CISC-102 FALL 2018

HOMEWORK 8
SOLUTIONS

(1) New parents wish to give their new baby one, two, or three different names. They have a book containing 500 names that they will choose from. How many different ways can this baby be named? Note the baby may NOT be named Alice, Alice, Alice. That is the parents may NOT use the same name more than once.

For one name there are 500 choices.
For two names there are $\frac{500!}{498!}$ choices.
And for three names there are $\frac{500!}{497!}$ choices.
We use the sum rule to obtain the total number of names to get:

$$500 + \frac{500!}{498!} + \frac{500!}{497!}$$

choices.

(2) How many passwords can be chosen under the condition that the password is exactly 7 symbols long using upper and lower case characters as well as digits from $0 \ldots 9$, such that at least one of the symbols is a digit.

There are exactly 52^7 passwords that use only upper case and lower case letters and no digits. The number of passwords that use upper and lower case letters and at least one digit is $62^7 - 52^7$.

(3) A skip straight is 5 cards that are in consecutive order, skipping every second rank (for example 3-5-7-9-J). How many 5 card hands are there (unordered selection from a standard 52 card deck) that form a skip straight?

One card defines a skip straight. That is if you know that the smallest value in the skip straight is a 3, then the other four values are fixed. A skip straight's lowest value comes from the set {A, 2, 3, 4, 5, 6 }. (We assume that Ace can be either low = 1, or high = 14. Once the numbers are fixed each of the cards can be any of the four suits. So the total number of skip straights is $4^5 \times 6$. This can also be written as:

$$\binom{4}{1}^5 \binom{6}{1}$$

(4) What is the number of ways to colour n identical objects with 3 colours? What is the number of ways to colour n identical objects with 3 colours so that each colour is used at least once?

The number of ways to colour n objects with 3 colours can be viewed as counting the number of binary strings with n 0's and 2 1's. This yields the expression:

$$\frac{(n+2)!}{n!2!} = \binom{n+2}{2} = \binom{n+2}{n}$$

To ensure that each colour is used at least once we pre-assign one object per colour leaving $n-3$ objects to be coloured with no further restrictions. We map this problem to counting binary strings with $n-3$ 0's and 2 1's. This yields the expression:

$$\frac{(n-3+2)!}{(n-3)!2!} = \binom{n-1}{2} = \binom{n-1}{n-3}$$

(5) From 100 used cars siting on a lot, 20 are to be selected for a test designed to check safety requirements. These 20 cars will be returned to the lot, and again 20 will be selected for testing for emission standards.

(a) In how many ways can the cars be selected for safety requirement testing?
$\binom{100}{20}$

(b) In how many ways can the cars be selected for emission standards testing?
$\binom{100}{20}$

(c) In how many different ways can the cars be selected for both tests?
$\binom{100}{20}\binom{100}{20}$

(d) In how many ways can the cars be selected for both tests if exactly 5 cars must be tested for safety and emission?

$\binom{100}{5}\binom{95}{15}\binom{80}{15}$

(6) Use the binomial theorem to expand the product $(x+y)^6$.

Recall: The binomial theorem can be stated as:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

So for this question we have:

$$(x+y)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5y + \binom{6}{2}x^4y^2 + \binom{6}{3}x^3y^3 + \binom{6}{4}x^2y^4 + \binom{6}{5}xy^5 + \binom{6}{6}y^6.$$

(7) Show that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + \binom{n}{n} = 0$$

HINT: Use the Binomial theorem.

Note that this equation can also be written as follows:

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i = 0$$

The binomial theorem with $a = 1$ and $b = -1$ can be written as:

$$0 = (1-1)^n = \sum_{i=0}^{n} \binom{n}{i} (1^{n-i})(-1)^i$$

And this proves the result.