Geometry and Harmony

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Abstract

A musical scale can be viewed as a subset of notes or pitches taken from a chromatic universe. Given integers (N, K) $N > K$ we use particular integer partitions of N into K parts to construct distinguished sets, or scales. We show that a natural geometric realization of these sets have maximal area, so we call them maximal area sets. We then discuss properties of maximal area sets for the integer pairs (12,5) (12,6) (12,7) and (12,8) with the obvious relevance to our normal chromatic collection of 12 pitches.

1 Introduction

Geometry and music are intertwined in many different ways. Music notation uses shape and space to convey pitch and time information. Guitar players visualize harmonic structures such as scales, arpeggios and chords, as geometric shapes on the fretboard. The origins of our musical system of seven note scales chosen form a collection of 12 pitches may be described in terms vibrating strings of various lengths. Combinatorics is another branch of mathematics that is used in music analysis. Inevitably combinatorial insight is supported by a picture, that is, a geometric representation.

Consider a circle with twelve equidistant points spread out on the boundary of a circle. The twelve points represent the 12 pitches we use. From these 12 points we choose a subset of points. Musically some subsets of at least five of the twelve pitches are called scales. Some of these subsets are more prominent than others.

Consider the examples shown in Figure 1. Upon selecting a subset of points we connect them in sequence to construct a convex polygon. We consider distinct polygons up to rotation. This corresponds to the notion that different modes from the same scale are not different scales.

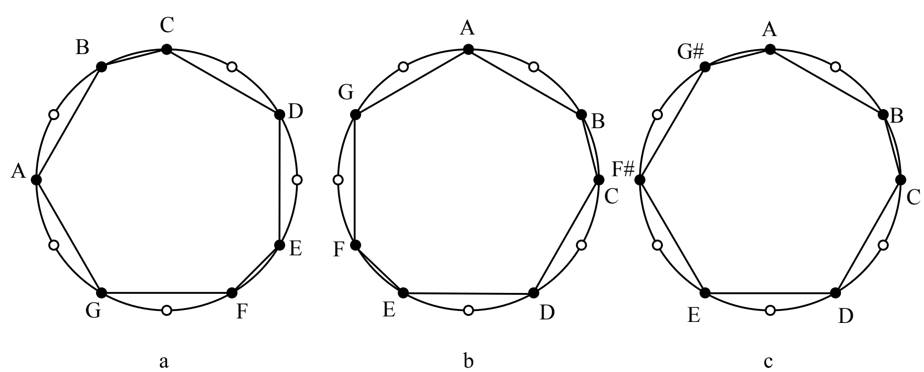

Figure 1: *The subsets in a) and b) represent two modes of the diatonic scale, Ionian and Aeolian, also known as the major and natural minor scales. For our purposes these two scales are considered to be equivalent. The diagram of part c) represents the ascending melodic minor scale and is distinct from a) and b).*

Since there are twelve equally spaced markings on the circle it makes sense to call these diagrams *clock diagrams*. Representing the notes of a scale by a polygon appears in a paper published in 1937 by E. Krenek [8], so sometimes these diagrams are called *Krenek diagrams* as in the paper by McCartin [10]. However, in an account by Nolan [11], Heinrich Vincent used this very same representation in his paper published in 1862 [15]. When looking at the notes of the usual diatonic scale, we soon observe that they are spread out evenly amongst the twelve chromatic pitches. Clough and Douthett [1] define a set to be maximally even if every interval obtainable from the set comes in one of two ”flavours”. Enumerating the intervals in the diatonic set we have the two flavour intervals : major and minor seconds, thirds, sixths, and sevenths. Then we have the perfect fifth and fourth and the augmented fourth or diminished fifth to complete the enumeration. This property is generalized for any choice for pairs of integers (N, K) with $K < N$, and are collectively called *maximally even sets*. The maximally even sets are unique (up to rotation) and include the some of the most widely used scales in Western music, the common anhemitonic pentatonic scale, the six note whole note scale, and the eight note diminished scale. The symmetric augmented triad and the diminished seventh chords are also maximally even.

When maximally even sets are represented by a clock diagram, then those points are subsets which uniquely maximize the sum of inter-point distances [3, 2]. A similar continuous case of this phenomenon is described by Fejes Tóth [12]. In that paper it is shown that a finite set of N points that maximize the sum of inter-point distances are located on the vertices of a regular convex N-gon. That is, the points are spread out as evenly as possible on the circumference of a circle.

In his book on harmony for the improvising jazz musician Levine [9] describes four fundamental scales that are useful for jazz improvisation. These four scales are the major scale (and its seven modes) the melodic minor scale, and the symmetric whole-tone and diminished scales. In jazz termology the term melodic minor almost always denotes the ascending melodic minor scale, and we follow this convention.

Three of these four scales are maximally even, the exception being the melodic minor scale which is not. Thus given pairs (12,8) (12,6) and (12,7) we may ask whether there is a mathematical characterization that exactly describes Levine’s four fundamental scales. In this note we arrive at such a characterization.

The paper is organized as follows. In the next section we provide a mathematical discussion on a class of subsets of K elements chosen from N. This characterization is both combinatorial and geometric. We begin by describing the so called maximal area sets, and prove some mathematical properties of these sets. The maximal area sets are interesting in their own right, but do not quite satisfy the goals mentioned above, as this characterization includes subsets of (12,8) and (12,7) that are not from the four fundametal scales. We then define and analyze complementary maximal area sets and show that this characterization satisfies our requirements.

2 Maximal Area Sets

A common misconception is that the prefix *di* in diatonic refers to the number *two*, signifying the characteristic that there are two step sizes in the usual diatonic set. However, the truth is the prefix *dia* refers to *from the tonic* [5]. However, this definition is the ideal spring board from which we can launch an exploration of scales that satisfy this property, that is collections of subsets of 7 pitches from 12, so that the spaces between consecutive pitches are either whole tones or semitones. This results in three distinct scales. Using clock diagrams we show these three distinct scales in Figure 2. In (a) we recognize the standard diatonic scale, (b) represents the ascending melodic minor (also called jazz minor) and in (c) we have the symmetric whole note scale plus a note, or the Neapolitan major scale.

It is not hard to verify that the polygons representing the scales have an equal area, and this area is maximized for any choice of seven points from twelve. Thus we will refer to these scales as *maximal area scales*, or more generically *maximal area subsets* which we abbreviate as MA sets.

We generalize this notion for any subset of K chosen from N. It will be most convenient to define our subsets in terms of integer partitions.

An *integer partition* of a natural number N is a way of writing N as an unordered sum of natural numbers. In [7] Keith points out the connection between integer partitions and musical scales.

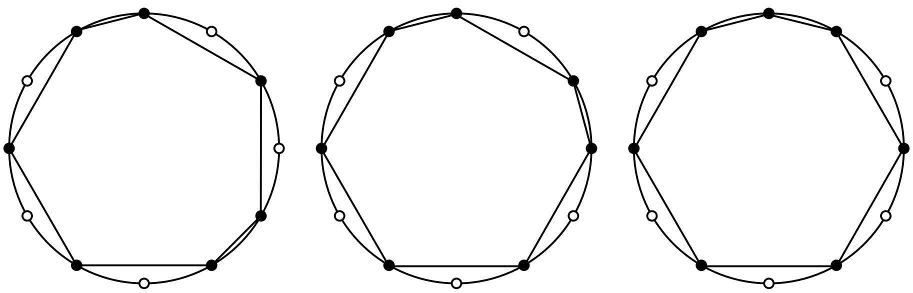

Figure 2: *Clock diagrams of the three MA scales. The interval structure of these scales are (a) the diatonic scale, (b) the ascending melodic minor (c) the Neapolitan major scale*

Consider positive integers N, K $K < N$ and a partition of N using exactly K positive integer summands, that is, $N = a_1 + a_2 + \cdots + a_k$. Furthermore, if we require that the summands differ by at most one, that is, $|a_i - a_j| \leq 1$, then we obtain our desired generalization for MA sets.

The following theorem provides a mathematical foundation for constructing and anylizing MA sets.

Theorem 2.1 *Given integers N, K with $K < N$, there exist unique integers u and m such that $N = mu + (K - m)(u + 1)$.*

Observe that for N, K, U, m as defined above we have the integer partition $N = a_1 + a_2 + \cdots + a_K$ with $a_i = u,$ for $i = 1 \ldots m$ and $a_i = u + 1,$ for $i = m + 1 \ldots K$. With the understanding that $i = m + 1 \ldots K$ is the empty set in the event that $m = K$, that is K divides N.

Proof: Let

$$u = \left\lfloor \frac{N}{K} \right\rfloor \text{ and } v = \left\lceil \frac{N}{K} \right\rceil .$$

Note that if K divides N then $v = u$, otherwise $v = u + 1$.

For the case $v = u$ we have $N = Ku$. Considering the case where $v = u + 1$ we have the equality $(N - Ku)v + (Kv - N)u = N(v - u) = N$. Thus $m = Kv - N = K(u + 1) - N$. Since u implies m it suffices to show that u is the unique value satisfying the required conditions. When K divides N uniqueness follows from the division algorithm [6]. When K does not divide N we enumerate the cases of using a number larger or smaller than u. Thus let w be an integer and $w > \lfloor \frac{N}{K} \rfloor$. However, this implies that $Kw > N$ so w cannot be greater than u. A similar symmetric argument can be used to show that $w < \lfloor \frac{N}{K} \rfloor$ leads to a contradiction.

Thus we have shown that u is unique, completing our proof. □

Theorem 2.2 *If a set is ME then it must also be MA.*

Proof: Since all intervals of ME sets differ by at most one, afortiori the single step interval differs by at most one. The uniqueness of u as proved in theorem 2.1 completes the argument. □

As was shown by the example illustrated in Figure 2 that although for any N, K there are unique values of u, m, one can possibly obtain more than one scale with step sizes u and $u + 1$ by reordering the positions of the u steps with respect to the $u + 1$ steps.

We now turn to the question of the area of the representative polygons. Referring to Figure 3 the area of the seven-gon is obtained by summing triangle areas.

Assuming that the seven-gon representing these scales is circumscribed by a circle with radius one, an equation giving the polygon area is: $2\sin(\pi/12)\cos(\pi/12) + 3\sin(2\pi/12)\cos(2\pi/12)$. In general given

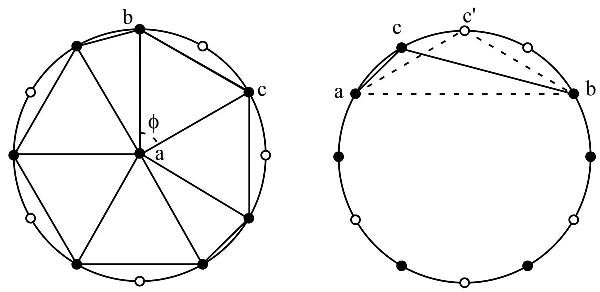

Figure 3: *One can obtain the area of the seven-gon by summing the areas of triangles as suggested above. Assuming that the radius of the circumscribing circle is 1, the sides ab and ac are of length 1 and the length of the side bc is $2\sin(\phi/2)$. Thus, referring to the diagram on the left, we can deduce that the area of the triangle a, b, c is given by $2\sin(\phi/2)\cos(\phi/2)$. On the right, observe that the area of triangle $abc' > abc$ illustrating lemma 2.3.*

(N, K, u, m) and m at least 3 the area of the representative polygons is: $m\sin(u\pi/N)\cos(u\pi/N) + K - m\sin((u+1)\pi/N)\cos((u+1)\pi/N)$.

We claim that all of these 7-gons are area maximizing 7-gons. This is easy enough to verify for this example. We prove the result for the general case in the next lemma.

Lemma 2.3 *Given (N, K, u, m) all polygon representations of these MA sets have equal area and that area is maximized.*

Proof: Consider for the sake of contradiction a K-gon P' from the N clock that is not from the set of MA scales. Since the K-gon is not from the MA set the difference between the smallest and largest interval sizes is greater than one. Using w to denote the smaller interval size the area contributions of these two intervals is found by taking the area of two triangles which are $\sin(w\pi/N)\cos(w\pi/N)$ and $\sin((w+k)\pi/N)\cos((w+k)\pi/N)$ where $k > 1$. The first derivative of $\sin(x)\cos(x)$, namely $cos^2(x) - sin^2(x)$ is positive and strictly decreasing in the range $0 \leq x \leq \pi/4$. Let $X = \sin(w\pi/N)\cos(w\pi/N) + \sin((w+k)\pi/N)\cos((w+k)\pi/N)$ and $Y = \sin((w+1)\pi/N)\cos((w+1)\pi/N) + \sin((w+k-1)\pi/N)\cos((w+k-1)\pi/N)$. And we have the inequality $X < Y$.

Thus P' cannot have maximal area. □

An intuitive geometric demonstration of the proof of Lemma 2.3 shows the difference in area obtained by increasing the size of the smallest interval by one and decreasing the size of the largest interval by one. In fact for each K-gon there is an equivalence class of K-gons with equal area, that is all K-gons that have the same K partition of N. We can choose to realize the K-gon with a polygon that has the smallest and largest intervals of the K-gon are adjacent. Since the K-gon is not from an MA set the difference between the largest and smallest interval sizes is more than one. See Figure 3 for an illustration. The rest of the argument follows by using simple geometry. Again referring to Figure 3 we obtain polygon P' from polygon P by moving the point c to c', that is, we decrease the size of the largest interval by one and increase the size of the smallest interval by one. The difference in area between the initial K-gon P and a newly constructed P' can be completely characterized by the difference in areas of the triangles abc and abc'.

3 Complementary Maximal Area Sets.

We consider the scales that are discussed by Levine [9] in his book on Jazz harmony. The most important 5 note scales are the common anhemitonic pentatonic scales. These scales can be characterized as the

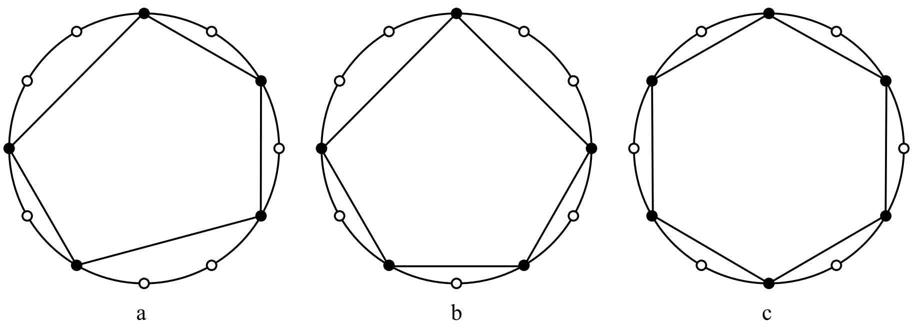

Figure 4: *The five and six note maximal area scales.*

complement of the diatonic set. Their use is ubiquitous and forms the basis of improvisation in much of popular music. The familiar pentatonic scales are known to be ME, and thus by Theorem 2.2 the pentatonic scales are also MA. There is one more collection of 5 notes that are MA and they are shown in Figure 4. To my knowledge this scale is not widely used. There is a single MA set (12,6) the symmetric wholetone scale, also shown in Figure 4. The seven note MA sets have already been discussed. There are 10 eight note MA sets, as described in [7, p. 31], too many to illustrate here. However, let us instead consider all MA sets of of size eight whose complementary set is also MA. We now have just a single set. In Figure 5 we show the single MA set of size eight whose complement is also MA. Let us call these two sets complementary MA sets. The six note MA set is its own complement. Let us now consider complementary MA sets of size seven and five.

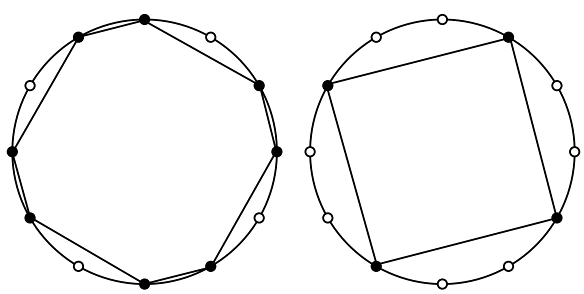

Figure 5: *The single complementary maximal area octatonic scale (diminished scale) with its complement.*

We see that there are exactly two sets of size 7 that are complementary MA, and these are the diatonic set, and the melodic minor. Their complements are shown in Figure 4 a) and b) respectively.

Thus we have been able to capture a mathematical property that characterizes Levine's four fundamental scales.

We showed that ME sets are MA. Using CMA to denote complementary maximal area sets, it is easy to argue that ME sets are also CMA. This follows immediately from the fact that the complement of an ME set is also ME [1].

4 Discussion

We have shown that a particular integer partition of N into K parts leads to maximal area polygons when these sets are represented with a clock diagram. These so called maximal area sets are computationally easy to find. However, a classification that seems more interesting uses the complementary maximal area sets. We have demonstrated that the complementary maximal area sets for (12,6) (12,7) and (12,8) contain the four fundamental scales as defined by Levine in his book on Jazz improvisation. These fundamental four scales by no means exhaust the large number of scales that are regularly used by jazz musicians. The common anhemitonic pentatonic scale is widely used. It is interesting to note that there are two complementary maximal area sets derived from (12,5). The five note scale depicted in Figure 4 (b) is new to me. It would be interesting to know whether there are any useful melodic musical properties of this scale. If we consider a rhythmic analog of the clock diagrams, that is, selected points represent onsets of beats, then the MA set shown in Figure 4 (b) represents the hand Flamenco clapping pattern used in the Solea, Buleria, and Guajira, see [4]

I am pleased to acknowledge the papers of Godfried Toussaint, [13, 14] as my first encounter with the use of clock diagrams to represent musical pitch and/or musical rhythm.

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