CISC-102 FALL 2019

HOMEWORK 9
SOLUTIONS

(1) Consider the equation

$$\sum_{i=0}^{2} \binom{3}{i}\binom{2}{2-i} = \binom{5}{2}. \qquad (1)$$

(a) Use algebraic manipulation to prove that the left hand and right hand sides of equation (1) are in fact equal.

First we work out the left hand side of equation (1).

$$\begin{aligned}\sum_{i=0}^{2} \binom{3}{i}\binom{2}{2-i} &= \binom{3}{0}\binom{2}{2} + \binom{3}{1}\binom{2}{1} + \binom{3}{2}\binom{2}{0} \\ &= 1+6+3 \\ &= 10\end{aligned}$$

And the right hand side of equation (1).

$$\begin{aligned}\binom{5}{2} &= \frac{5!}{3!2!} \\ &= \frac{5\times 4}{2} \\ &= 10\end{aligned}$$

(b) Use a counting argument to prove that the left hand and right hand sides of equation (1) are in fact equal.

On the right we count the number of ways of selecting 2 balls from a bag of 5 different balls without regard to ordering.

On the left we have two bags one with 3 balls and the other with 2 balls, which we call the 3bag and 2bag respectively. We now sum the products of selecting, without ordering, 0 from the 3bag times 2 from the 2bag, 1 from the 3bag and 1 from the 2bag, and 2 from the 3 bag and 0 from the 2 bag.

(2) Now consider a generalization of the previous equation.

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}. \tag{2}$$

Use a counting argument to prove that the left hand and right hand sides of equation (2) are in fact equal.

On the right we count the number of ways of selecting k balls without ordering from a bag of $m+n$ balls. On the left we count selections from two bags one with m balls and the other with n balls. We sum products of selecting $k-i$ and i balls from the two bags.

(3) In the notes for Week 9 you will find Pascal's triangle worked out for rows 0 to 8. The numbers in row 8 are 1 8 28 56 70 56 28 8 1. Work out the values of rows 9 and 10 of Pascal's triangle with the help of the equation:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}. \tag{3}$$

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row 8:     1   8   28   56   70   56   28    8    1
row 9:   1   9   36    84   126 126   84    36   9   1
row10:1   10 45   120   210 252   210   120 45 10   1
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(4) Show that $\binom{n}{0} = \binom{n-1}{0}$, and that $\binom{n-1}{n-1} = \binom{n}{n}$ by an algebraic argument as well as a counting argument.

Recall that $0! = 1$. So we have:

$$\binom{n}{0} = \frac{n!}{n!0!} = 1,$$

and

$$\binom{n-1}{0} = \frac{(n-1)!}{(n-1)!0!} = 1.$$

The counting argument is that choosing nothing from any number of items is always 1, and in particular for n items and $n-1$ items.

$$\binom{n}{n} = \frac{n!}{n!0!} = 1,$$

and

$$\binom{n-1}{n-1} = \frac{(n-1)!}{(n-1)!0!} = 1.$$

The counting argument is that there is only one way to choose all items, and in particular for n items and $n-1$ items.

(5) Prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0$$

Note that this equation can also be written as follows:

$$\sum_{i=0}^{n} \binom{n}{i} (-1^i) = 0$$

HINT: This can be viewed as a special case of the binomial theorem.

Observe that by the binomial theorem we have:

$$0 = (1-1)^n = \sum_{i=0}^{n} \binom{n}{i} (-1^i) 1^{n-i}$$

And this proves the result.

(6) Prove (using mathematical induction on n) that:

$$\sum_{m=0}^{n} \binom{m+1}{m} = \binom{n+2}{n}$$

is true for all $n \in \mathbb{N}$.

Base: When $n = 1$ we have $\binom{1}{0} + \binom{2}{1} = \binom{3}{1}$

Induction Hypothesis:

$$\sum_{m=0}^{k} \binom{m+1}{m} = \binom{k+2}{k}$$

Induction Step

$$\begin{aligned}\sum_{m=0}^{k+1}\binom{m+1}{m} &= \sum_{m=0}^{k}\binom{m+1}{m}+\binom{k+2}{k+1}\\ &= \binom{k+2}{k}+\binom{k+2}{k+1}\text{(using the induction hypothesis)}\\ &= \binom{k+3}{k+1}\end{aligned}$$

Therefore by the principle of mathematical induction we have shown that

$$\sum_{m=0}^{n}\binom{m+1}{m}=\binom{n+2}{n}$$

is true for all $n \in \mathbb{N}$. $\square$

(7) Use a truth table to verify that the proposition $p \vee \neg(p \wedge q)$ is a tautology, that is, the expression is true for all values of p and q.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

(8) Use a truth table to verify that the proposition $(p \wedge q) \wedge \neg(p \vee q)$ is a contradiction, that is, the expression is false for all values of p and q.

p	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$	$(p \wedge q) \wedge \neg(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

(9) Use a truth table to show that $p \vee q \equiv \neg(\neg p \wedge \neg q)$.

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
T	T	F	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	F	T	F

(10) Show that the following argument is valid.

$$p \to q, \neg q \vdash \neg p$$

We need to show that $[(p \to q) \wedge \neg q] \to \neg p$ is a tautology, and we do so using a truth table as follows:

$\neg p$	p	q	$\neg q$	$p \to q$	$(p \to q) \wedge \neg q$	$[(p \to q) \wedge \neg q] \to \neg p$
F	T	F	T	F	F	T
F	T	T	F	T	F	T
T	F	T	F	T	F	T
T	F	F	T	T	T	T

(11) Let A = {1,2,3,4,5}. Determine the truth value of each of the following statements.

(a) $(\exists x \in A)(x + 2 = 7)$
This is true with $x = 5$.

(b) $(\forall x \in A)(x + 2 < 8)$
This is true, because

$$(1 + 2 < 8) \wedge (2 + 2 < 8) \wedge (3 + 2 < 8) \wedge (4 + 2 < 8) \wedge (5 + 2 < 8).$$

(c) $(\exists x \in A)(x + 3 < 2)$
This is false because:

$$(1 + 3 \not< 2) \wedge (2 + 3 \not< 2) \wedge (3 + 3 \not< 2) \wedge (4 + 3 \not< 2) \wedge (5 + 3 \not< 2).$$

(d) $(\forall x \in A)(x + 3 \leq 9)$
This is true, because

$$(1 + 3 \leq 9) \wedge (2 + 3 \leq 9) \wedge (3 + 3 \leq 9) \wedge (4 + 3 \leq 9) \wedge (5 + 3 \leq 9).$$