

Contents

Contents	i
1 Introduction	1
2 Eigenvalues and Eigenvectors	2
2.1 Eigenvalues and Eigenvectors	4
2.2 Characteristic Polynomials and Characteristic Equations	4
2.3 Eigenfacts: Matrix Properties and Eigenvalues	6
2.4 Eigenvectors	7
2.5 Extra Notes: Extensions to Linear Algebra	7
2.6 Extra Notes: Characterization of a Real Symmetric Matrix	9
3 Graphs: Adjacency Matrix and Laplacian Matrix	10
3.1 Relevant Definitions	12
3.2 Adjacency Matrix of a Graph	13
3.3 Weighted Adjacency Matrix of a Graph	17
3.4 Laplacian Matrix of a Graph	17
3.5 Eigenvalues of a Laplacian Matrix	20
3.6 The Fiedler Vector of a Graph	22
3.7 Extra Notes: Incidence Matrix of a Graph	23
3.8 Extra Notes: Properties of a Laplacian Matrix	25
4 Vector Spaces	26
4.1 Vector Space: Properties	26
4.1.1 Vector Space: Interpretation	27
4.1.2 Vector Space: Examples	28
4.2 Matrices and Vector Spaces	30
4.3 Echelon Forms of a Matrix $A \in \mathbb{R}^{m \times n}$	31
4.4 Null Space and the RREF	33

5	Spanning Sets and Basis Vectors	38
5.1	Orthogonal Basis and Orthonormal Basis	40
5.2	Rank-Nullity Theorem	41
5.3	Orthogonal Subspaces	43
6	Review For Quiz #1: Graphs and Spectral Decomposition	44
7	Diagonalizable Matrices	45
7.1	Similar Matrices	45
7.2	Eigenvectors as a Basis	46
7.3	Eigenvector Basis	47
7.4	Orthogonal Matrices	47
7.5	Symmetric Matrices	49
7.6	Skew-Symmetric Matrices	50
7.7	Nondiagonalizable Matrices	51
7.8	Extra Notes: Matrix Powers	53
7.9	Extra Notes: Small Perturbations	54
8	Spectral Decomposition and Positive [Semi–]Definite Matrices	56
8.1	The Spectral Decomposition	56
8.2	Positive-Definite and Positive-Semidefinite Matrices	58
8.3	Quadratic Form of a Symmetric Matrix	58
8.4	Example: Product of a Full-Rank Matrix and its Transpose	59
8.5	Statistics of Vectors: Means and Variance	60
8.6	Example: Covariance Matrix in Statistics	61
8.7	Example: Linear Elastic Structures	63
9	Review For Test #1: Basic Linear Analysis	65
10	Design Matrix and Standardized Data	67
10.1	Data Matrix and Design Matrix	69
10.2	Standardized Data or Z Score	70

11 Orthogonal Projection	73
11.1 Projecting to a 1D Subspace: Vector to Vector	73
11.2 Projecting to a 2D Subspace: Vector to Basis	75
12 Patterns – Linear Regression	78
12.1 Residual Error	79
12.2 Linear Regression	80
12.3 Example: Hooke’s Law	80
12.4 Linear Regression – One Dependent Variable, Plus Intercept	81
12.5 Data Matrix For Linear Regression	82
12.6 Assessment Of Linear Regression	84
12.7 Extra Notes For Derivations of Linear Regression	85
13 Cross-Validating Linear Regression	88
13.1 Training and Testing	88
13.2 Linear Regression and the Design Matrix X	89
13.3 Measuring Error in Linear Regression	89
13.4 Cross-Validation of Linear Regression: Leaving Data Out	90
13.5 Cross-Validation of Linear Regression: K-Fold Analysis	91
13.6 Cross-Validation of Linear Regression: Monte Carlo Methods	92
13.7 Example: 13 Data With 2 Outliers	92
14 SVD – Singular Value Decomposition	95
14.1 Eigenvectors of the matrix $[A^T A]$	95
14.2 Eigenvectors of the matrix $[A A^T]$	96
14.3 The Singular Value Decomposition, or SVD	97
14.4 Using the SVD	98
15 Review for Quiz #2: Matrices and Linear Regression	99

16	Orthonormal Basis Vectors and the SVD	100
16.1	SVD of a Square Matrix	100
16.2	SVD of a Non-Square Matrix	102
16.3	The SVD as an Approximate Basis for a Vector Space	103
16.4	Some SVD Properties	104
17	Principal Components Analysis – PCA	106
17.1	Motivation, by Example	106
17.2	Zero-Mean Data Matrix	107
17.3	Principal Components Analysis as an SVD	108
17.4	Using the SVD to Compute PCA Scores	109
17.5	Matrix Norms	111
17.6	Eigenvalues and Singular Values	112
17.7	Extra Notes: A Matrix As A Series	113
17.8	Extra Notes: Eckart-Young Theorem	114
18	Review for Test #2: Matrices and Linear Regression	116
19	PCA – Matrix Algebra and Dimensionality Reduction	118
19.1	Scatter Matrix, SVD, and PCA	119
19.2	PCA and Low-Rank Approximation	120
19.3	PCA for Dimensionality Reduction	121
19.4	Approximations and The Scree Plot	121
20	Unsupervised Learning – K-Means Clustering	123
20.1	Supervised and Unsupervised Classification	123
20.2	Unsupervised Classification: Data Clustering	123
20.3	Clustering – Iris Data	124
20.4	K-Means Clustering	125
20.4.1	A K-Means Algorithm	125
20.4.2	Example: Fisher’s Iris Data	126

21 Classification – Linear Separability	130
21.1 Separation of Two Clusters	130
21.2 Separation of Three or More Clusters	133
21.3 Hyperplane Specification With a Unit Normal Vector	136
22 Classification – Assessment With Confusion Matrix	137
22.1 Relative Confusion Matrix	139
23 Classification – Assessment With ROC Curve	141
23.1 Receiver Operator Characteristic – ROC	141
23.2 Example: Two Variants of a Virus	143
24 Review for Quiz #3: SVD, PCA, And Dimensionality Reduction	147
25 Odds of Occurrence and Probability	148
25.1 Logistic Function – Some Properties	152
25.2 Hyperplane Classification and Pseudo-Distance	153
26 Elementary Numerical Optimization	154
26.1 Scalar Argument – Fixed-Stepsize Search	155
26.2 Vector Argument – Fixed-Stepsize Search	158
27 Review for Test #3: SVD, PCA, And Dimensionality Reduction	160
28 Artificial Neuron – Learning Weights	162
28.1 History – The Perceptron Rule	163
28.2 Logistic Activation of an Artificial Neuron	163
28.3 The Method of Steepest Descent	164
28.4 Extra Notes: Gradient of Squared-Error Objective	166
29 Classification – Logistic Regression	168
29.1 Semilinear Activation – Logistic Function As A Sigmoid Curve	169
29.2 Models Of Residual Error	170
29.3 Implementations Of Logistic Activation	172
29.3.1 Example – Fisher’s Iris Data	172

30 Nonlinear Separation – Embeddings and Gram Matrix	175
30.1 Linear Separation Using an Embedding	176
30.2 Kernel Functions and the Gram Matrix	178
30.3 Some Kernel Functions for Row Spaces	179
30.4 Extra Notes for The Gram Matrix and Kernel Functions	181
31 Nonlinear Separation – Kernel PCA	182
31.1 Principal Components Analysis And Scatter Of Observations	182
31.2 Kernel Functions and the Gram Matrix	185
31.3 Example – Kernel PCA For Fisher’s Iris Data	186
32 Spectral Clustering Of Data	188
32.1 Multiple Clusters In Graphs	188
32.1.1 Example: 30 Vertices In 3 Clusters	189
32.2 Graph Representation Of Simple Data	190
32.3 Weighted Graphs	194
32.4 Extra Notes on The Laplacian Matrix	194
33 Preparation for Quiz #4: k-Means, Assessment, and Odds of Occurrence	197
34 Course Summary – Linear Data Analysis	198
35 Preparation for Test #4: k-Means, Assessment, and Odds of Occurrence	202
References	204