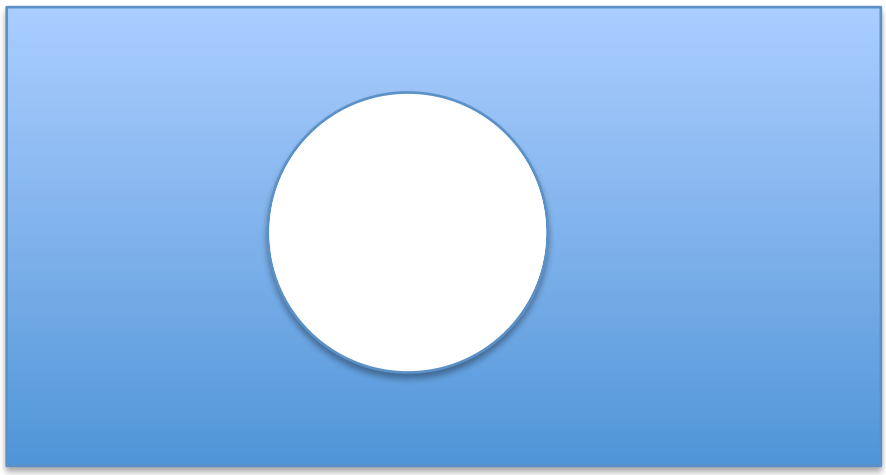
Note the rectangle surrounding the circles denotes the Universe **U**.



(a) $A \subseteq B$ (b) A and B are disjoint (c)

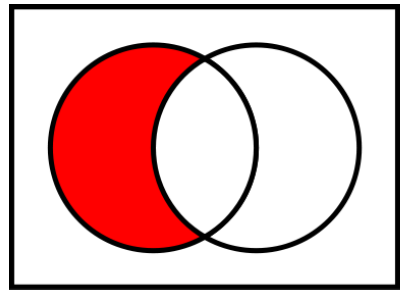
The complement of a set A written A^c is defined as:

$$A^c = \{x | x \notin A\}$$



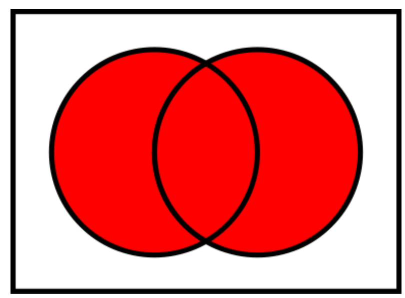
The relative complement of a set B with respect to A, sometimes called the difference

$$A \backslash B = \{x \mid x \in A, x \notin B\}$$

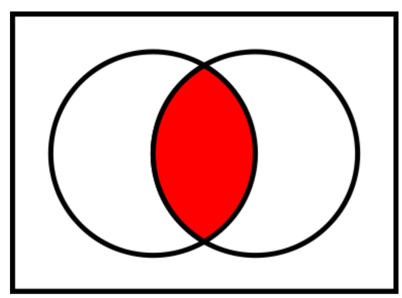


(The relative complement is sometimes written as A – B.)

$$A \cup B = \{x : x \in A \vee x \in B\}$$



$$A \cap B = \{x : x \in A \wedge x \in B\}$$

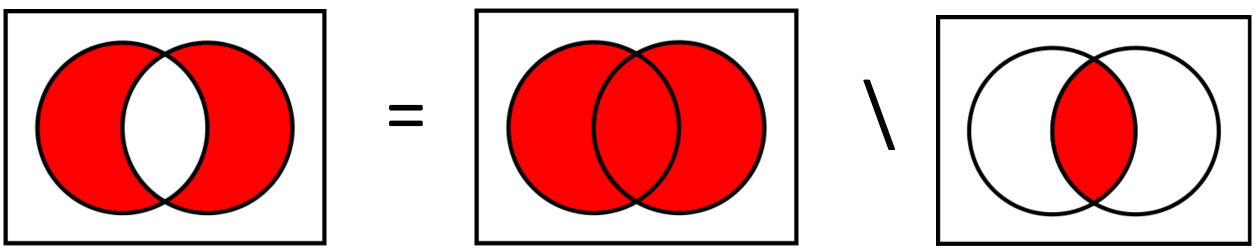


The symmetric difference of sets A and B:

$$A \oplus B = (A \cup B) \backslash (A \cap B) \quad \text{or} \quad A \oplus B = (A \backslash B) \cup (B \backslash A)$$

The symmetric difference consists of elements that are in A or in B but not in both.

$$A \oplus B = (A \cup B) \backslash (A \cap B)$$



$$A \oplus B = (A \backslash B) \cup (B \backslash A)$$