CISC-102
Winter 2019
Week 3

Principle of Mathematical Induction

A *proposition* is defined as a statement that is either true or false. We will at times make a declarative statement as a proposition and then proceed to prove that it is true. Alternately we may provide an example (called a *counterexample*) showing that the proposition is false.

Let P be a proposition defined on the positive integers $\mathbb{N}$; that is, P(n) is either true or false for each $n \in \mathbb{N}$. Suppose P has the following two properties:
(i) P(1) is true.
(ii) P(k+1) is true whenever P(k) is true.

Then by the principle of Mathematical Induction P is true for every positive integer $n \in \mathbb{N}$.

Mathematical induction is by far the most useful tool for proving results in computing.

Note: Step (i) may be replaced by any integer b and then the principle of mathematical induction would hold for all integers greater than or equal to b.

Example:

$$2^{n-1} + 2^{n-2} + \cdots + 2^1 + 2^0 = 2^n - 1$$

for all natural numbers n .

Recall from last weeks notes we reviewed binary numbers.

Consider the binary number 1111.

It is equal to:

$1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 15 = 2^4 - 1.$

1	1	1	1
1 × 8	1 × 4	1 × 2	1 × 1
1×2^3	1×2^2	1×2^1	1×2^0

We can verify that the equation holds for small values of n, say $n = 1,2, 3$. However this does not prove that the equation is true for all natural numbers n.

Let P(n) be the proposition that the equation above is correct for the natural number n. We will use mathematical induction to prove that P(n) is true for every $n \in \mathbb{N}$.

The steps to using induction have been described as:

(i)*P(1) is true.*
(ii) P(k+1) is true whenever P(k) is true.

I will label step (i) the Base. Step (ii) will be split into two parts labelled the Induction Hypothesis, and the Induction Step.

$2^{n-1} + 2^{n-2} + \cdots + 2^1 + 2^0 = 2^n - 1$ for all natural numbers n .

Base: We show that P(1) is true, that is,

for $n = 1$, we have $2^{1-1} = 2^1 - 1 = 1.$

Induction Hypothesis : Assume that P(k) is true, for some fixed natural number k, such that $k \geq 1$. That is,

$$2^{k-1} + 2^{k-2} + \cdots + 2^1 + 2^0 = 2^k - 1$$

Induction Step: We prove that P(k+1) is true using the assumption that P(k) is true.

$$2^k + 2^{k-1} + 2^{k-2} + \cdots + 2^1 + 2^0 = 2^k + 2^k - 1$$

The equality holds because we use the induction hypothesis to replace

$$2^{k-1} + 2^{k-2} + \cdots + 2^1 + 2^0$$

with

$$2^k - 1.$$

Now we do a bit of arithmetic.

$$2^k + 2^k - 1 = 2(2^k) - 1 = 2^{k+1} - 1$$

If you follow the chain of equalities we have:

$$2^k + 2^{k-1} + 2^{k-2} + \cdots + 2^1 + 2^0 = 2^{k+1} - 1$$

and that P(k+1) is true.

Therefore by the principle of mathematical induction we conclude that P(n) is true for all natural numbers n. $\square$

Observe that the sum:

$$2^{n-1} + 2^{n-2} + \cdots + 2^1 + 2^0$$

can be written as:

$$\sum_{i=0}^{n-1} 2^i$$

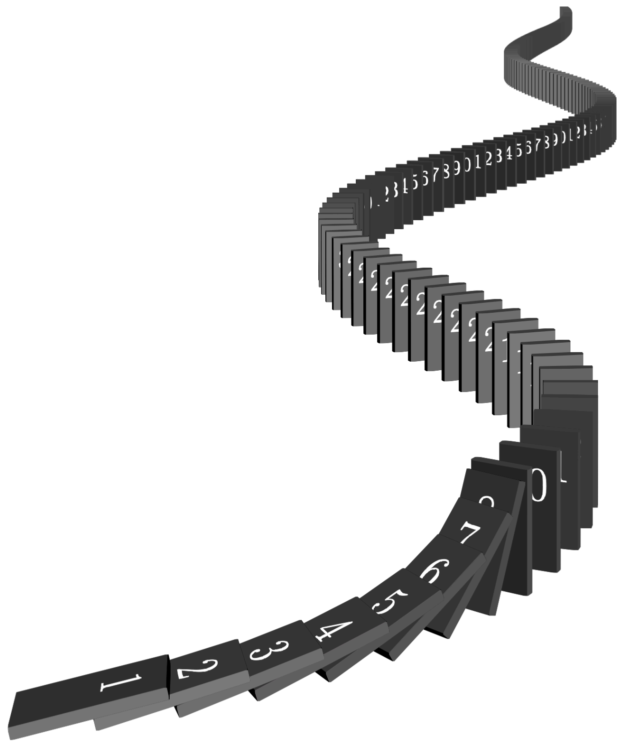
Following mathematics convention the sequence of equalities in the induction step can be written as follows:

$$\sum_{i=0}^{k} 2^i = \sum_{i=0}^{k-1} 2^i + 2^k$$

$$= 2^k - 1 + 2^k \text{ (using the Ind. Hyp.)}$$

$$= 2^{k+1} - 1$$

Induction



As an analogy think of an unending sequence of dominoes. You can be sure that all will fall if:

1. The first one falls. (P(1))
2. And if the k^{th} one falls it will knock over the $k+1^{st}$, that is, P(k) true implies that P(k+1) is also true.

What is the value of the sum:

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}, n \in \mathbb{N}, n \geq 1 \text{ ?}$$

For $n = 1$ the sum is $1/2$,
for $n = 2$ the sum is $3/4$,
for $n = 3$ the sum is $7/8$.

These observations lead us to the equation:

$$\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$$

We can now prove this using mathematical induction.

Let P(n) denote the proposition

$$\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$$

We prove, using mathematical induction, that P(n) is true for all natural numbers n.

Base: For n = 1, we have 1/2 = 1 - 1/2.

Induction Hypothesis: P(k) is true for a fixed $k \in \mathbb{N}$, $k \geq 1$, that is: $\sum_{i=1}^{k} \frac{1}{2^i} = 1 - \frac{1}{2^k}$.

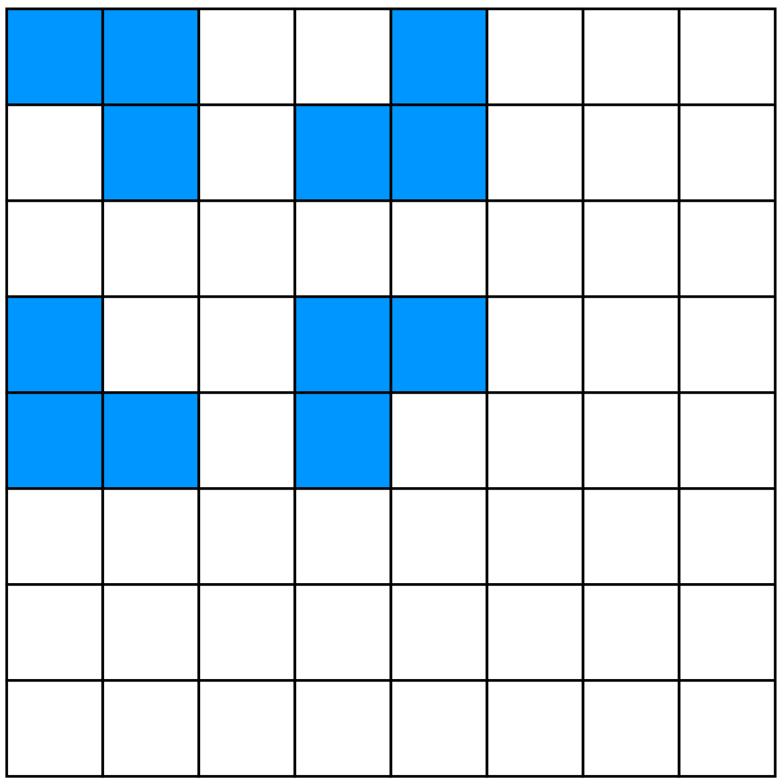
Induction Step: We show that P(k) true implies that P(k+1) is true.

$$\begin{aligned}\sum_{i=1}^{k+1} \frac{1}{2^i} &= \sum_{i=1}^{k} \frac{1}{2^i} + \frac{1}{2^{k+1}} \\ &= 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \text{ (using the Ind. Hyp.)} \\ &= 1 - \frac{1}{2^{k+1}}\end{aligned}$$

Therefore by the principle of mathematical induction we conclude that P(n) is true for all natural numbers n. □

The next example does not involve sums or generalized set operations, but we will still be able to use induction.

Consider a $2^n \times 2^n$ square grid, as shown below for $n = 3$.

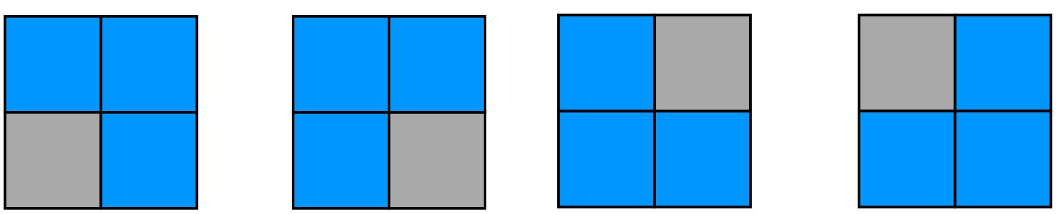


An L-shaped piece is the union of three squares in the shape of an L. An L-shaped piece can be aligned and oriented over the square grid in four ways as shown above.

A tiling of the grid with L-shaped pieces is an arrangement of the pieces so no two overlap (pairwise empty intersection) and the pieces cover the entire grid (union of the pieces is the entire grid) So a tiling can also be described as a partition of the grid into L-shaped pieces.

We can prove, using mathematical induction, that it is impossible to tile a $2^n \times 2^n$ square grid with L-shaped pieces. But we will leave that for later on in the course. Today, we will prove that L-shaped pieces can tile a $2^n \times 2^n$ square grid with <u>one square missing,</u> for all natural numbers n.

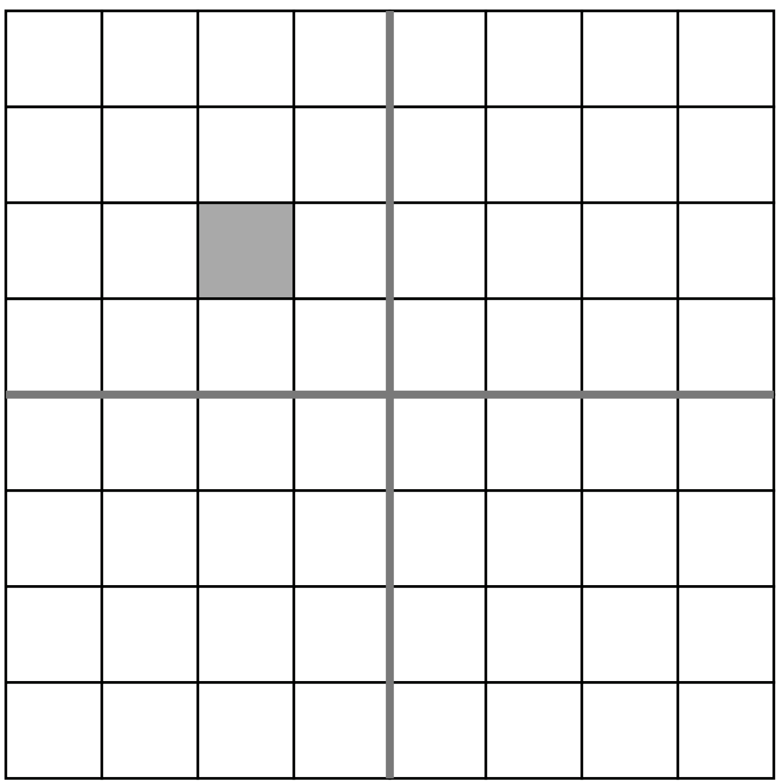
Base: P(1) is the proposition the a 2×2 square grid with one piece missing can be tiled with L-shaped pieces. The illustration below exhaustively enumerates the ways in which this can be done.



Induction Hypothesis: We assume that P(k) is true, for some fixed k, $k \in \mathbb{N}$, that is a $2^k \times 2^k$ square grid with one piece missing can be tiled with L-shaped pieces.

Induction Step: We show that P(k) true implies that P(k+1) is true.

Consider a $2^{k+1} \times 2^{k+1}$ square grid with one piece removed as shown below for $k = 2$.



The $2^{k+1} \times 2^{k+1}$ square grid with one piece removed can be partitioned into four $2^k \times 2^k$ square grids. One of the $2^k \times 2^k$ square grids has a square removed.

Place an L-shaped piece at the boundary of the four $2^k \times 2^k$ square grids, so as to avoid the the $2^k \times 2^k$ square grid with one piece removed. This in effect creates a collection of four $2^k \times 2^k$ square grids each having one square removed. Now apply the induction hypothesis to complete the tiling.

Therefore, we have shown that the proposition P(k) true implies that P($k+1$) is true. So by the principle of mathematical induction we have P(n) is true for all natural numbers. □

Example: The sum of the first n odd numbers is n^2.

Let's try it for some small values of n.
$n = 1$ $(1 = 1^2)$, $n = 2$ $(1+ 3 = 4 = 2^2)$,
$n = 3$ $(1 + 3 + 5 = 9 = 3^2)$
This is NOT A PROOF! These simply show that the propositions P(1), P(2) and P(3) are true.

Preliminaries: The k^{th} odd number can be written as $2k-1$.
e.g.[1] $1 = 2 \times 1-1$, $3 = 2\times 2-1$, $5=2\times 3-1$ *etc*[2]. This fact will be useful for proving that the sum of the first n odd numbers is n^2.

At this point let's take a closer look at what is meant by an odd number and define it precisely.

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[1]exempli gratia are Latin words meaning "for example"

[2] et cetera are Latin words meaning "and so on"

Let n be a natural number.

Even Natural numbers

If 2 divides n, that is n/2 is a natural number then we say that n is even. For example 2/2 = 1 so 2 is even, 4/2 = 2 so 4 is even. Every even natural number can then be expressed as a multiple of 2. For example $2 \times 1 = 2$, and $2 \times 2 = 4$. So if k is a natural number $2k$ is even.

Odd Natural numbers

When we study integers and integer arithmetic we will be better equipped to formally define what is meant be an odd number. For now we can simply define an odd natural number as any natural number that is 1 less than an even number, that is $2k$-1.

Theorem: The proposition P(n), the sum of the first n odd numbers is n^2 for all natural numbers n.

Proof:

Base: P(1) $1 = 1^2$, so P(1), the base case is true.

Induction Hypothesis: Assume P(k) is true where k is any arbitrary integer greater than or equal to 1.

That is, $1 + 3 + 5 + ... + 2k-1 = k^2$.

Induction Step: Consider the sum of the first k+1 odd numbers.

$$\underline{1 + 3 + 5 \ldots + 2k-1} + 2k+1 = \underline{k^2} + 2k+1 \quad \text{(Ind. Hyp.)}$$
$$= (k+1)(k+1) \text{ (factor}$$
$$= (k+1)^2$$

Therefore, we have shown that the proposition P(k) true implies that P($k+1$) is true. So by the principle of mathematical induction we conclude that P(n) is true for all natural numbers n. □

Some of you may have learned to resolve this type of sequence of equations as follows.

RHS	LHS
$1 + 3 + \ldots + 2K+1 =$	$(k+1)^2$
$k^2 + 2k + 1 \ =$	$(k+1)\ (k+1)$
$(k+1)(k+1) \ =$	$(k+1)(k+1)$

You can use this as a preliminary step but it is an abuse of notation. Once you have worked this preliminary step you can re-write the sequence by going down the right hand side, and then up the left hand side (omitting repeats).

Theorem: The proposition P(n), the sum of the first n even natural numbers is $n^2 + n$ for all natural numbers n.

Proof:

Base: P(1)**:** $2 = 1^2 + 1$, so P(1), the base case is true.

Induction Hypothesis: Assume P(k) is true where k is any arbitrary integer greater than or equal to 1. That is, $2 + 4 + 6 + ... + 2k = k^2 + k$.

Our goal in the Induction step is to show that

$$2 + 4 + \ldots + 2k + 2(k + 1) = (k+1)^2 + k+1.$$

Tip: $(k+1)^2 + k + 1 = k^2 + 2k + 1 + k + 1$

$$= k^2 + 3k + 2$$

Induction Step:

$$2 + 4 \;\ldots + 2k + 2(k + 1) = k^2 + k + 2k + 2$$

$$= k^2 + 3k + 2$$

Therefore, we have shown that the proposition P(k) true implies that P(k+1) is true. So by the principle of mathematical induction we conclude that P(n) is true for all natural numbers n. □

You may be tempted to think that it is enough to just enumerate a few cases to convince yourself that a proposition is true.

Let P(n) be the proposition that $3n < 1000$ for all natural numbers n.

$3 \times 1 < 1000$
$3 \times 2 < 1000$
.
.
.
$3 \times 333 < 1000$

So P(n) must be true. (Obviously NOT!)

Here's another example where the first few cases lead to a false conclusion.

Let P(n) be the proposition that $n! < 2^n$ is true for all $n \in \mathbb{N}$.

Observe that:

$1! = 1 < 2$
$2! = 2 < 4$
$3! = 6 < 8$

However, if we check one additional case,

$4! = 24 > 16.$

In fact we can use induction to prove that

$n! \geq 2^n$, is true for all $n \in \mathbb{N}$, $n \geq 4$.

Students sometimes find proving results using inequalities (that is relations like $\leq$, $<$, $\geq$, $>$) hard to grasp. Don't worry if you don't get this the first time you read it. If you persist you should eventually understand this.

Theorem: $n! \geq 2^n$, for $n \in \mathbb{N}$, $n \geq 4$.

Proof: Let P(n) be the proposition $n! \geq 2^n$, for $n \geq 4$.

Base: P(4) is true because $4 \times 3 \times 2 \times 1 \geq 2^4$

Induction Hypothesis: P(k) is true for $k \geq 4$.

Induction Step: $(k + 1)\,! = k!\,(k+1)$

$\geq 2^k\,(k+1)$ (because P(k) is true)

$\geq 2^k\,(2)$(because $k \geq 4$)

$\geq 2^{k+1}$

Therefore, we have shown that the proposition P(k) true implies that P($k+1$) is true. So by the principle of mathematical induction we have P(n) is true for all natural numbers $n \geq 4$. □

Yet Another Example

Observe that if we have sets A_1, A_2, B_1, B_2 such that

$$A_1 \subseteq B_1, A_2 \subseteq B_2$$

then $(A_1 \cup A_2) \subseteq (B_1 \cup B_2)$.

This should be apparent because $A_1 \subseteq B_1, A_2 \subseteq B2$ implies that every element of $(A_1 \cup A_2)$ is also an element of B_1 or B_2

We will make use of this fact to prove:

P(n) the proposition that if $A_1, A_2, \ldots A_n, B_1, B_2, \ldots, B_n$ are sets such that $A_i \subseteq B_i$ for all i, $1 \leq i \leq n$,
then $\cup_{i=1}^{n} A_i \subseteq \cup_{i=1}^{n} B_i$ for all natural numbers n, $n \geq 2$.

P(n) the proposition that if $A_1, A_2, \ldots A_n, B_1, B_2, \ldots, B_n$ are sets such that $A_i \subseteq B_i$ for all i, $1 \le i \le n$,
then $\cup_{i=1}^{n} A_i \subseteq \cup_{i=1}^{n} B_i$ for all natural numbers n, $n \ge 2$.

Base: We have previously argued that A_1, A_2, B_1, B_2 such that $A_1 \subseteq B_1, A_2 \subseteq B_2$ then $(A_1 \cup A_2) \subseteq (B_1 \cup B_2)$.
Induction Hypothesis: We assume that P(k) is true, that is for some fixed k $\in \mathbb{N}$, such that $A_i \subseteq B_i$ for all i, $1 \le i \le k$, then $\cup_{i=1}^{k} A_i \subseteq \cup_{i=1}^{k} B_i$.
Induction Step: We show that P(k) true implies P($k+1$) true.
Observe that $\displaystyle\bigcup_{i=1}^{k+1} A_i = \bigcup_{i=1}^{k} A_i \cup A_{k+1}$
By the induction hypothesis we have $\cup_{i=1}^{k} A_i \subseteq \cup_{i=1}^{k} B_i$. Furthermore $A_{k+1} \subseteq B_{k+1}$. Therefore we conclude that

$$\cup_{i=1}^{k+1} A_i \subseteq \cup_{i=1}^{k+1} B_i.$$

Therefore, we have shown that the proposition P(k) true implies that P($k+1$) is true. So by the principle of mathematical induction we have P(n) is true for all natural numbers $n \ge 2$. □