Chapter 51

The Aquarium Keeper's Problem*

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Abstract

We solve the problem of computing the shortest closed path inside a given polygon which visits every edge at least once (*Aquarium Keeper's Tour*). For convex polygons, we present a linear-time algorithm which uses the reflection principle and shortest-path maps. We then generalize that method by using relative convex hulls to provide a linear algorithm for polygons which are not convex.

1 Introduction

Although variously attributed to Fagnano [11] and Steiner [5],[9],[15],[14], many sources insist that, over one hundred years ago, Schwarz not only solved the problem of computing the minimum perimeter triangle with one vertex on each edge of a given triangle but that he also posed the problem [4],[8],[12],[13]. In any case, there is consensus that Schwarz used the *reflection principle* to show that the foot points of the altitudes of an acute triangle are the vertices of the minimum inscribed polygon. For obtuse triangles, the minimum perimeter inscribed triangle is realized by twice the shortest altitude, *i.e.* it is degenerate with two vertices coinciding with the obtuse vertex of the input triangle. The earliest problem solved with the reflection principle was even simpler than the triangle: given a line and two points on one side of it, find the shortest path between the two points via the line. This problem was solved by Heron of Alexandria in 100 AD while extending Euclid's work on optics [8].

In 1985, Klotzler and Rudolph [9] used a semi-infinite simplex-method to determine the minimum perimeter polygon with one vertex on each edge of a given convex polygon, or the minumum perimeter *in-polygon*. In 1986, Focke [5] presented an algorithm for computing the optimum inpolygon, using a finite descent method resulting from Schwarz' reflection principle [13] and coordinate-wise descent. He presents four examples to illustrate efficient performance of the algorithm, but provides no formal complexity analysis. In addition, his classification of correct solutions omits one realizable type and it is unclear whether his algorithm would in fact detect the correct solution.

At a recent workshop, Toussaint [16] posed the problem somewhat differently: he asked for the shortest closed path inside a simple polygon (*Aquarium Keeper's Tour*) which visits every edge at least once. If the polygon is convex, then the optimum path is, in fact, the minimum perimeter inpolygon, and we present a linear-time algorithm which uses the reflection principle and shortest-path maps. We also generalize our method by using relative convex hulls to provide a linear algorithm for polygons which are not convex. Chin and Ntafos have used a somewhat similar combination of the reflection principle and shortest-paths to solve a related problem on rectilinear polygons [3].

The Aquarium Keeper's Problem is related to another problem Chin and Ntafos [2], the *Zoo Keeper's Problem*. In the Zoo Keeper's Problem, given a simple polygon P of n vertices, k convex polygons (cages) attached to edges of P, and an entry point x on the boundary of P, the goal is to find the minimum perimeter tour in P and not in the interior of any cage, starting and ending at x, that visits every cage. The paper [2] contains an $\mathcal{O}(n \log^k n)$ time algorithm and refers to a $\mathcal{O}(n^2)$ algorithm for the problem as well. If we consider the cages as edges of P (*i.e.*, the edges of P represent the front glass plates of a series of aquariums) and have n cages, then the Zoo Keeper's Problem reduces to a

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simplified version of the Aquarium Problem in which a fixed starting point is given. Note that even without this restriction, our complexity is appreciably less than either $\mathcal{O}(n \log^n n)$ or $\mathcal{O}(n^2)$.

2 Optimum Keeper's Tour in a Convex Aquarium

Given a convex polygon P with vertices $v_0, v_1, ..., v_{n-1}$ given in clockwise order, let e_i represent the edge oriented in clockwise order which originates at v_i. The goal is to determine a sequence of points, $w_0, w_1, ..., w_{k-1}$ such that at least one of the w_i lies on each of the e_i and such that the length of the closed path $w_0, w_1, ..., w_{k-1}, w_0$ is a minimum. Since a point w_i may coincide with a vertex v_j of P and thus lie on two distinct edges, it is possible that $k < n$. Since P is convex, there can be no advantage in visiting an edge more than once, so $k \leq n$.

LEMMA 2.1. *If the aquarium polygon is convex, then the optimum keeper's tour visits the sides of the polygon in order. In other words, the optimum keeper's tour is exactly the optimum inpolygon.*

Proof. Assume that the optimum tour does not visit the edges in order, and thus that edges $w_i w_{i+1}$ and $w_j w_{j+1}$ cross each other, for $i < j$. Then $w_i w_j w_{i+1} w_{j+1}$ define a convex quadrilateral and *from the fact that in every triangle the length of one side is less than the sum of the lengths of the other two sides, it follows that* the sum of lengths of the diagonals exceeds the sum of the lengths of both pairs of opposite sides. Thus $w_1, ..., w_i, w_j, w_{j-1}, ..., w_{i+2}, w_{i+1}, w_{j+1}, w_{j+2}, ..., w_k, w_1$ describes a shorter tour, providing a contradiction.

If n is even, then one candidate tour consists of the vertices $v_1, v_3, v_5, ..., v_{n-1}, v_1$. In many instances, this tour will be optimal, but not necessarily so. If the tour does visit a side of the polygon in its interior, then the angle of incidence equals the angle of reflection.

LEMMA 2.2. *If a point w_i of the optimum tour lies in the interior of edge e_j, then angle $w_{i-1} w_i v_j$ must equal angle $v_{j+1} w_i w_{i+1}$.*

Proof. Fix points w_{i-1} and w_{i+1}. Let L represent the line containing e_j. Then a unique ellipse with foci w_{i-1} and w_i is tangent to L at a point q which minimizes the sum of the distances from w_{i-1} and w_{i+1}. If q does not lie on e_j, then w_i will coincide with the endpoint of e_j closer to q. If $q \in e_j$, then $w_i = q$. But the radii of an ellipse to a point of tangency intersect the tangent line in equal angles [8].

The segment of tour between a point on e_i and a point on e_{i+1} lies in the triangle $\Delta v_i v_{i+1} v_{i+2}$, whether both points are identically equal to v_{i+1} or the two points are disjoint. Thus, it is possible to unwrap the

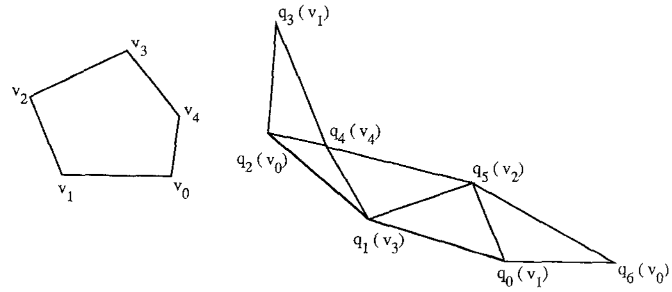

Figure 1: A convex polygon P and its triangulated unfolding Q.

tour as follows, producing a new polygonal structure Q of $n+2$ vertices. Let Q_1 represent triangle $\Delta v_0 v_1 v_2$ in its original position in P. To build Q_2, reflect $\Delta v_1 v_2 v_3$ around edge $v_1 v_2$ and attach it to Q_1 at v_1 and v_2. To build Q_3, translate and rotate $\Delta v_2 v_3 v_4$ until vertices v_2 and v_3 are aligned with the current position of v_2 and v_3 in Q_2. In general, to build Q_{2i}, reflect $\Delta v_{2i-1} v_{2i} v_{2i+1}$ around edge $v_{2i-1} v_{2i}$ and translate and rotate it until vertices v_{2i-1} and v_{2i} are aligned with v_{2i-1} and v_{2i} in Q_{2i-1}. To build Q_{2i+1}, translate and rotate $\Delta v_{2i} v_{2i+1} v_{2i+2}$ until vertices v_{2i} and v_{2i+1} are aligned with v_{2i} and v_{2i+1} in Q_{2i}. Finally, perform the appropriate transformation of of $\Delta v_{n-1} v_0 v_1$ and attach it to Q_{n-1} to form Q. Label these second instances of v_0 and v_1 as v_n and v_{n+1}. Thus, the vertices of Q in clockwise order can be identified as follows: $q_i = v_{2i+1}$ for $i = 0...\lfloor n/2 \rfloor$; $q_i = v_{2(n-i+1)}$ for $i = \lfloor n/2 \rfloor + 1...n+1$ (see fig. 1). Thus, the directed edge $v_0 v_1$ in P corresponds to the pair of edges $q_{n+1} q_0$ and $q_{\lfloor n/2 \rfloor + 1} q_{\lfloor n/2 \rfloor}$(resp. $q_{\lfloor n/2 \rfloor} q_{\lfloor n/2 \rfloor + 1}$) in Q if n is even(resp. odd).

Q is a polygon composed of a chain of triangles: every triangle has one side on the boundary of Q, except for the first and last which have two such sides each. The triangulated polygon Q may be non-simple (see fig. 2), but it does not matter whether it is or not. We focus on the triangulation we have been given.

LEMMA 2.3. *The optimum keeper's tour in P which starts and ends at a point on $v_0 v_1$ corresponds to the shortest path from the corresponding point on $q_{n+1} q_0$ to its image on $q_{\lfloor n/2 \rfloor} q_{\lfloor n/2 \rfloor + 1}$ which passes through every triangle of Q.*

Proof. Every path R from a point on $q_{n+1} q_0$ to its image on $q_{\lfloor n/2 \rfloor} q_{\lfloor n/2 \rfloor + 1}$ which passes through every

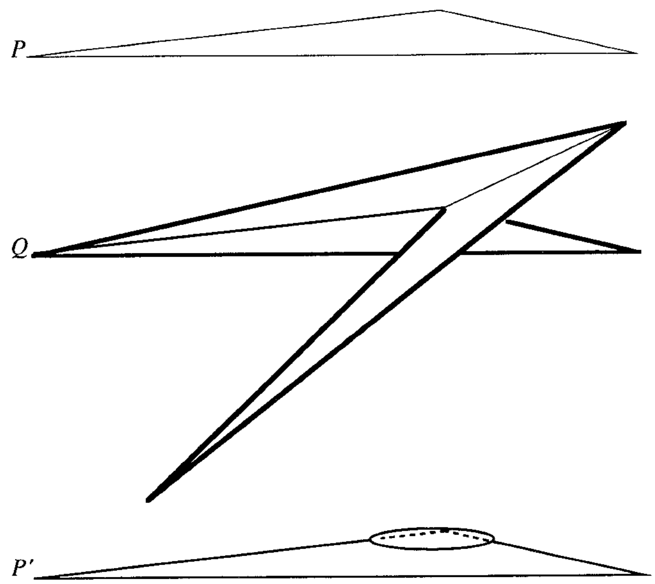

Figure 2: A non-simple triangulated polygon Q is obtained from the triangle P. Any odd number of vertices can be placed in the marked region to createa polygon P' whose associated triangulated polygon Q' will be non-simple.

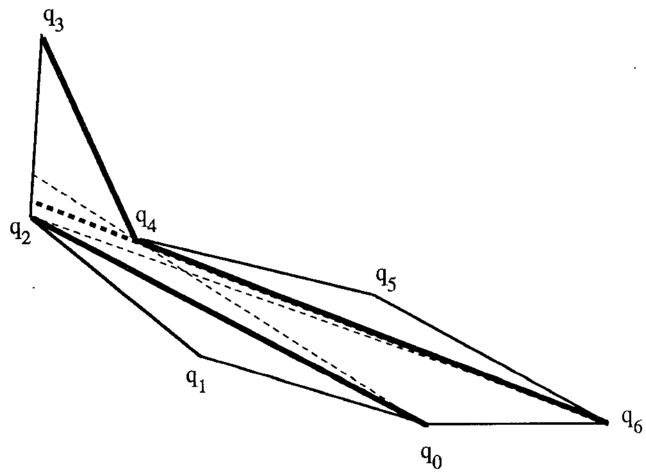

Figure 3: In unfolded polygon Q, the shortest paths from q_0 and q_{n+1} to $q_{\lfloor n/2 \rfloor}$ and $q_{\lfloor n/2 \rfloor+1}$ and vice versa and the induced subdivisions of the line segments q_0q_{n+1} and $q_{\lfloor n/2 \rfloor}q_{\lfloor n/2 \rfloor+1}$.

triangle of Q corresponds to a candidate keeper's tour of P. Divide R at the boundaries of all triangles it crosses. Reflect first, and then translate and rotate the portion of the path within $\Delta q_iq_{n-i}q_{i+1}$ (corresponding to $\Delta v_{2i-1}v_{2i}v_{2i+1}$) so that its endpoints lie at the corresponding points on e_{2i-1} and e_{2i} in P. Translate and rotate the portion of R within $\Delta q_{n-i+1}q_iq_{n-i}$ (corresponding to $\Delta v_{2i}v_{2i+1}v_{2i+2}$) until its endpoints lie at the corresponding points on e_{2i} and e_{2i+1} in P. The tour T formed by the union of these pieces has exactly the same length as R. By reversing the process, every candidate tour T in P starting and finishing at a point $tv_0 + (1-t)v_1$ for some t, $0 \leq t \leq 1$ corresponds to a path R in Q of exactly the same length from point $tq_{n+1} + (1-t)q_0$ to $tq_{\lfloor n/2 \rfloor+1} + (1-t)q_{\lfloor n/2 \rfloor}$ (resp. $tq_{\lfloor n/2 \rfloor} + (1-t)q_{\lfloor n/2 \rfloor+1}$) if n is even (resp. odd). Given the $1-1$ correspondence between tours and paths, the optimum tour T must correspond to the shortest path R.

THEOREM 2.1. *The optimum keeper's tour in a convex polygon P of n vertices can be computed in $O(n)$ time.*

Proof. Create the triangulated polygon Q as described above in linear time. In linear time [7], create the shortest path inside Q from q_0 to $q_{\lfloor n/2 \rfloor}$ and from q_{n+1} to $q_{\lfloor n/2 \rfloor+1}$, denoted $S(Q(q_0, q_{\lfloor n/2 \rfloor}))$ and $S(Q(q_{n+1}, q_{\lfloor n/2 \rfloor+1}))$, respectively. These paths are inward convex chains. In linear time [7], it is also possible to extend the edges of these chains as well as the inner common tangents of the chains so as to subdivide edges q_0q_{n+1} and $q_{\lfloor n/2 \rfloor}q_{\lfloor n/2 \rfloor+1}$. All points x on a single segment C_i of q_0q_{n+1} share the same shortest path to $q_{\lfloor n/2 \rfloor}$ (resp. $q_{\lfloor n/2 \rfloor+1}$) with the exception of the straight segment from the particular point to the first vertex on the path. That vertex is denoted $left\text{-}anchor(C_i)$ (resp. $right\text{-}anchor(C_i)$). Similarly, for a point x on the corresponding oriented segment D_i on $q_{\lfloor n/2 \rfloor}q_{\lfloor n/2 \rfloor+1}$, let $left\text{-}anchor(D_i)$ (resp. $right\text{-}anchor(D_i)$ equal the first vertex on $S(Q(x, q_0)$ (resp. $S(Q(x, q_{n+1})$ (see fig. 3). Below we use a type of case analysis on the left- and right-anchors similar to that used by Melissaratos and Souvaine [10] in computing the minimum length area-separator or the minimum area or perimeter triangle inscribed in a simple polygon.

To compute the shortest path from each segment $C_i = c_ic_{i+1}$ of v_0v_1 to its image $D_i = d_id_{i+1}$ on v_nv_{n+1}, we begin by examining the left- and right-anchors of the two segments. In constant time, we can decide whether C_i and D_i are entirely visible from each other in Q, entirely invisible, or partially visible. In the first case, the goal is to find the value of t that minimizes the straight-line distance from $tc_i+(1-t)c_{i+1}$ to $td_i + (1-t)d_{i+1}$. The square of this distance can be represented as a quadratic function of t which can be minimized in constant time. It is interesting to note that if C_i and D_i have opposite orientation, then the shortest segment from C_i to D_i either joins two endpoints or is perpendicular to the bisector of the angle formed by the lines containing C_i and D_i.

In the second case, every shortest path from a point $c \in C_i$ to a point $d \in D_i$ inside Q passes through at least one vertex. Let C (D) be the vertex closest to c (resp. d). The path from C to D is independent of starting (ending) point on c (resp. d). To determine these two points, reflect, if necessary, and then translate and rotate the triangle defined by D_i and D until D_i is aligned with C_i and C and D are on opposite sides of C_i. The shortest path from C to D in this new figure is either a straight segment or a pair of segments joined at an endpoint of c, and thus can be computed easily in constant time.

In the third case, first perform the same procedure adopted in the second case, possibly twice, using the appropriate left- and/or right-anchors for C and D. Then use the same procedure as in the first case with the added constraint that both left-anchors lie on one side of the chosen segment and that both right-anchors lie on the other side. Keep the optimum path determined by either technique.

There are at most a linear number of segment pairs (C_i, D_i), an optimum path can be found for each in constant time, and the shortest of all these paths corresponds to the optimum tour.

3 Optimum Keeper's Tour in a Non-Convex Aquarium

Given a simple, but non-convex, polygon $\mathcal{P}$ with vertices $v_0, v_1, ..., v_{n-1}$ given in clockwise order, let e_i represents the edge oriented in clockwise order which originates at v_i. Certainly, all reflex vertices $r_0, r_1, ..., r_{h-1}$ can be identified in linear time. Let i_j represent the index of r_j among the original vertices of P, *i.e.* $v_{i_j} = r_j$, for all $0 \leq j \leq h-1$. The goal is to determine a sequence of points, $w_0, w_1, ..., w_{n-1}$ such that at least one of the w_i lies on each of the e_i and such that the length of the closed path $w_0, w_1, ..., w_{n-1}, w_0$ is a minimum. Since a point w_i may coincide with a vertex v_j of $\mathcal{P}$ and thus lie on two distinct edges, it is possible that $k < n$. Since $\mathcal{P}$ is non-convex, then a single reflex vertex may be visited more than once in order to touch all edges of the polygon. We say that two edges of a polygonal path *cross* if the edges intersect in their interiors at a single point. Given a simple polygon P and two points in its interior a and b, we use $S(P(a, b))$ to denote the shortest path from a to b that lies inside P.

LEMMA 3.1. *No two edges of an optimum keeper's tour of a simple aquarium cross.*

Proof. Assume for the sake of contradiction that there is an optimum keeper's tour Q of a simple aquarium P with the edges (w_i, w_{i+1}) and (w_j, w_{j+1}) that cross at the point x. Eliminate the crossing edges (w_i, w_{i+1}) and (w_j, w_{j+1}) from the keeper's tour and replace them with $S(P(w_i, w_j))$ and $S(P(w_{i+1}, w_{j+1}))$ to obtain a new tour Q'. Observe that Q' is a keeper's tour. The path $S(P(w_i, w_j))$ is convex and lies within the triangle $\Delta(w_i, w_j, x)$, because w_i is visible from w_{i+1} and w_j is visible from w_{j+1}. Therefore, $S(P(w_i, w_j))$ is shorter than the sum of the lengths of the edges (w_i, x) and (w_j, x). Similarly $S(P(w_{i+1}, w_{j+1}))$ is shorter than the sum of the lengths of (w_{i+1}, x) and (w_{j+1}, x). We have shortened a tour we assumed to be optimum, thus obtaining a contradiction.

COROLLARY 3.1. *Every optimal keeper's tour of a simple aquarium partitions the plane into two equivalence classes, a closed and bounded interior and an open unbounded exterior. The edges of the tour can be directed so that the exterior lies to the left and the interior lies to the right of a clockwise traversal.*

LEMMA 3.2. *Every optimum keeper's tour visits every reflex vertex of a simple aquarium P at least once.*

Proof. Suppose that there exists an optimum keeper's tour Q that leaves a reflex vertex v unvisited. Let L be a horizontal line passing through v. Assume, without loss of generality, that the line segment l, the connected subset of L containing v and lying in P, contains v in its interior. If this is not the case, then we can easily make it so by rotating P by a suitable radial angle. Thus, there exist points of intersection between Q and l immediately to the *left* and to the *right* of v, which we label a and b, respectively. Let w_i and w_j respectively denote the points of contact of Q to P that are encountered immediately before a and immediately after b when traversing Q in a clockwise orientation. It follows from the corollary of lemma 3.1 that the part of Q from a to b keeps the interior of Q to its right. Thus we can shorten the tour by replacing the path $w_i, ..., w_j$ by the path $w_i, ..., a, b, ..., w_j$ in Q. Since the line segment ab is shorter than the path $a, ..., b$ in Q, we have demonstrated that any tour that leaves a reflex vertex unvisited must be suboptimal.

We use $Q(r_1, r_2)$ to denote the part of the optimum keeper's tour that passes from r_1 to (the first occurrence of) r_2, two adjacent reflex vertices of a simple aquarium. Let the intermediary edges of P between r_1 and r_2 be denoted as $I(P(r_1, r_2))$.

LEMMA 3.3. *$Q(r_1, r_2)$ of a simple aquarium P, touches every edge of $I(P(r_1, r_2))$ in order, and is constrained to lie in a region bounded by $I(P(r_1, r_2))$ and $S(P(r_1, r_2))$.*

Proof. Assume that one or more edges of $I(P(r_1, r_2)$ are not visited by $Q(r_1, r_2)$. Let e denote

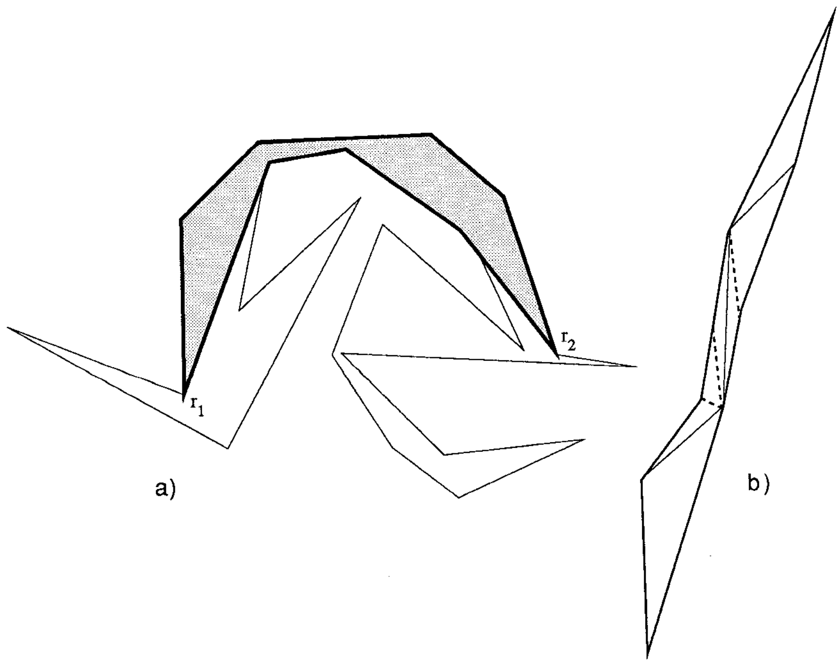

Figure 4: a) A simple aquarium with shaded region P^1 defined by consecutive reflex vertices r_1 and r_2. b) The structure Q^1 obtained from P^1.

one such edge. Let a and b denote the points of contact of $Q(r_1, r_2)$ and $I(P(r_1, r_2))$ immediately preceding and succeeding the edge e in a clockwise traversal.

Thus all of e lies to the left of the part of $Q(r_1, r_2)$ when traversed from a to b. Since Q must visit e there is at least one point of e in the interior of Q, a contradiction. Thus we can assume that the shortest path between r_1 and r_2 visits every edge of $I(P(r_1, r_2))$. If r_1 sees r_2, that is, $S(P(r_1, r_2))$ is the edge (r_1, r_2), then $Q(r_1, r_2)$ is the shortest path from r_1 to r_2 visiting every intermediary edge in a convex polygon. As shown in sect. 2, this path must visit edges in order. On the other hand r_1 may not necessarily see r_2. We claim that the path $Q(r_1, r_2)$, since it is constrained to lie within P, must lie in a region bounded by $S(P(r_1, r_2))$ and the intermediary edges $I(P(r_1, r_2))$. If $Q(r_1, r_2)$ crosses the path $S(P(r_1, r_2))$ then it must cross $S(P(r_1, r_2))$ an even number of times. The parts of $Q(r_1, r_2)$ between the odd and even numbered crossings can be replaced by parts of $S(P(r_1, r_2))$ thus making the entire path shorter. Thus $Q(r_1, r_2)$ must visit every edge in $I(P(r_1, r_2))$ in order on the boundary of P, and must lie in the region bounded by $Q(r_1, r_2)$ and $S(P(r_1, r_2))$.

The first step, then, is to compute the boundaries for all of the subproblems. To tackle the problem naively and compute the shortest path tree from every reflex vertex could take time $\mathcal{O}(n^2)$ in the worst case.

LEMMA 3.4. *The polygon R consisting of the reflex vertices of P and the shortest paths joining adjacent reflex vertices in P has size $\mathcal{O}(n)$ and can be computed in $\mathcal{O}(n)$ time. R may be non-simple as vertices of the boundary may be repeated, but each region of finite area defined by R lies to the right of the oriented edges bounding it.*

Proof. The polygon R is in fact the *relative-convex-hull* (also *geodesic convex hull*) of the set of reflex vertices of P with respect to P [17]. In [17], an $\mathcal{O}(n \log n)$ time and $\mathcal{O}(n)$ space algorithm is presented for computing the relative convex hull of a set S of n points in s simple polygon P. The optimality of the algorithm follows because S can lie anywhere in P. However, if S consists of a set of vertices of P, then the algorithm in [17] simplifies considerably and is dominated only by triangulating P. Since P can be triangulated in $\mathcal{O}(n)$ time with Chazelle's new algorithm [1], the result follows.

The global problem of computing the optimum Aquarium tour now reduces to h subproblems of finding the shortest path from one fixed vertex r_j to another r_{j+1} that touches every edge of the boundary convex chain joining r_j and r_{j+1} without crossing a second convex chain which joins the same two vertices. Let us say that these two convex chains define a polygon P^j (where P^j is in fact a *spiral* polygon).

Since P^j is nearly convex, we can solve each of these subproblems using essentially the same procedure we described in section 2. To create the polygonal structure Q^j, however, we no longer reflect and translate only triangles. The segment of tour between a point on e_i and a point on e_{i+1} still lies in the triangle $\Delta v_i v_{i+1} v_{i+2}$, but since the diagonal from v_{i-1} to v_{i+1} may no longer lie in the interior of the polygon, we may need a greater restriction. In fact, we need to focus on the starshaped polygons determined by e_{i-1}, e_i and the inward convex chain $S(P(v_{(i-1) mod\ n}, v_{(i+1) mod\ n}))$ which represents the shortest path from $v_{(i-1) mod\ n}$ to $v_{(i+1) mod\ n}$ for every i along each individual convex chain. As all vertices in this polygons are visible from v_i, diagonals from v_i triangulate the polygon and can be added in the process of creating Q^j (see fig. 4).

LEMMA 3.5. *For each of the h subproblems on a polygon P^j composed of two nested convex chains, the polygonal structure Q^j can be formed in time and space linear in the size of P^j.*

THEOREM 3.1. *The optimum keeper's tour in a simple polygon P of n vertices can be computed in $\mathcal{O}(n)$ time.*

Proof. If P should be convex, then the procedures of sect. 2 apply. Otherwise, P has at least one reflex vertex which must lie on the shortest tour. Thus the problem reduces to finding the shortest path from the image of each reflex vertex r_j to the image of the succeeding vertex r_{j+1} (possibly equal to r_j!) in the triangulated polygonal structure Q^j. For each j, this can be done in time linear in the size of Q^j. But the sum of the sizes of all the Q^j is $\mathcal{O}(n)$. Thus, the tour can be computed in $\mathcal{O}(n)$ time.

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