CISC-102 FALL 2018

HOMEWORK 9
SOLUTIONS

(1) In the notes for Week 11 you will find Pascal's triangle worked out for rows 0 to 8. The numbers in row 8 are 1 8 28 56 70 56 28 8 1. Work out the values of rows 9 and 10 of Pascal's triangle with the help of the equation:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}.$$

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row 8:     1  8  28  56  70  56  28   8   1
row 9:  1   9  36   84  126 126  84   36  9  1
row10:1  10 45  120  210 252  210  120 45 10  1
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(2) Show that $\binom{n}{0} = \binom{n-1}{0}$, and that $\binom{n-1}{n-1} = \binom{n}{n}$ by an algebraic argument as well as a counting argument.

Recall that $0! = 1$. So we have:

$$\binom{n}{0} = \frac{n!}{n!0!} = 1,$$

and

$$\binom{n-1}{0} = \frac{(n-1)!}{(n-1)!0!} = 1.$$

The counting argument is that choosing nothing from any number of items is always 1, and in particular for n items and $n-1$ items.

$$\binom{n}{n} = \frac{n!}{n!0!} = 1,$$

and

$$\binom{n-1}{n-1} = \frac{(n-1)!}{(n-1)!0!} = 1.$$

The counting argument is that there is only one way to choose all items, and in particular for n items and $n-1$ items.

(3) Prove (using mathematical induction on n) that:

$$\sum_{m=0}^{n} \binom{m+1}{m} = \binom{n+2}{n}$$

is true for all $n \in \mathbb{N}$.

Base: When $n = 1$ we have $\binom{1}{0} + \binom{2}{1} = \binom{3}{1}$
Induction Hypothesis:

$$\sum_{m=0}^{k} \binom{m+1}{m} = \binom{k+2}{k}$$

Induction Step

$$\begin{aligned}
\sum_{m=0}^{k+1} \binom{m+1}{m} &= \sum_{m=0}^{k} \binom{m+1}{m} + \binom{k+2}{k+1} \\
&= \binom{k+2}{k} + \binom{k+2}{k+1} \text{(using the induction hypothesis)} \\
&= \binom{k+3}{k+1}
\end{aligned}$$

Therefore by the principle of mathematical induction we have shown that

$$\sum_{m=0}^{n} \binom{m+1}{m} = \binom{n+2}{n}$$

is true for all $n \in \mathbb{N}$. □

I will now redo the induction step using $k-1$ for the induction hypothesis and k for the induction step. This makes the arithmetic a bit neater.

Induction Hypothesis:

$$\sum_{m=0}^{k-1} \binom{m+1}{m} = \binom{k+1}{k-1}$$

Induction Step

$$\begin{aligned}
\sum_{m=0}^{k} \binom{m+1}{m} &= \sum_{m=0}^{k-1} \binom{m+1}{m} + \binom{k+1}{k} \\
&= \binom{k+1}{k-1} + \binom{k+1}{k} \text{(using the induction hypothesis)} \\
&= \binom{k+2}{k}
\end{aligned}$$

Therefore by the principle of mathematical induction we have shown that

$$\sum_{m=0}^{n} \binom{m+1}{m} = \binom{n+2}{n}$$

is true for all $n \in \mathbb{N}$. $\square$

(4) Use a truth table to verify that the proposition $p \vee \neg(p \wedge q)$ is a tautology, that is, the expression is true for all values of p and q.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

(5) Use a truth table to verify that the proposition $(p \wedge q) \wedge \neg(p \vee q)$ is a contradiction, that is, the expression is false for all values of p and q.

p	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$	$(p \wedge q) \wedge \neg(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

(6) Use a truth table to show that $p \vee q \equiv \neg(\neg p \wedge \neg q)$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
T	T	F	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	F	T	F

(7) Show that the following argument is valid.

$$p \rightarrow q, \neg q \vdash \neg p$$

We need to show that $[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$ is a tautology, and we do so using a truth table as follows:

$\neg p$	p	q	$\neg q$	$p \rightarrow q$	$(p \rightarrow q) \wedge \neg q$	$[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$
F	T	F	T	F	F	T
F	T	T	F	T	F	T
T	F	T	F	T	F	T
T	F	F	T	T	T	T