CISC-102
Winter 2020
Week 5

Relations (See chapter 2. of SN)

Functions are mappings from one set to another with specific additional properties.

Recall: A function must map every element of the Domain set to a single element in the Range set.

Mappings without these additional properties are also valid entities in mathematics.

An ordered pair of elements a,b is written as (a,b).
NOTE: Mathematical convention distinguishes between "( )" brackets -order is important – and "{ }" -- not ordered.
Example: $\{1,2\} = \{2,1\}$, but $(1,2) \neq (2,1)$.

Product Sets

Let A and B be two arbitrary sets. The set of all ordered pairs (a,b) where $a \in A$ and $b \in B$ is called the product or Cartesian product or cross product of A and B.
The cross product is denoted as:

$$A \times B = \{(a,b) : a \in A \text{ and } b \in B \}$$

and is pronounced “A cross B”.
It is common to denote $A \times A$ as A^2.

A “famous” example of a product set is , $\mathbb{R}^2$, that is, the product of the Reals, or the two dimensional real plane or Cartesian plane -- x and y coordinates.

Relations

Definition: Let A and B be arbitrary sets. A *binary relation*, or simply a *relation* from A to B is a subset of
$A \times B$.
(We study relations to continue our exploration of mathematical definitions and notation.)

Example: Suppose $A = \{1,3,6\}$ and $B = \{1,4,6\}$
$A \times B = \{(a,b) : a \in A, \text{ and } b \in B \}$
$= \{(1,1),(1,4),(1,6),(3,1),(3,4),(3,6),(6,1),(6,4)(6,6)\}$

Example: Consider the relation $\leq$ on $A \times B$ where A and B are defined above.
The subset of $A \times B$ in this relation are the pairs:
$\{(1,1),(1,4),(1,6),(3,4),(3,6),(6,6)\}$

That is, a pair (a,b) is in the relation $\leq$ whenever $a \leq b$.

Consider a relation from the set
S = {A, B, C, D, F,G} to the set T= {1, 2, 3, 4, 5, 6, 7}

	1	2	3	4	5	6	7
A							
B							
C							
D							
E							
F							
G							1

A 1 in table entry (s,t) denotes that the pair (s,t) is in the relation, otherwise we leave the table entry blank.

How would you describe the relation if

I. There are 1's along the main diagonal.
II.Every row has exactly one 1.
III. Every row and every column has exactly one 1.

Functions as relations

A function can be viewed as a special case of relations.

A relation R from A to B is a function if every element $a \in A$ belongs to a unique ordered pair (a,b) in R.

Let A = {a,b,c, …, z} and let S = {1,2,3, …, 26}. We define a relation R from A to S as:
$R = \{(x,y) \in A \times S$: letter x is the y^{th} letter of the alphabet.}

We can verify that R is a function be observing that for every letter $x \in A$, there is a single value $y \in S$, such that $(x,y) \in R$

In fact R is a bijection, that is, a one-to-one and onto function. Why?

Also observe that $|A| = |S|$.

Vocabulary

When we have a relation on $S \times S$ (which is a very common occurrence) we simply call it a relation *on* S, rather than a relation on $S \times S$.

Let $A = \{1,2,3,4\}$, we can define the following relations on A.

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$R_3 = \{(1,3), (2,1)\}$

$R_4 = \varnothing$

$R_5 = A \times A = A^2$ (How many elements are there in R_5?)

Properties of relations on a set A

Reflexive: A relation R is <u>reflexive</u>
if $(a,a) \in R$ for all $a \in A$.

Symmetric: A relation R is *<u>symmetric</u>*
if whenever $(a_1, a_2) \in R$ then $(a_2, a_1) \in R$.

Antisymmetric: A relation R is *<u>antisymmetric</u>*
if whenever $(a_1, a_2) \in R$ and $(a_2, a_1) \in R$ then $a_1 = a_2$.

NOTE: There are relations that are neither symmetric nor antisymmetric or both symmetric and antisymmetric.

Transitive: A relation R is *<u>transitive</u>*
if whenever $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ then $(a_1, a_3) \in R$.

Let A = {1,2,3,4}, we can define the following relations on A.

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$
NOT reflexive, NOT symmetric, antisymmetric, transitive

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$
reflexive, symmetric, NOT antisymmetric, transitive

$R_3 = \{(1,3), (2,1)\}$
NOT reflexive, NOT symmetric, antisymmetric, NOT transitive

$R_4 = \varnothing$
NOT reflexive, symmetric, antisymmetric, transitive

$R_5 = A \times A = A^2$ (How many elements are there in R_5 ?)
reflexive, symmetric, transitive.

Consider the relation

$R_6 = \{(1,1), (1,2), (2,1), (2,3),(2,2), (3,3)\}$
NOT reflexive, NOT symmetric, NOT antisymmetric, NOT transitive

Consider the relations $<$, $\leq$, and $=$ on the Natural numbers. (less than, less than or equal to, equal to)

The relation $<$ on the Natural numbers

$\{(a,b) : a,b \in N, a < b\}$ is:

NOT reflexive, NOT symmetric, antisymmetric, transitive

The relation $\leq$ is on the Natural numbers

$\{(a,b) : a,b \in N, a \leq b\}$ is:

reflexive, NOT symmetric, antisymmetric, transitive

The relation $=$ on the Natural numbers

$\{(a,b) : a,b \in N, a = b\}$ is:

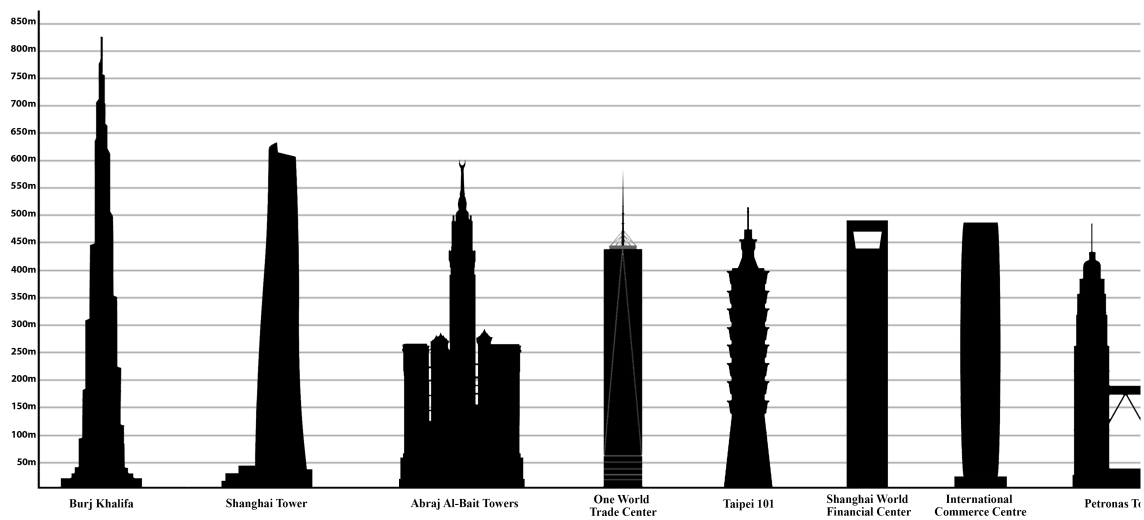
reflexive, symmetric, antisymmetric, transitive

Partial orders and equivalence relations

A relation R is called a *partial order* if R is reflexive, antisymmetric, and transitive.

Partial order relations can be used when we want to compare and order things.

NOTE: The relation $\leq$ is on the Natural numbers
$\{(a,b) : a,b \in N, a \leq b\}$ is a partial order relation.



We can order the tallest buildings in the world by height.

Let P(S) denote the power set of the set S, and let R be a relation on P(S) defined as:

$R = \{(s,t) \in P(S) \times P(S) : s \subseteq t \}$

Observe that R is a partial order, because:
$(s,s) \in R$ for all sets $s \in P(S)$, therefore R is reflexive.

Whenever $s \subseteq t$ and $t \subseteq s$, then $s = t$, therefore R is antisymmetric

Whenever $s \subseteq t$ and $t \subseteq w$, then $s \subseteq w$, therefore R is transitive.

A relation R is called an *equivalence relation* if R is reflexive, symmetric, and transitive.

Equivalence relations can be used when we want to compare and classify things.
The relation = on the Natural numbers
$\{(a,b) : a,b \in N, a = b\}$ is an equivalence relation.

We can partition fruit into *equivalence classes* using an equivalence relation.

Suppose R is an equivalence relation on a set S. For each element $s \in S$, let $[s] = \{t \in S : (s,t) \in R\}$. We call [s] an <u>*equivalence class*</u> of S.

For example let S be the set {A,B,C, a,b,c,1,2,3} and let R be the relation $\{(s,t) \in S \times S$: s and t are both upper case, both lower case, or both digits}.

Thus, R partitions S into 3 equivalence classes,
[a] = {a,b,c}, [A] = {A,B,C}, [1] = {1,2,3}.

Observe that:
$(s,s) \in R$ so R is reflexive.
Whenever $(s,t) \in R$, then $(t,s) \in R$.
Whenever $(s,t) \in R$ and $(t,v) \in R$, then $(s,v) \in R$.

So R is an equivalence relation. Furthermore, note that

$[a] \cap [A] = \emptyset$, $[a] \cap [1] = \emptyset$, $[A] \cap [1] = \emptyset$, and that
$[a] \cup [A] \cup [1] = S$.

That is the equivalence classes partition the set S.

Consider the relation W defined as

$W = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$

We will show that W is an equivalence relation.

Reflexive: $x - x = 0 \in \mathbb{Z}$ for all $x \in \mathbb{R}$.
Symmetric : Let $a, b \in \mathbb{R}$, then $a - b = -(b - a)$,
so if $a - b \in \mathbb{Z}$ then $b - a \in \mathbb{Z}$
Transitive: Let $a, b, c \in \mathbb{R}$ if $a - b \in \mathbb{Z}$ and $b - c \in \mathbb{Z}$
then $a - c \in \mathbb{Z}$ because $a - b + b - c \in \mathbb{Z}$.

Consider a fixed value $k \in \mathbb{R}$. The equivalence class denoted by $[k]$ is defined as
$[k] = \{y \in R : k - y \in \mathbb{Z}\}$.

For example suppose $k = 4.2$, then some elements of $[k]$ could be 1.2, 42.2, 96.2 etc…

Observe that $\mathbb{R} = \bigcup_{x \in \mathbb{R}} [x]$

Consider the relation V on the set of all binary bit strings, defined as:

V = {(s,t): s,t, are binary strings that contain the same number of 1's}

For example (111, 1010100) are in V.

Show that V is an equivalence relation.

Reflexive:

Symmetric :

Transitive:

A standard notation that can be used to denote binary strings of arbitrary length is $\{0,1\}^*$.

Let [n] denote the equivalence class with respect to V as all binary strings with n 1's.

Observe that $\{0,1\}^* = \bigcup_{n \in \mathbb{N}} [n]$