

CISC-102
Winter 2020
Week 2

Idempotent laws:	(1b) $A \cap A = A$
Associative laws:	(2b) $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws:	(3b) $A \cap B = B \cap A$
Distributive laws:	(4b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws:	(5b) $A \cap \mathbf{U} = A$
	(6b) $A \cap \emptyset = \emptyset$

Verify these properties with $\mathbf{U} = \{1,2,3,4,5,6,7\}$,
 $A = \{1,2,3\}$, $B = \{2, 4, 6\}$ $C = \{4,5\}$.

Idempotent laws:	(1a) $A \cup A = A$
Associative laws:	(2a) $(A \cup B) \cup C = A \cup (B \cup C)$
Commutative laws:	(3a) $A \cup B = B \cup A$
Distributive laws:	(4a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Identity laws:	(5a) $A \cup \emptyset = A$
	(6a) $A \cup U = U$

Verify these properties with $U = \{1,2,3,4,5,6,7\}$,
 $A = \{1,2,3\}$, $B = \{2, 4, 6\}$.

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Duality

Suppose E is an equation of set algebra. The dual E^* of E is the equation obtained by replacing each occurrence of \cup , \cap , \mathbf{U} and \emptyset in E by \cap , \cup , \emptyset , and \mathbf{U} , respectively.

Involution laws:	(7) $(A^C)^C = A$
Complement laws:	(8a) $A \cup A^C = \mathbf{U}$
	(9a) $\mathbf{U}^C = \emptyset$
DeMorgan's laws:	(10a) $(A \cup B)^C = A^C \cap B^C$

Verify these properties with $U = \{1,2,3,4,5,6,7\}$,
 $A = \{1,2,3\}$, $B = \{2, 4, 6\}$.

Complement laws:	(8b) $A \cap A^C = \emptyset$
	(9b) $\emptyset^C = U$
DeMorgan's laws:	(10b) $(A \cap B)^C = A^C \cup B^C$

Verify these properties with $U = \{1,2,3,4,5,6,7\}$,
 $A = \{1,2,3\}$, $B = \{2, 4, 6\}$.

Definition: A set S is said to be *finite* if S is empty or if S contains exactly m elements where m is a positive integer; otherwise S is *infinite*.

Some finite sets:

$$A = \{1, 2, 3, 4\},$$

$$B = \{x : x \in \mathbb{N}, x \geq 1, x \leq 6\}$$

Some infinite sets:

$$C = \{1, 2, 3, 4, \dots\},$$

$$D = \{x : x \in \mathbb{R}, x \geq 1, x \leq 6\}$$

Notation: We use vertical bars $| |$ to denote the size or cardinality of a finite set.

$$|A| = 4.$$

$$B = \{1, 2, 3, 4, 5, 6\}, \text{ so } |B| = 6.$$

Definition: The set of all (different) subsets of S is the power set of S , which we denote as $P(S)$.

For example: Suppose $S = \{a, b, c\}$, then the power set of S is:

$$P(S) = \{\{\}, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}.$$

(How many proper subsets are there of the set S ?)

If S is a finite set we can prove that:

$$|P(S)| = 2^{|S|}.$$

Some examples:

\emptyset : the empty set has 0 elements, and 1 subset.

$$\text{So } |P(\emptyset)| = 2^0.$$

$\{a\}$: has 1 element and 2 subsets.

$$\text{So } |P(\{a\})| = 2^1.$$

$\{a,b\}$: has 2 elements and 4 subsets.

$$\text{So } |P(\{a,b\})| = 2^2.$$

Exercise: What is the power set of the sets:

$\{\text{red, blue}\}$, $\{1,2,3\}$, $\{a,b,c,d\}$

$\{a,b,c\}$: Let's keep track of the subsets of $\{a,b,c\}$ by using a binary string counter.



Corollary: There are 2 types of people in the world, those who can spell “those” and “don’t” correctly and those that can’t.

a b c
1 1 1 denotes {a,b,c}

a b c
1 0 1 denotes {a,c}

a b c
0 0 0 denotes \emptyset

We can keep track of the subsets of {a,b,c} by using a 0 or 1 in a 3 bit binary string to denote the presence or absence of the symbol in the subset.

For elements of {a,b,c} we say that a is element 1, b is element 2, c is element 3

We also say that the binary bits are numbered 1,2,3 from left to right.

To map a subset to a binary number set bit i to

0 if the element i **is not in** the subset

1 if the element i **is in** the subset

Claim: Every different 3 bit binary string denotes a different subset of $\{a,b,c\}$, and every subset of $\{a, b, c\}$ is represented uniquely by a 3 bit binary string.

If we can count 3 bit binary strings then we can also count subsets of $\{a,b,c\}$.

There are 2 choices for the left bit (bit 1).

There are 2 choices for the middle bit (bit 2).

There are 2 choices for the right bit (bit 3).

The total number of choices are:

$$2 \times 2 \times 2 = 2^3 = 8.$$

This coincides with the claim that:

$$|\mathcal{P}(\{a,b,c\})| = 2^3 .$$

A standard “trick” in mathematics is to obtain a mapping from a new problem to one where the solution is known. When done properly (the mapping has to be one-to-one and onto, also known as a *bijection*) the known solution can be used to solve a new problem.

Humans use the decimal system or numbers base 10, using decimal digits 0,1,2,3,4,5,6,7,8,9.

Consider the decimal number 2019.

It is equal to:

$$2 \times 10^3 + 0 \times 10^2 + 1 \times 10^1 + 9 \times 10^0.$$

2	0	1	9
2×1000	0×100	1×10	9×1
2×10^3	0×10^2	1×10^1	9×10^0

The maximum value that can be written using 4 decimal digits is 9999. The total number of values from 0 ... 9999 is 10000 or 10^4 .

Binary numbers are numbers base 2, using binary bits 0,1.

Consider the binary number 1011.

It is equal to:

$$1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 .$$

1	0	1	1
1×8	0×4	1×2	1×1
1×2^3	0×2^2	1×2^1	1×2^0

The maximum value that can be written using 4 binary bits is 1111. The total number of values from 0 ... 1111(binary) is 10000 (binary) or 2^4 .

Consider the binary number

11111100011

What is the corresponding value in decimal?

1	1	1	1	1	1	0	0	0	1	1
1×2^{10}	1×2^9	1×2^8	1×2^7	1×2^6	1×2^5	0×2^4	0×2^3	0×2^2	1×2^1	1×2^0
1024	512	256	128	64	32	0	0	0	2	1

$$1024+512+256+128+64+32+2+1 = 2019$$

Exercise: Write 2018, 2020, 2047 in binary.

Consider the set expression.

$$P(A) \cup P(B) = P(A \cup B)$$

Is this expression true for any two sets A and B?

Letting $A = \emptyset$ and $B = \{1\}$ we have:

$$P(A) \cup P(B) = \{ \emptyset \} \cup \{ \emptyset, \{1\} \} = \{ \emptyset, \{1\} \}$$

and

$$P(A \cup B) = P(B) = \{ \emptyset, \{1\} \}$$

Now consider $A = \{1\}$ and $B = \{2\}$ we have

$$\begin{aligned} P(A) \cup P(B) &= \{ \emptyset, \{1\} \} \cup \{ \emptyset, \{2\} \} \\ &= \{ \emptyset, \{1\}, \{2\} \} \end{aligned}$$

and

$$P(A \cup B) = P(\{1,2\}) = \{ \emptyset, \{1\}, \{2\}, \{1,2\} \}$$

Therefore: We conclude that in general

$$P(A) \cup P(B) \neq P(A \cup B)$$

Now consider the set expression:

$$P(A) \cap P(B) = P(A \cap B)$$

Is this expression true for any two sets A and B?

Consider $A = \{1\}$ and $B = \{2\}$ we have

$$\begin{aligned} P(A) \cap P(B) &= \{ \emptyset, \{1\} \} \cap \{ \emptyset, \{2\} \} \\ &= \{ \emptyset \} \end{aligned}$$

and

$$P(A \cap B) = P(\emptyset) = \{ \emptyset \}.$$

By now you should realize that this does not prove anything. However, we can make an argument to show that

$$(P(A) \cap P(B)) = P(A \cap B)$$

for any two sets A and B.

The strategy that can be used is to show that $(P(A) \cap P(B)) \subseteq P(A \cap B)$, and that $P(A \cap B) \subseteq (P(A) \cap P(B))$

Thus,

let x be an arbitrary element of $P(A) \cap P(B)$, and let y be an arbitrary element of $P(A \cap B)$.

Properties of x :

$x \in P(A) \cap P(B)$, so $x \in P(A)$, and $x \in P(B)$, therefore $x \subseteq A$ and $x \subseteq B$, so $x \subseteq (A \cap B)$ and therefore $x \in P(A \cap B)$.

Therefore: $(P(A) \cap P(B)) \subseteq P(A \cap B)$

Properties of y :

$y \in P(A \cap B)$, so $y \subseteq (A \cap B)$ therefore $y \subseteq A$ and $y \subseteq B$ and $y \in (P(A) \cap P(B))$.

Therefore: $P(A \cap B) \subseteq (P(A) \cap P(B))$

Definition: Let S be a non-empty set. A partition of S consists of pairwise disjoint non-empty subsets of S whose union is S .

For example: Let S be the set $\{1,2,3,4,5,6\}$:

$$E = \{x \in S : x \text{ is even}\}$$

$$O = \{x \in S : x \text{ is odd}\}$$

Observe that $E \cap O = \emptyset$ and $E \cup O = S$, so the set (of sets) $\{E, O\}$ is a partition of S .

This $\{\{1,2\}, \{4,3,6\}, \{5\}\}$ is another partition of S .

When solving algorithmic problems it is sometimes useful to partition the problem into cases.

A collection or a class of sets can be defined using an index as follows:

Let A_i denote the set $\{x : x \in \mathbb{Z}, x \geq i\}$, for all $i \in \mathbb{N}$. So we have:

$A_1 = \{ 1, 2, 3, \dots \}$, $A_2 = \{ 2, 3, 4, \dots \}$, $A_3 = \{ 3, 4, 5, \dots \}$
etc.

A shorthand notation can be used to denote multiple unions and intersection operations.

For example:

$$A_1 \cup A_2 \cup A_3$$

can be written as:

$$\bigcup_{i=1}^3 A_i$$

Let A_i denote the set $\{x : x \in \mathbb{Z}, x \geq i\}$, for all $i \in \mathbb{N}$. This defines infinitely many sets. Using the generalized union and intersection operators we can express the union or intersection of an arbitrary finite number of sets.

So for union we have:

$$\bigcup_{i=1}^n A_i = \{x : x \in A_i \text{ for any } i = 1, \dots, n\}$$

What is the value of $\bigcup_{i=1}^n A_i$?

And for intersection:

$$\bigcap_{i=1}^n A_i = \{x : x \in A_i \text{ for every } i = 1, \dots, n\}$$

What is the value of $\bigcap_{i=1}^n A_i$?

We can also state properties, or perform union and intersection operations on infinitely many sets.

1. $A_i \subseteq A_j$ if $i \geq j$ for all $i \in \mathbb{N}$.

2. $\bigcup_{i \in \mathbb{N}} A_i = \mathbb{N}$

3. $\bigcap_{i \in \mathbb{N}} A_i = \emptyset$

To justify 3. suppose that there is at least one element k in the “intersection”. So k must be a positive integer.

However by definition there is a set A_ℓ such that $\ell > k$ and $k \notin A_\ell$. This contradicts the assumption that there is at least one element in the “intersection” and therefore we conclude that the “intersection” is equal to \emptyset .

Notation

We will make extensive use of a generalized sum notation throughout the course. It uses a large greek letter Σ (pronounced “sigma”).

Recall that the binary number 1111 is equal to:

$$2^3 + 2^2 + 2^1 + 2^0$$

This can be written concisely using sigma notation as:

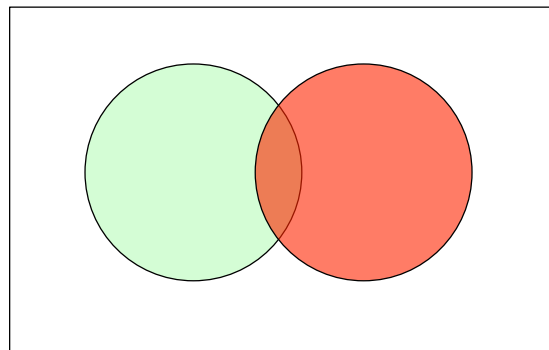
$$\sum_{i=0}^3 2^i$$

Principle of Inclusion and Exclusion

Ginger owns a peculiar music store where all of the instruments are either red or if the instrument is not red it is a guitar.

There are 15 red instruments and 18 guitars. How many instruments are there at Ginger's?

Here's a Venn diagram modelling the inventory at Ginger's.



The diagram clearly shows that we also need to know how many of the guitars at Ginger's are red to determine the total number of instruments in the store. So suppose there are 8 red guitars at Ginger's.

Let R denote the set of red instruments at Ginger's and G the set of guitars. We then have:

$$|R| = 15, |G| = 18, |R \cap G| = 8.$$

The total number of instruments is:

$$|R \cup G| = |R| + |G| - |R \cap G| = 15 + 18 - 8 = 25.$$

The Principle of Inclusion and Exclusion can be stated as follows:

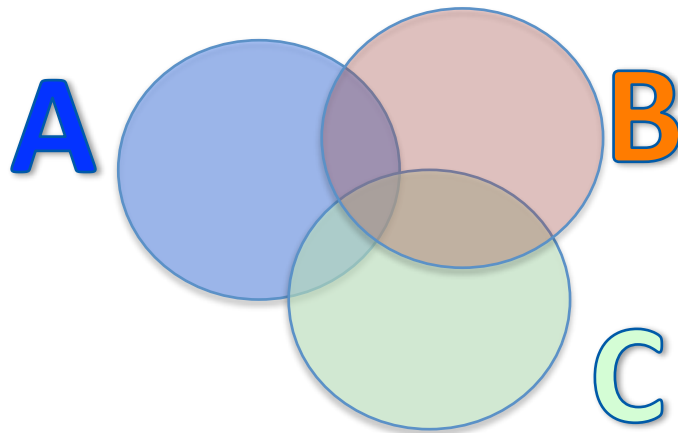
Theorem: Suppose A and B are finite sets. Then:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

This generalizes to a formula for determining the cardinality of the union of three sets.

Corollary: Suppose A , B , and C are finite sets. Then:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Consider a collection of 40 people where each of them is wearing something that is red or blue or green such that:

- 20 wear something blue,
- 20 wear something red,
- 20 wear something green
- 10 wear red and blue, 10 wear red and green, 10 wear blue and green.

How many people in the class are wearing all 3 colours?

We will come back to the Principle of inclusion and exclusion when we look at more counting problems.