Planar Tree Transformation through Flips.

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Abstract

A flip or edge-replacement is considered as a transformation by which one edge e of a geometric object is removed and an edge f ($f \neq e$) is inserted such that the resulting object belongs to the same class as the original object. Here, we consider planar trees as geometric objects. In this paper, we present a technique for transforming a given planar tree into another one for a set S of n points in general position in the plane. We show that any planar tree can be transformed into another planar tree by at most $2n-k-s-2$ ($O(n)$)flips ($k > 0$ and $s > 0$ are defined later) which is an improvement of the result in [3].

1 Introduction

The problem of transformations of a certain class of geometric objects consisting of straight line segments and points in the plane by applying small changes or flips (e.g., removing an edge and replacing it with another) in the object has been studied extensively [1], [2], [3], [5]. Given two objects, two usual questions that are studied are whether the two objects can be transformed to each other and how many transformations are required. In this paper, we study the transformation of planar trees using flips for a set of points in in general position in the plane and also determine bounds on the number of transformations needed.

Although flips are studied extensively in triangulations, there are a number of other examples in computational geometry that illustrate this particular computational problem of transformation related to flips. For instance, flips have been used in other classes of objects, such as spanning trees, Euclidean matchings [4], linked-edge lists, pseudo-triangulations etc. Algorithms for such transformation as well as lower and upper bounds of achieving transformation results can be found in [2]-[5]. One of the best-known results in the case of planar tree transformation is by Avis and Fukuda [3] who showed that for a point set in general position every planar tree can be transformed into another planar tree by means of at most $2n-4$ flips.

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1.1 The Meta Graph and its Connectedness

One can then define a mathematical model in terms of transformations for all the objects of a class of whether any two geometric objects of a certain class (satisfying some property) are reachable from one to another via a finite sequence of flips. Here we define a graph, called the *meta graph*, that serves as the mathematical model to describe the relationships among the objects of that graph. The meta graph can be defined as the graph having the set of objects in the class as its vertex set, and a pair of vertices is connected by an edge if the objects represented by the vertices differ by a small change. In this way, one can develop adjacencies among nodes of the meta graph.

In most of the cases, generating, examining and establishing relationships among all the objects is not feasible due to the amount of time needed to construct a graph containing such a huge number of vertices. There is a way that makes it possible to establish polynomial-size descriptions among the objects even though the size of the graph is exponential in most of the cases. The procedure of incorporating local modifications on some initial object to visit new objects allows us to study the essential characteristics such as connectedness, reachability, diameter of the graph and so on. The existence of a path between two vertices in the graph means transformability of the corresponding trees represented by the vertices into one another by means of repeated application of the local transformations. The length of the shortest path between two vertices corresponds to the distance between the two trees in terms of the number of transformations.

1.2 Flips in Trees

A considerable amount of similar study has been conducted for general graphs. The meta graph of trees was introduced in [8], in connection with the study of electrical networks. A characteristic of the meta graph, specifically that the graph of trees is Hamiltonian is proved in [8]. Graph-theoretical versions of the problem of geometric object transformations have been largely studied in [7] for tree graphs and was shown that tree graphs have maximum connectivity (a directed graph is said to be maximally connected if it is k-connected and k is the minimum in or out degree of all vertices). Motivated by the question of enumerating the set of all planar spanning trees for points in general position, Avis and Fukuda [3] showed that the corresponding tree graph is connected and has a diameter bounded by $2n-4$. To the best of our knowledge this is the only known result for a general point set. In this paper, we present an improved bound of $2n-k-s-2$ (where $k, s \geq 1$) which in some cases produces a better result.

The paper is organized as follows. In section 2, we provide the definitions and terminologies that will be used throughout the paper. The main result of the paper is presented in section 3 followed by an example. The description and formation of the meta graph consisting of vertices which is the set of planar trees of a set of points are provided in this section. We conclude in section 4.

2 Definitions and Terminologies

A number of definitions are provided in this section which are given below.

A **graph** $G=(V,E)$ consists of V, a set of vertices, and an edge set E, $E=\{(v_i,v_j)|v_i,v_j \in V\}$. A graph G is called **planar** if it can be drawn in the plane so that no two edges intersect, except at a common vertex. If $(v_i,v_j) \in E$, then v_i and v_j are **adjacent** or **neighbors**. We define $N(v_j)$ to denote the set of vertices which are adjacent to vertex v_j.

Assume $P=\{v_0,v_1,v_2,\cdots,v_{n-1}\}$ is a set of n points in general position (no three points are collinear) in the two-dimensional Euclidean plane. We assume that trees are drawn in the plane. Vertices (V) and edges (E) of a tree are represented by points of P and straight line-segments. A **planar rooted tree** $T=(V,E)$ is a connected acyclic planar graph in which a vertex is designated as the **root** of the tree. Here, we designate the root of a tree $T=(V,E)$ is the vertex with minimal $x-$coordinate (represented by v_0). The vertex with smallest $y-$coordinate will be designated as the root if there are ties. A **canonical tree** $T^*_{v_0}$ is a planar rooted tree where all vertices are adjacent to v_0, that is, in $T^*_{v_0}$, $v_i \in N(v_0), \forall_i, i \neq 0$. Two vertices v_i and v_j, $v_i \neq v_j$ in an embedding of G are **visible** to each other if the straight line segment $(v_i,v_j) \in E$ between them does not intersect any of the edges in G. A **flip** in a tree T_1 is the operation of removal an edge e and addition of a edge f so that $T_2 = T_1 \backslash \{e\} \cup \{f\}$ is a tree.

Let $\mathcal{T}(P)$ denote the set of all trees of P and the geometric tree graph $T_G(P)$ denote the graph having $\mathcal{T}(P)$ as vertex set. Two trees $T_1, T_2 \in \mathcal{T}(P)$ are adjacent if $T_2 = T_1 \backslash \{e\} \cup \{f\}$ for some edges e and f.

In the rest of the paper, it is assumed that a tree is planar unless otherwise mentioned.

3 Tree Transformation

Let $T'=(V,E')$ and $T''=(V,E'')$ be any two trees belonging to $\mathcal{T}(P)$. It is required to construct T'' by applying a sequence of flips one by one to T'. In general, we say that T'' can be transformed from T' by p flips if there is a set of trees $T_0, T_1, \cdots, T_p$ where $T'=T_0$ and $T''=T_p$ such that T_{t+1} can be obtained from T_t by a single flip. This implies that for any t, T_t and T_{t+1} are adjacent in $T_G(P)$.

Consider the Fig. 1 where the tree T'' is obtained from T' by a sequence of transformation.

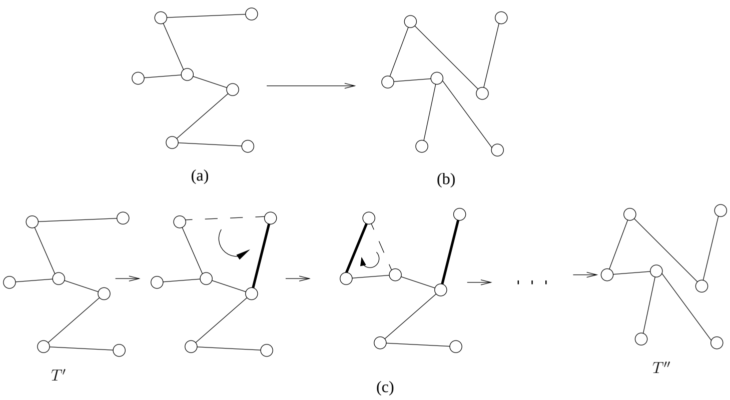


Figure 1: (a) A tree $T'=(V,E')$ (b) A tree $T''=(V,E'')$ (c) Transformations (shown with thick edges) applied on T' to construct T''.

In this paper, we improve the upper bound $(2n-4)$ for tree transformation proposed in [3]. In the following section, it is proved that two trees $T', T'' \in \mathcal{T}(P)$ can be transformed to each other with at most $2n-k-s-2$ flips, where $k>0$ and $s>0$ are the number of neighbors of the roots of T' and T'', respectively. Instead of proving directly that a tree can be transformed into another tree, our strategy is to show that it is always possible to transform the given tree into a unique canonical tree. Then, one can perform the reverse operations that transform the target tree into the same canonical tree.

3.1 Transformation via Canonical Tree

The following lemma provides a useful direction towards the main result:

Lemma 1 *At least one vertex of $T \in \mathcal{T}(P)$ and $T \neq T^*_{v_0}$ is visible to the root, v_0 of T.*

Proof. Since the tree $T=(V,E)$ is not in its canonical form, $\exists i$ such that $v_i \notin N(v_0)$. Let N_1 and N_2 be the set of neighbors of v_0 such that $deg(v_p)=1$ for $v_p \in N_1$ and $deg(v_q)>1$ for $v_q \in N_2$ with $0<p,q \leq n-1$. According to the definition, v_0 can see neither the vertices of N_1 nor those of N_2.

Let ℓ_1 be a line segment connecting v_0 to any of the vertices $v_r \notin N(v_0)$. Denote by $\angle \ell_1$ the angle made by ℓ_1 with v_0 along the vertical line that goes up through v_0. If ℓ_1 does not cross any edge of E, this means v_r is visible to v_0 and the proof is done. So we may assume that ℓ_1 crosses some edges of T. Denote the set of intersected edges E_{int_1}.

Select the edge $e_1 \in E_{int_1}$ whose intersection with ℓ_1 is nearest to v_0.

Now at least one of the vertices of $e_1 = (v_1, v_2)$ must not be a neighbor of v_0. Assume v_1,$v_2 \notin N(v_0)$. In this case, we can arbitrarily choose v_1 or v_2 to connect to v_0. If one of them is a neighbor then we select the other vertex which is not a neighbor of v_0 and connect it to v_0. Let ℓ_2 denote a line segment joining v_0 to one of the vertices v_1 or v_2 . Now check whether the edge ℓ_2 crosses any edges of T. If there is no crossing then the other vertex on ℓ_2 is visible to v_0. If this is not the case, i.e., $|E_{int_2}| > 0$, then choose the closest intersected edge e_2 from E_{int_2} from the root and follow the above procedure of connecting v_0 to the vertex of e_2 which is not neighbor of v_0. Repeatedly applying the procedure described above gives rise to the following cases:

Case 1: Case 1 deals with the situation where $\angle\ell_1 < \angle\ell_2 < \angle\ell_3 \cdots < \angle\ell_a$ (here $a \leq n-2$). Since there is a finite number of vertices in the graph, by following the above procedure we can reach to a vertex v_s where the edge ℓ_a (one end vertex is v_s and the other one is v_0) will not cross any edges. This ensures us that in the worst case ($a = n-2$) the line segment ℓ_a makes the largest angle joining the vertex v_s to v_0. Then v_s is the vertex visible to v_0.

Case 2: Case 2 is the reverse of case 1 ($\angle\ell_1 > \angle\ell_2 > \angle\ell_3 \cdots > \angle\ell_a$) and can be similarly resolved.

Case 3: In Case 3 we consider the following:

There exists some i such that (i) $\angle\ell_i < \angle\ell_{i+1} > \angle\ell_{i+2}$ or

(ii) $\angle\ell_i > \angle\ell_{i+1} < \angle\ell_{i+2}$

Without loss of generality consider (i) $\angle\ell_i < \angle\ell_{i+1} > \angle\ell_{i+2}$. Now the line segment ℓ_{i+2} must lie between ℓ_{i+1} and ℓ_i and e_i. Let v_i and v_{i+1} be the end vertices of the line segments ℓ_i and ℓ_{i+1} respectively. Any ℓ_j (for $j > i+2$) must lie in triangle formed by the line segments ℓ_i,ℓ_{i+1} and e_i because at each step we are taking the closest intersected edges to v_0. Also this implies that at every iteration the number of vertices to be considered for visibility to v_0 are reduced (only those vertices inside the triangles) as the area of successive triangles are decreased gradually. Since the number of vertices are finite and the number is becoming smaller and smaller at each step, we must end up with a single vertex v_s inside the smallest of such triangles (in the worst case) that will be visible to v_0. Fig. 2. shows such a scenario.

□

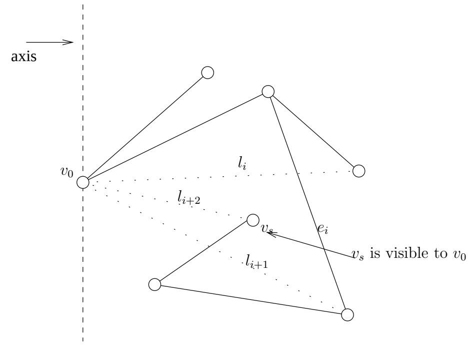


Figure 2: A tree T depicting case 3. Only the solid line segments represent the edges of the tree.

Lemma 2 *Any tree $T \in \mathcal{T}(P)$ and $T \neq T^*_{v_0}$ can be transformed into its canonical form $T^*_{v_0}$ through flipping of edges at most $(n-k-1)$ steps, where $k = |N(v_0)|$.*

Proof. Begin with any vertex visible to v_0. Assume v_0 can see v_j. Since adding a line segment (v_0, v_j) makes a unique cycle $v_0 \cdots v_i v_j v_0$ in T, break the cycle by eliminating the edge (v_i, v_j). This flip increases the degree of v_0 by one. Then continue the same process with another vertex visible to v_0 until $\forall v_i \in N(v_0)$ and $v_i \neq v_0$. An illustration is Fig. 3 shows the transformation of T into canonical tree $T^*_{v_0}$ through successive flips.

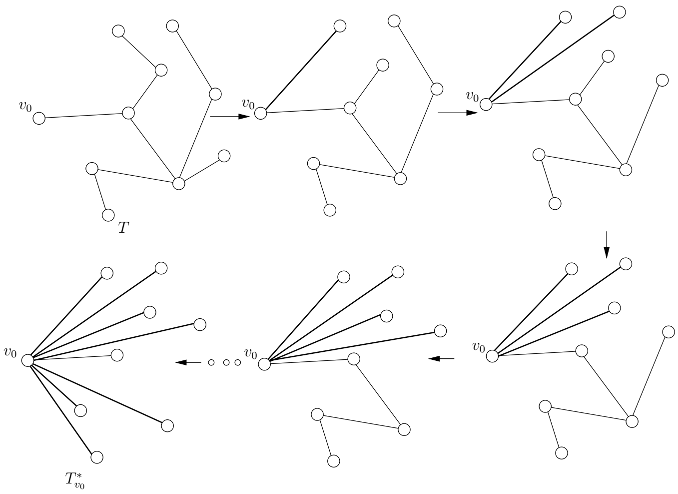
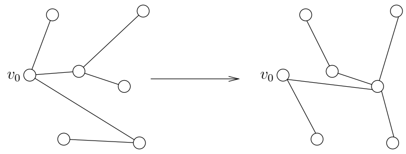


Figure 3: Showing the transformation of tree T into canonical tree $T^*_{v_0}$. Thick edges represent the edges that are flipped.

Since for any tree there can be $n-k-1$ edges possible which are not incident to v_0, the total number of flips required to produce $T^*_{v_0}$ is $n-k-1$.

□

Given the tree, T in Fig. 3 with $n = 10$ and $k = 1$ note that the number of flips to obtain $T^*_{v_0}$ from T is, $n-k-1 = 10-1-1 = 8$.



(a) T' (b) T''

Figure 4: Showing an example for the Corollary 4.

Now we will prove the main results of this paper, i.e., two trees $T', T'' \in \mathcal{T}(P)$ can be transformed into one another with only a linear number of flips. Instead of directly proving that a tree can be transformed into another tree our strategy is that we transform T' into the unique canonical tree $T^*_{v_0}$. And then we can reverse the operations that transform T'' into $T^*_{v_0}$. This is shown in the following lemma:

Lemma 3 *Two trees $T', T'' \in \mathcal{T}(P)$ can be transformed to each other with at most $2n - k - s - 2$ flips where $k \geq 1$ and $s \geq 1$ are the number of neighbors of the roots of T' and T'' respectively.*

Proof. Given the tree T', one can transform it into canonical tree $T^*_{v_0}$ using at most $n-k-1$ steps where k is the number of neighbors of v_0. Similarly, the tree T'' can be transformed into canonical tree $T^*_{v_0}$ with at most $n-s-1$ flips where s is the number of neighbors of v_0 in T''. Since the canonical trees for T' and T'' are same, it takes $2n - k - s - 2$ flips to transform T' into T''. This completes the proof.

□

Corollary 4 *The number of flips required to transform a tree T' into another tree T'', where $T', T'' \in \mathcal{T}(P)$ is $2n - p - 2$ where $p = max_{v \in CH(V)}$(degree v in T' + degree v in T'').*

Suppose that T' and T'' are two trees with $n = 7$ as shown in Fig. 4 where the neighbors of the roots of T' and T'' are 3 and 2 respectively. It takes 3 flips for T' to be modified to $T^*_{v_0}$ and 4 flips for T'' to be transformed to $T^*_{v_0}$. Hence, we need $2n - k - s - 2 = 14 - 3 - 2 - 2 = 7$ flips to transform T' to T''.

The above result can be expressed in terms of the meta graph $T_G(P)$ whose vertices are the non-crossing set of trees, $\mathcal{T}(P)$. Two vertices of the meta graph are adjacent if the trees represented by them can be obtained from the other by a single edge replacement. According to the proof of the above lemma, we can state that $T_G(P)$ is connected and the diameter of $T_G(P)$ which is the shortest distance between any two furthest vertices, can be at most $2n - k - s - 2$. We improve the result of [3] and show that our result produces a better result when the values for k or s are greater than one. Even in the worst case when both the parameters k and s have the value equal to one our result is as good as theirs.

4 Conclusion

In this paper, we present a method for tree transformation through flips when the points are in general position. In this method, a definition of a canonical tree is given for a fixed set of points in the plane. With the use of the canonical tree we obtained the necessary transformation from one tree to another. Later, description and formation of the meta graph $\mathcal{G}_\mathcal{T}$ consisting of vertices which are the non-crossing set of trees $\mathcal{T}(P)$ of a point set P is provided. This helps derive the bound of the diameter of the meta graph in terms of the number flips, which is the shortest longest distance between any two vertices of the meta graph. The diameter of $\mathcal{G}_\mathcal{T}$ is also determined to be bounded by $2n - k - s - 2$, where $k, s \geq 1$.

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