

Greedy Bidirectional Polymorphism

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30 August 2009

2009 ML Workshop
Edinburgh

Why Bidirectional Typechecking?

- Type inference (without annotations) often undecidable
 - Intersection types
 - Dependent types
 - Impredicative polymorphism (instantiating \forall with \forall)

Bidirectional Typechecking

$\Gamma \vdash e \Uparrow A$ e synthesizes type A

$\Gamma \vdash e \Downarrow A$ e checks against type A

- Many typing problems now decidable (with annotations)
- Can report type errors earlier

Why Polymorphism?

- Am I in the right room?

Why Polymorphism?

- Am I in the right room?
- Embarrassingly absent from my dissertation
- Adding polymorphism from scratch
 - First-class polymorphism accidental!

Annotations

- In its **simplest formulation**, bidirectional typing needs annotations on **redexes**: mostly, `let fun` and $(\lambda x. e_1) e_2$

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- Annotations are good documentation.
- Thinking about the types is good for you.

Annotations

- In its **simplest formulation**, bidirectional typing needs annotations on **redexes**: mostly, `let fun` and $(\lambda x. e_1) e_2$
- **Objection:** I don't want to!
- Annotations are good documentation.
- Thinking about the types is good for you.
- Just because the **type system** needs an annotation doesn't mean the user must write it.
When inference is decidable, a tool could write it, keeping complexity out of the type system.

Outline

- Bidirectional simple typing
- Oracular polymorphic typing
- Greedy polymorphism with ordered contexts
- Completeness
- (Hints)
- (Intersections and unions)
- Future work
- Related work

Simply-Typed Language: $\mathbf{1}, \rightarrow$

Contexts $\Gamma ::= \cdot \mid \Gamma, x:A$

Types $A, B, C ::= \mathbf{1} \mid A \rightarrow B$

Terms $e ::= x \mid () \mid \lambda x. e \mid e_1 e_2 \mid (e : A)$

Values $v ::= x \mid () \mid \lambda x. e \mid (v : A)$

Bidirectional Typing

- Two judgment forms

Synthesis $\Gamma^+ \vdash e^+ \uparrow A^-$

Checking $\Gamma^+ \vdash e^+ \downarrow A^+$

+ input

- output

Bidirectional Typing

- Using assumption

$$\frac{\Gamma(x) = A}{\Gamma \vdash x : A} \text{ var}$$

Bidirectional Typing

- Using assumption

$$\frac{\Gamma(x) = A}{\Gamma \vdash x \uparrow A} \text{ var}$$

Bidirectional Typing: \rightarrow

- Introduction rules **check**,
elimination rules **synthesize**

$$\frac{\Gamma, x:A \vdash e : B}{\Gamma \vdash \lambda x. e : A \rightarrow B} \rightarrow\text{I}$$

$$\frac{\Gamma \vdash e_1 : A \rightarrow B \quad \Gamma \vdash e_2 : A}{\Gamma \vdash e_1 e_2 : B} \rightarrow\text{E}$$

- e_1 usually a variable or another application

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- e_1 usually a variable or another application

Change of Direction

- In syntax-directed rules, intros check, elims synthesize
- Checking a synthesizing form:

$$\frac{\Gamma \vdash e \uparrow A' \quad A' = A}{\Gamma \vdash e \Downarrow A} \text{sub}$$

(Trivial =-“subsumption”, for now!)

- Synthesizing a checking form: annotation

$$\frac{\Gamma \vdash e \Downarrow A}{\Gamma \vdash (e : A) \uparrow A} \text{anno}$$

Synthesizing and Checking

- Synthesizing terms: $x \ e_1 \ e_2$ (elim)
- Checked terms: $\lambda x. e \ ()$ (intro)
- Synthesizing positions: $e_1 \ e_2$
- Checked positions: $e_1 \ e_2 \ \lambda x. e \ (e : A)$
- Annotations only where a checked term is used in a synthesizing position, i.e. on the function part of a redex

$(\lambda x. e_1) e_2$

and on function declarations.

Simply-Typed Bidirectional Rules

$$\frac{\Gamma(x) = A}{\Gamma \vdash x \uparrow A} \text{var}$$

$$\frac{\Gamma \vdash e \uparrow A' \quad A' = A}{\Gamma \vdash e \Downarrow A} \text{sub} \quad \frac{\Gamma \vdash e \Downarrow A}{\Gamma \vdash (e : A) \uparrow A} \text{anno}$$

$$\frac{}{\Gamma \vdash () \Downarrow \mathbf{1}} \mathbf{1I}$$

$$\frac{\Gamma, x:A \vdash e \Downarrow B}{\Gamma \vdash \lambda x. e \Downarrow A \rightarrow B} \rightarrow I \quad \frac{\Gamma \vdash e_1 \uparrow A \rightarrow B \quad \Gamma \vdash e_2 \Downarrow A}{\Gamma \vdash e_1 e_2 \uparrow B} \rightarrow E$$

Polymorphic: $\mathbf{1}$, \rightarrow , \forall

Type Variables α, β, γ

Contexts $\Gamma ::= \cdot \mid \Gamma, x:A \mid \alpha$

Types $A, B, C ::= \mathbf{1} \mid A \rightarrow B \mid \alpha \mid \forall\alpha. A$

Terms $e ::= x \mid () \mid \lambda x. e \mid e_1 e_2 \mid (e : A)$

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Type Variables α, β, γ

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Terms $e ::= x \mid () \mid \lambda x. e \mid e_1 e_2 \mid (e : A)$

$$\frac{\Gamma, \alpha \vdash v \Downarrow A}{\Gamma \vdash v \Downarrow \forall\alpha. A} \forall I \qquad \frac{\Gamma \vdash e \Uparrow \forall\alpha. A \quad \Gamma \vdash A' \text{ wf}}{\Gamma \vdash e \Uparrow [A'/\alpha]A} \forall E$$

But we don't know A' ...and we're not magicians.

Subtyping for \forall

- “ $A \leq B$ ” read “A is at least as polymorphic as B”

$$\frac{}{\Gamma \vdash \mathbf{1} \leq \mathbf{1}} \mathbf{1} \leq \quad \frac{\Gamma \vdash B_1 \leq A_1 \quad \Gamma \vdash A_2 \leq B_2}{\Gamma \vdash A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2} \rightarrow \leq$$

$$\frac{}{\Gamma \vdash \alpha \leq \alpha} \alpha \text{Refl} \quad \frac{\Gamma \vdash [A'/\alpha]A \leq B}{\Gamma \vdash \forall \alpha. A \leq B} \forall L \leq \quad \frac{\Gamma, \beta \vdash A \leq B}{\Gamma \vdash A \leq \forall \beta. B} \forall R \leq$$

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$$\frac{}{\Gamma \vdash \alpha \leq \alpha} \alpha \text{Refl} \quad \frac{\Gamma \vdash [A'/\alpha]A \leq B}{\Gamma \vdash \forall \alpha. A \leq B} \forall L \leq \quad \frac{\Gamma, \beta \vdash A \leq B}{\Gamma \vdash A \leq \forall \beta. B} \forall R \leq$$

- Change sub:

$$\frac{\Gamma \vdash e \uparrow A \quad \Gamma \vdash A \leq B}{\Gamma \vdash e \downarrow B} \text{sub}$$

Example

$$\begin{array}{c}
 \forall \alpha. (\alpha \rightarrow \text{bool}) \\
 \rightarrow \alpha \text{ list} \\
 \Gamma \vdash \text{filter} \uparrow \rightarrow \alpha \text{ list} \quad \Gamma \vdash \text{int wf} \\
 \hline
 (\text{int} \rightarrow \text{bool}) \quad \forall \text{E} \\
 \rightarrow \text{int list} \quad \Gamma \vdash f \uparrow \text{int} \rightarrow \text{bool} \quad \Gamma \vdash \text{int} \rightarrow \text{bool} \\
 \leq \text{int} \rightarrow \text{bool} \quad \text{sub} \\
 \Gamma \vdash \text{filter} \uparrow \rightarrow \text{int list} \quad \Gamma \vdash f \Downarrow \text{int} \rightarrow \text{bool} \\
 \hline
 \rightarrow \text{E}
 \end{array}$$

$$\begin{array}{c}
 \Gamma \vdash \text{filter } f \uparrow \text{int list} \rightarrow \text{int list} \quad \Gamma \vdash \chi s \Downarrow \text{int list} \\
 \hline
 \Gamma \vdash \text{filter } f \chi s \uparrow \text{int list} \quad \rightarrow \text{E}
 \end{array}$$

Greed

- The “greedy” method [Cardelli '93]:
Generate existential variable $\hat{\alpha}$, something like

$$\frac{\Gamma \vdash e \uparrow \forall \alpha. A}{\Gamma \vdash e \uparrow [\hat{\alpha}/\alpha]A} \quad \frac{\Gamma \vdash [\hat{\alpha}/\alpha]A \leq B}{\Gamma \vdash \forall \alpha. A \leq B}$$

then take the first solution.

Ordered Contexts

- Track existential variables explicitly

$$\Gamma ::= \Gamma, x:A \mid \Gamma, \alpha$$

$$\mid \Gamma, \hat{\alpha}$$

Unsolved existential type var

$$\mid \Gamma, \hat{\alpha}=A$$

Solved existential type var

$$\mid \Gamma, \blacktriangleleft \hat{\alpha}$$

Scope marker

- Thread Γ through:

$$\frac{\Gamma_1 \vdash e \uparrow \forall \alpha. A \vdash \Gamma_2}{\Gamma_1 \vdash e \uparrow [\hat{\alpha}/\alpha]A \vdash \Gamma_2, \hat{\alpha}} \forall E \hat{\alpha} \quad \frac{\Gamma_1, \blacktriangleleft \hat{\alpha}, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]A \leq B \vdash \Gamma_2, \blacktriangleleft \hat{\alpha}, \Gamma_Z}{\Gamma_1 \vdash \forall \alpha. A \leq B \vdash \Gamma_2} \forall L \hat{\alpha} \leq$$

$$\frac{\Gamma_1 \vdash e_1 \uparrow A \rightarrow B \vdash \Gamma_2 \quad \Gamma_2 \vdash e_2 \downarrow A \vdash \Gamma_3}{\Gamma_1 \vdash e_1 e_2 \uparrow B \vdash \Gamma_3} \rightarrow E$$

- Greedily instantiate existential variables, e.g.

$$\frac{\Gamma_1 \vdash B \text{ wf}}{\Gamma_1, \hat{\alpha}, \Gamma_2 \vdash \hat{\alpha} \leq B \dashv \Gamma_1, \hat{\alpha}=B, \Gamma_2}$$

$$\frac{\Gamma_1 \vdash A \text{ wf}}{\Gamma_1, \hat{\beta}, \Gamma_2 \vdash A \leq \hat{\beta} \dashv \Gamma_1, \hat{\beta}=A, \Gamma_2}$$

Synthesis $\Gamma_1^+ \vdash e^+ \uparrow A^- \dashv \Gamma_2^-$

Checking $\Gamma_1^+ \vdash e^+ \downarrow A^+ \dashv \Gamma_2^-$

+ input - output

Input and output contexts the same modulo $\hat{\alpha}$ and $\hat{\alpha}=A$:

Input

Output

$x:1$

$x:1, \hat{\alpha}$

✓

$\hat{\alpha}, x:\hat{\alpha}, \hat{\beta}$

$\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}=\hat{\alpha}_1 \rightarrow \hat{\alpha}_2, x:\hat{\alpha}, \hat{\beta}=\hat{\alpha}$

✓

$\hat{\alpha}, x:\hat{\alpha}, \hat{\beta}$

$\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}=\hat{\alpha}_1 \rightarrow \hat{\alpha}_2, x:\hat{\alpha}_1 \rightarrow \hat{\alpha}_2, \hat{\beta}=\hat{\alpha}, y:\hat{\beta}$

✗

We don't replace $\hat{\alpha}$ in declarations.

Example

$(\alpha \rightarrow \text{bool})$
 $\rightarrow \alpha \text{ list}$

$\Gamma \vdash \text{filter} \uparrow \forall \alpha. \rightarrow \alpha \text{ list} \vdash \Gamma$

 $\forall E \hat{\alpha}$

$\Gamma \vdash \text{filter} \uparrow$

⋮

	$\Gamma \vdash \text{int wf}$ <hr/> $\hat{\alpha} \leq \text{int}$	$\hat{\alpha} = L \leq$	bool $\leq \text{bool}$	
				$\rightarrow \leq$
	$f \uparrow \text{int} \rightarrow \text{bool}$ <hr/> $f \downarrow$	$\text{int} \rightarrow \text{bool}$ $\leq \hat{\alpha} \rightarrow \text{bool}$	sub	
				$\rightarrow E$
	$\Gamma \vdash \text{filter } f$			

Example

$(\alpha \rightarrow \text{bool})$
 $\rightarrow \alpha \text{ list}$

$\Gamma \vdash \text{filter} \uparrow \forall \alpha. \rightarrow \alpha \text{ list} \vdash \Gamma$

 $\forall E \hat{\alpha}$

$(\hat{\alpha} \rightarrow \text{bool})$
 $\rightarrow \hat{\alpha} \text{ list}$

$\Gamma \vdash \text{filter} \uparrow \rightarrow \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha}$

⋮

$\Gamma \vdash \text{int wf}$

$\hat{\alpha} = L \leq$

bool

$\hat{\alpha} \leq \text{int}$

$\leq \text{bool}$

$\rightarrow \leq$

$\text{int} \rightarrow \text{bool}$

$f \uparrow \text{int} \rightarrow \text{bool}$

$\leq \hat{\alpha} \rightarrow \text{bool}$

sub

$f \downarrow$

$\rightarrow E$

$\Gamma \vdash \text{filter } f$

Example

$$(\alpha \rightarrow \text{bool})$$

$$\rightarrow \alpha \text{ list}$$

$$\frac{\Gamma \vdash \text{filter} \uparrow \forall \alpha. \rightarrow \alpha \text{ list} \vdash \Gamma}{\quad} \forall E \hat{\alpha}$$

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$$\Gamma \vdash \text{filter} \uparrow \rightarrow \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha}$$

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$$\Gamma \vdash \text{int wf}$$

$$\hat{\alpha} = L \leq$$

$$\hat{\alpha} \leq \text{int}$$

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$$\leq \text{bool}$$

$$\rightarrow \leq$$

$$\Gamma, \hat{\alpha} \vdash f \uparrow \text{int} \rightarrow \text{bool}$$

$$\text{int} \rightarrow \text{bool}$$

$$\leq \hat{\alpha} \rightarrow \text{bool}$$

$$\text{sub}$$

$$\Gamma, \hat{\alpha} \vdash f \downarrow \hat{\alpha} \rightarrow \text{bool}$$

$$\rightarrow E$$

$$\Gamma \vdash \text{filter } f$$

Example

$(\alpha \rightarrow \text{bool})$
 $\rightarrow \alpha \text{ list}$

$\Gamma \vdash filter \uparrow \forall \alpha. \rightarrow \alpha \text{ list} \vdash \Gamma$

$\forall E \hat{\alpha}$

$(\hat{\alpha} \rightarrow \text{bool})$
 $\rightarrow \hat{\alpha} \text{ list}$

$\Gamma \vdash filter \uparrow \rightarrow \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha}$

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$\Gamma \vdash \text{int wf}$

$\hat{\alpha} \leq \text{int}$

$\hat{\alpha} = L \leq$

bool

$\leq \text{bool}$

$\rightarrow \leq$

$\Gamma, \hat{\alpha} \vdash f \uparrow \text{int} \rightarrow \text{bool} \vdash \Gamma, \hat{\alpha}$

$\text{int} \rightarrow \text{bool}$

$\leq \hat{\alpha} \rightarrow \text{bool}$

sub

$\Gamma, \hat{\alpha} \vdash f \downarrow \hat{\alpha} \rightarrow \text{bool}$

$\rightarrow E$

$\Gamma \vdash filter f$

Example

$$(\alpha \rightarrow \text{bool})$$

$$\rightarrow \alpha \text{ list}$$

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$$\Gamma \vdash \text{filter} \uparrow \rightarrow \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha}$$

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$$\Gamma \vdash \text{int wf}$$

$$\hat{\alpha} = L \leq$$

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$$\rightarrow \leq$$

$$\text{int} \rightarrow \text{bool}$$

$$\Gamma, \hat{\alpha} \vdash f \uparrow \text{int} \rightarrow \text{bool} \vdash \Gamma, \hat{\alpha} \quad \Gamma, \hat{\alpha} \vdash \leq \hat{\alpha} \rightarrow \text{bool}$$

$$\text{sub}$$

$$\Gamma, \hat{\alpha} \vdash f \downarrow \hat{\alpha} \rightarrow \text{bool}$$

$$\rightarrow E$$

$$\Gamma \vdash \text{filter } f$$

Example

$$(\alpha \rightarrow \text{bool}) \\ \rightarrow \alpha \text{ list}$$

$$\Gamma \vdash \text{filter} \uparrow \forall \alpha. \rightarrow \alpha \text{ list} \dashv \Gamma$$

$$\forall E \hat{\alpha}$$

$$(\hat{\alpha} \rightarrow \text{bool}) \\ \rightarrow \hat{\alpha} \text{ list}$$

$$\Gamma \vdash \text{filter} \uparrow \rightarrow \hat{\alpha} \text{ list} \dashv \Gamma, \hat{\alpha}$$

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$$\text{int} \rightarrow \text{bool}$$

$$\Gamma, \hat{\alpha} \vdash f \uparrow \text{int} \rightarrow \text{bool} \dashv \Gamma, \hat{\alpha}$$

$$\Gamma, \hat{\alpha} \vdash \leq \hat{\alpha} \rightarrow \text{bool}$$

$$\text{sub}$$

$$\Gamma, \hat{\alpha} \vdash f \downarrow \hat{\alpha} \rightarrow \text{bool}$$

$$\rightarrow E$$

$$\Gamma \vdash \text{filter} f$$

Example

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 $\rightarrow \alpha \text{ list}$

$\Gamma \vdash \text{filter} \uparrow \forall \alpha. \rightarrow \alpha \text{ list} \vdash \Gamma$

 $\forall E \hat{\alpha}$

$(\hat{\alpha} \rightarrow \text{bool})$
 $\rightarrow \hat{\alpha} \text{ list}$

$\Gamma \vdash \text{filter} \uparrow \rightarrow \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha}$

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$\Gamma \vdash \text{int wf}$

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 $\rightarrow \leq$

$\Gamma, \hat{\alpha} \vdash f \uparrow \text{int} \rightarrow \text{bool} \vdash \Gamma, \hat{\alpha}$ $\Gamma, \hat{\alpha} \vdash \leq \hat{\alpha} \rightarrow \text{bool}$

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$\Gamma \vdash \text{filter} f$

 $\rightarrow E$

Example

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$$\frac{\Gamma \vdash \text{filter} \uparrow \forall \alpha. \rightarrow \alpha \text{ list} \dashv \Gamma}{\quad} \forall E \hat{\alpha}$$

$$(\hat{\alpha} \rightarrow \text{bool}) \\ \rightarrow \hat{\alpha} \text{ list}$$

$$\Gamma \vdash \text{filter} \uparrow \rightarrow \hat{\alpha} \text{ list} \dashv \Gamma, \hat{\alpha}$$

⋮

$$\frac{\frac{\Gamma \vdash \text{int wf}}{\Gamma, \hat{\alpha} \vdash \hat{\alpha} \leq \text{int} \dashv \Gamma, \hat{\alpha} = \text{int}} \hat{\alpha} = L \leq \quad \text{bool}}{\Gamma, \hat{\alpha} = \text{int} \vdash \leq \text{bool} \dashv \Gamma, \hat{\alpha} = \text{int}} \rightarrow \leq$$

$$\frac{\Gamma, \hat{\alpha} \vdash f \uparrow \text{int} \rightarrow \text{bool} \dashv \Gamma, \hat{\alpha} \quad \Gamma, \hat{\alpha} \vdash \leq \hat{\alpha} \rightarrow \text{bool} \dashv \Gamma, \hat{\alpha} = \text{int}}{\Gamma, \hat{\alpha} \vdash f \downarrow \hat{\alpha} \rightarrow \text{bool}} \text{sub}$$

$$\frac{\Gamma \vdash \text{filter} f}{\quad} \rightarrow E$$

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$\forall E \hat{\alpha}$

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$\Gamma \vdash \text{filter} \uparrow \rightarrow \hat{\alpha} \text{ list} \dashv \Gamma, \hat{\alpha}$

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$\Gamma \vdash \text{int wf}$

$\hat{\alpha} = L \leq$

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 $\Gamma, \hat{\alpha} = \text{int} \vdash \leq \text{bool} \dashv \Gamma, \hat{\alpha} = \text{int}$

$\rightarrow \leq$

$\text{int} \rightarrow \text{bool}$

$\Gamma, \hat{\alpha} \vdash f \uparrow \text{int} \rightarrow \text{bool} \dashv \Gamma, \hat{\alpha}$ $\Gamma, \hat{\alpha} \vdash \leq \hat{\alpha} \rightarrow \text{bool} \dashv \Gamma, \hat{\alpha} = \text{int}$

sub

$\Gamma, \hat{\alpha} \vdash f \downarrow \hat{\alpha} \rightarrow \text{bool} \dashv \Gamma, \hat{\alpha} = \text{int}$

$\rightarrow E$

$\Gamma \vdash \text{filter} f$

Example

$$(\alpha \rightarrow \text{bool}) \\ \rightarrow \alpha \text{ list}$$

$$\frac{\Gamma \vdash \text{filter} \uparrow \forall \alpha. \rightarrow \alpha \text{ list} \dashv \Gamma}{\text{}} \quad \forall E \hat{\alpha}$$

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$$\frac{\Gamma \vdash \text{int wf} \quad \Gamma, \hat{\alpha} \vdash \hat{\alpha} \leq \text{int} \dashv \Gamma, \hat{\alpha} = \text{int} \quad \Gamma, \hat{\alpha} = \text{int} \vdash \leq \text{bool} \dashv \Gamma, \hat{\alpha} = \text{int}}{\Gamma, \hat{\alpha} \vdash \leq \hat{\alpha} \rightarrow \text{bool} \dashv \Gamma, \hat{\alpha} = \text{int}} \quad \rightarrow \leq$$

$$\frac{\Gamma, \hat{\alpha} \vdash f \uparrow \text{int} \rightarrow \text{bool} \dashv \Gamma, \hat{\alpha} \quad \Gamma, \hat{\alpha} \vdash \leq \hat{\alpha} \rightarrow \text{bool} \dashv \Gamma, \hat{\alpha} = \text{int}}{\Gamma, \hat{\alpha} \vdash f \downarrow \hat{\alpha} \rightarrow \text{bool} \dashv \Gamma, \hat{\alpha} = \text{int}} \quad \text{sub}$$

$$\frac{\Gamma, \hat{\alpha} \vdash f \downarrow \hat{\alpha} \rightarrow \text{bool} \dashv \Gamma, \hat{\alpha} = \text{int}}{\Gamma \vdash \text{filter} f \uparrow \hat{\alpha} \text{ list} \rightarrow \hat{\alpha} \text{ list} \dashv \Gamma, \hat{\alpha} = \text{int}} \quad \rightarrow E$$

Example, cont.

$$\begin{array}{c}
 \frac{\frac{\frac{\text{int} \leq \text{int}}{\text{int} \leq \hat{\alpha}}{\text{int list} \leq \hat{\alpha} \text{ list}}}{\text{xs} \uparrow \text{int list}}}{\Gamma \vdash \text{filter } f \uparrow \hat{\alpha} \text{ list} \rightarrow \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha} = \text{int}} \text{sub} \quad \text{xs} \downarrow \hat{\alpha} \text{ list} \\
 \hline
 \Gamma \vdash \text{filter } f \text{ xs} \rightarrow \text{E}
 \end{array}$$

Example, cont.

ExSubstR \leq

$$\text{int} \leq$$

$$\text{int} \leq \hat{\alpha}$$

$$\text{int list} \leq \hat{\alpha} \text{ list}$$

$$xs \uparrow \text{int list}$$

sub

$$\Gamma \vdash \text{filter } f \uparrow \hat{\alpha} \text{ list} \rightarrow \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha} = \text{int}$$

$$\Gamma, \hat{\alpha} = \text{int} \vdash xs \downarrow \hat{\alpha} \text{ list}$$

\rightarrow E

$$\Gamma \vdash \text{filter } f \text{ } xs$$

Example, cont.

$\Gamma, \hat{\alpha} = \text{int} \vdash$
 $\text{xs} \uparrow \text{int list}$

$\text{int} \leq$

$\text{int} \leq \hat{\alpha}$

$\text{int list} \leq \hat{\alpha} \text{ list}$

ExSubstR \leq

sub

$\Gamma \vdash \text{filter } f \uparrow \hat{\alpha} \text{ list} \rightarrow \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha} = \text{int}$

$\Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \downarrow \hat{\alpha} \text{ list}$

$\Gamma \vdash \text{filter } f \text{ xs}$ →E

Example, cont.

$$\frac{\Gamma, \hat{\alpha} = \text{int} \vdash \quad \text{xs} \uparrow \text{int list} \dashv \Gamma, \hat{\alpha} = \text{int}}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \uparrow \text{int list} \dashv \Gamma, \hat{\alpha} = \text{int}} \text{sub}$$

$$\frac{\frac{\text{int} \leq}{\text{int} \leq \hat{\alpha}} \text{ExSubstR} \leq}{\text{int list} \leq \hat{\alpha} \text{ list}}$$

$$\frac{\Gamma \vdash \text{filter } f \uparrow \hat{\alpha} \text{ list} \rightarrow \hat{\alpha} \text{ list} \dashv \Gamma, \hat{\alpha} = \text{int} \quad \Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \downarrow \hat{\alpha} \text{ list}}{\Gamma \vdash \text{filter } f \text{ xs}} \rightarrow E$$

Example, cont.

$$\begin{array}{c}
 \Gamma, \hat{\alpha} = \text{int} \vdash \\
 \text{xs} \uparrow \text{int list} \dashv \Gamma, \hat{\alpha} = \text{int} \quad \frac{\frac{\text{int} \leq \hat{\alpha}}{\text{int} \leq \hat{\alpha}} \text{ExSubstR} \leq}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int list} \leq \hat{\alpha} \text{ list}} \text{sub}
 \end{array}$$

$$\frac{\Gamma \vdash \text{filter } f \uparrow \hat{\alpha} \text{ list} \rightarrow \hat{\alpha} \text{ list} \dashv \Gamma, \hat{\alpha} = \text{int} \quad \Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \downarrow \hat{\alpha} \text{ list}}{\Gamma \vdash \text{filter } f \text{ xs}} \rightarrow \text{E}$$

Example, cont.

$$\frac{\Gamma, \hat{\alpha} = \text{int} \vdash \quad \frac{\frac{\text{int} \leq}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int} \leq \hat{\alpha}}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int list} \leq \hat{\alpha} \text{ list}} \text{ExSubstR} \leq}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \uparrow \text{int list} \dashv \Gamma, \hat{\alpha} = \text{int} \quad \Gamma, \hat{\alpha} = \text{int} \vdash \text{int list} \leq \hat{\alpha} \text{ list}} \text{sub}$$

$$\frac{\Gamma \vdash \text{filter } f \uparrow \hat{\alpha} \text{ list} \rightarrow \hat{\alpha} \text{ list} \dashv \Gamma, \hat{\alpha} = \text{int} \quad \Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \downarrow \hat{\alpha} \text{ list}}{\Gamma \vdash \text{filter } f \text{ xs}} \rightarrow \text{E}$$

Example, cont.

$$\frac{\Gamma, \hat{\alpha} = \text{int} \vdash \quad \frac{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int} \leq \text{int} \quad \Gamma, \hat{\alpha} = \text{int} \vdash \text{int} \leq \hat{\alpha}}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int} \leq \hat{\alpha}} \text{ExSubstR} \leq}{\Gamma, \hat{\alpha} = \text{int} \vdash \quad \Gamma, \hat{\alpha} = \text{int} \vdash \text{int list} \leq \hat{\alpha} \text{ list}} \text{sub}$$

$$\frac{\Gamma \vdash \text{filter } f \uparrow \hat{\alpha} \text{ list} \rightarrow \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha} = \text{int} \quad \Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \downarrow \hat{\alpha} \text{ list}}{\Gamma \vdash \text{filter } f \text{ xs}} \rightarrow \text{E}$$

Example, cont.

$$\frac{\Gamma, \hat{\alpha} = \text{int} \vdash \quad \frac{\frac{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int} \leq \text{int} \quad \Gamma, \hat{\alpha} = \text{int}}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int} \leq \hat{\alpha}} \text{ExSubstR} \leq}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int list} \leq \hat{\alpha} \text{ list}} \text{sub}}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \uparrow \text{int list} \quad \Gamma, \hat{\alpha} = \text{int}}$$

$$\frac{\Gamma \vdash \text{filter } f \uparrow \hat{\alpha} \text{ list} \rightarrow \hat{\alpha} \text{ list} \quad \Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \downarrow \hat{\alpha} \text{ list} \quad \Gamma, \hat{\alpha} = \text{int} \vdash \text{int list} \leq \hat{\alpha} \text{ list}}{\Gamma \vdash \text{filter } f \text{ xs}} \rightarrow \text{E}$$

Example, cont.

$$\frac{\Gamma, \hat{\alpha} = \text{int} \vdash \quad \frac{\frac{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int} \leq \text{int} \quad \Gamma, \hat{\alpha} = \text{int}}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int} \leq \hat{\alpha}} \text{ExSubstR} \leq}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int list} \leq \hat{\alpha} \text{ list}} \text{sub}}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \uparrow \text{int list} \quad \Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \downarrow \hat{\alpha} \text{ list}}$$

$$\frac{\Gamma \vdash \text{filter } f \uparrow \hat{\alpha} \text{ list} \rightarrow \hat{\alpha} \text{ list} \quad \Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \downarrow \hat{\alpha} \text{ list}}{\Gamma \vdash \text{filter } f \text{ xs}} \rightarrow \text{E}$$

Example, cont.

$$\frac{\Gamma, \hat{\alpha} = \text{int} \vdash \quad \frac{\frac{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int} \leq \text{int} \vdash \Gamma, \hat{\alpha} = \text{int}}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int} \leq \hat{\alpha} \vdash \Gamma, \hat{\alpha} = \text{int}} \text{ExSubstR} \leq}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \uparrow \text{int list} \vdash \Gamma, \hat{\alpha} = \text{int} \quad \Gamma, \hat{\alpha} = \text{int} \vdash \text{int list} \leq \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha} = \text{int}} \text{sub}}$$

$$\frac{\Gamma \vdash \text{filter } f \uparrow \hat{\alpha} \text{ list} \rightarrow \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha} = \text{int} \quad \Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \downarrow \hat{\alpha} \text{ list}}{\Gamma \vdash \text{filter } f \text{ xs}} \rightarrow E$$

Example, cont.

$$\frac{\Gamma, \hat{\alpha} = \text{int} \vdash \quad \frac{\frac{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int} \leq \text{int} \vdash \Gamma, \hat{\alpha} = \text{int}}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int} \leq \hat{\alpha} \vdash \Gamma, \hat{\alpha} = \text{int}} \text{ExSubstR} \leq}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \uparrow \text{int list} \vdash \Gamma, \hat{\alpha} = \text{int}} \quad \frac{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int list} \leq \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha} = \text{int}}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \uparrow \text{int list} \vdash \Gamma, \hat{\alpha} = \text{int}} \text{sub}}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \uparrow \text{int list} \vdash \Gamma, \hat{\alpha} = \text{int}}$$

$$\frac{\Gamma \vdash \text{filter } f \uparrow \hat{\alpha} \text{ list} \rightarrow \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha} = \text{int} \quad \Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \downarrow \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha} = \text{int}}{\Gamma \vdash \text{filter } f \text{ xs}} \rightarrow \text{E}$$

Example, cont.

$$\frac{\Gamma, \hat{\alpha} = \text{int} \vdash \quad \frac{\frac{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int} \leq \text{int} \vdash \Gamma, \hat{\alpha} = \text{int}}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int} \leq \hat{\alpha} \vdash \Gamma, \hat{\alpha} = \text{int}} \text{ExSubstR} \leq}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \uparrow \text{int list} \vdash \Gamma, \hat{\alpha} = \text{int}} \quad \frac{\Gamma, \hat{\alpha} = \text{int} \vdash \text{int list} \leq \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha} = \text{int}}{\Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \uparrow \text{int list} \vdash \Gamma, \hat{\alpha} = \text{int}} \text{sub}$$

$$\frac{\Gamma \vdash \text{filter } f \uparrow \hat{\alpha} \text{ list} \rightarrow \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha} = \text{int} \quad \Gamma, \hat{\alpha} = \text{int} \vdash \text{xs} \downarrow \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha} = \text{int}}{\Gamma \vdash \text{filter } f \text{ xs} \uparrow \hat{\alpha} \text{ list} \vdash \Gamma, \hat{\alpha} = \text{int}} \rightarrow \text{E}$$

Explicit Substitution Rules

$$\frac{\Gamma \vdash e \Downarrow \Gamma(\hat{\alpha}) \dashv \Gamma'}{\Gamma \vdash e \Downarrow \hat{\alpha} \dashv \Gamma'} \text{ExSubst}\Downarrow \quad \frac{\Gamma \vdash e \Uparrow \hat{\alpha} \dashv \Gamma'}{\Gamma \vdash e \Uparrow \Gamma'(\hat{\alpha}) \dashv \Gamma'} \text{ExSubst}\Uparrow$$

$$\frac{\Gamma \vdash \Gamma(\hat{\alpha}) \leq B \dashv \Gamma'}{\Gamma \vdash \hat{\alpha} \leq B \dashv \Gamma'} \text{ExSubstL}\leq \quad \frac{\Gamma \vdash A \leq \Gamma(\hat{\beta}) \dashv \Gamma'}{\Gamma \vdash A \leq \hat{\beta} \dashv \Gamma'} \text{ExSubstR}\leq$$

Example:

$$\frac{\hat{\alpha}_1 = \text{int}, \hat{\alpha}_2 = 1, \hat{\alpha} = \hat{\alpha}_1 \rightarrow \hat{\alpha}_2 \vdash x \Uparrow \hat{\alpha}}{\hat{\alpha}_1 = \text{int}, \hat{\alpha}_2 = 1, \hat{\alpha} = \hat{\alpha}_1 \rightarrow \hat{\alpha}_2 \vdash x \Uparrow \hat{\alpha}_1 \rightarrow \hat{\alpha}_2} \text{ExSubst}\Uparrow$$

Complete?

- Sound with respect to oracular system
- Is it complete?
 - Can we always find the A' used in $\forall E$ and $\forall L_{\leq}$?
 - Impredicative counterexample:

$f: \forall \alpha. \alpha \rightarrow \alpha \rightarrow 1, \quad x: (\text{int} \rightarrow \text{int}) \rightarrow 1, \quad y: (\forall \beta. \beta \rightarrow \beta) \rightarrow 1$

\vdash

$f \ x \ y \ \uparrow$

α becomes $\hat{\alpha}$, which is then solved: $(\text{int} \rightarrow \text{int}) \rightarrow 1$

Need to show: $(\forall \beta. \beta \rightarrow \beta) \rightarrow 1 \leq (\text{int} \rightarrow \text{int}) \rightarrow 1$

So need to show: $\text{int} \rightarrow \text{int} \leq \forall \beta. \beta \rightarrow \beta$

Completeness

Instantiation	\forall location	Complete?
Predicative A^{mono} / α	prenex $\forall \alpha. A \rightarrow B$	✓
Predicative A^{mono} / α	arbitrary-rank $(\forall \alpha. A \rightarrow B) \rightarrow C$	✓
Impredicative $(\forall \beta. \beta \rightarrow \beta) / \alpha$	arbitrary-rank $(\forall \alpha. A \rightarrow B) \rightarrow C$	✗

Hints

Annotations can instantiate, but somewhat verbosely:

```
(*[ val ordinarily_annotated
      : ( -all 'a- 'a → 'a → unit)
      → ((int → int) → unit)
      → ((-all 'b- 'b → 'b) → unit)
      → unit ]*)
```

```
fun ordinarily_annotated f x y =
  let val f_at = ( f : ((-all 'c- 'c → 'c) → unit)
                  → ((-all 'c- 'c → 'c) → unit)
                  → unit )
  in f_at x y
  end
```

Hints

Instead, use a **hint**:

```
(*[ val hinted : (-all 'a- 'a → 'a → unit)
      → ((int → int) → unit)
      → ((-all 'b- 'b → 'b) → unit)
      → unit ]*)
```

```
fun hinted f x y =
  hint (-all 'c- 'c → 'c) → unit in
  f x y
end
```

Implementation

- System decidable (see paper), but not syntax-directed

$$\frac{\Gamma, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]A \leq B \dashv \Gamma', \hat{\alpha}[\dots], \Gamma_Z}{\Gamma \vdash \forall\alpha. A \leq B \dashv \Gamma'} \forall L \hat{\alpha} \leq$$

$$\frac{\Gamma_1 \vdash A \text{ wf}}{\Gamma_1, \hat{\beta}, \Gamma_2 \vdash A \leq \hat{\beta} \dashv \Gamma_1, \hat{\beta} = A, \Gamma_2} \hat{\alpha} = R \leq$$

Either rule might derive $\Gamma_1, \hat{\beta}, \Gamma_2 \vdash \forall\alpha. A_0 \leq \hat{\beta}$

$\forall L \hat{\alpha} \leq$ instantiates $\hat{\beta}$ with $[\hat{\alpha}/\alpha]A_0$

$\hat{\alpha} = R \leq$ instantiates $\hat{\beta}$ with $\forall\alpha. A_0$

- Backtracking used

Intersection Types

- $v \Downarrow A \wedge B$ if $v \Downarrow A$ and $v \Downarrow B$
- If $e \Uparrow A \wedge B$ then $e \Uparrow A$ and $e \Uparrow B$

$\text{op}+ : (\text{int} \rightarrow \text{int} \rightarrow \text{int}) \wedge (\text{real} \rightarrow \text{real} \rightarrow \text{real})$

Subtyping Needs Unions

- Add “real” subtyping \leq

$$\text{empty} \leq \text{list} \quad \text{nonempty} \leq \text{list}$$

- Let $\text{choose} : \forall \alpha. \alpha \rightarrow \alpha \rightarrow \alpha$

$$\text{ne:nonempty}, \text{xs:list} \not\vdash \text{choose ne xs} \uparrow$$

- Let $\text{choose} : \forall \alpha_1, \alpha_2. \alpha_1 \rightarrow \alpha_2 \rightarrow (\alpha_1 \vee \alpha_2)$

$$\text{ne:nonempty}, \text{xs:list} \vdash \text{choose ne xs} \uparrow (\text{nonempty} \vee \text{list}) \quad \checkmark$$

Type Abuse

$compose : \forall \alpha, \beta, \gamma. (\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$

- Can't compose $g : (B_1 \rightarrow C_1) \wedge (B_2 \rightarrow C_2)$
and $f : A \rightarrow (B_1 \vee B_2)$
- So rewrite type, as we did for *choose*:

$$\begin{aligned} compose : \forall \alpha, \beta_1, \beta_2, \gamma_1, \gamma_2. \\ & ((\beta_1 \rightarrow \gamma_1) \wedge (\beta_2 \rightarrow \gamma_2)) \\ & \rightarrow (\alpha \rightarrow (\beta_1 \vee \beta_2)) \\ & \rightarrow \alpha \\ & \rightarrow (\gamma_1 \vee \gamma_2) \end{aligned}$$

- Doesn't scale

$$\frac{\Gamma_1 \vdash B \text{ wf}}{\Gamma_1, \hat{\alpha}, \Gamma_2 \vdash \hat{\alpha} \leq B \quad \vdash \Gamma_1, \hat{\alpha}^\wedge = B, \Gamma_2} \hat{\alpha}^{\text{=L}} \text{Auto}$$

$$\frac{\Gamma_1 \vdash A \text{ wf}}{\Gamma_1, \hat{\beta}, \Gamma_2 \vdash A \leq \hat{\beta} \quad \vdash \Gamma_1, \hat{\beta}^\vee = A, \Gamma_2} \hat{\alpha}^{\text{=R}} \text{Auto}$$

$$\frac{\Gamma_1 \vdash B \text{ wf}}{\Gamma_1, \hat{\alpha}^\wedge = A, \Gamma_2 \vdash \hat{\alpha} \leq B \quad \vdash \Gamma_1, \hat{\alpha}^\wedge = A \wedge B, \Gamma_2} \hat{\alpha}^\wedge \text{L Auto}$$

$$\frac{\Gamma_1 \vdash A \text{ wf}}{\Gamma_1, \hat{\beta}^\vee = A, \Gamma_2 \vdash A \leq \hat{\beta} \quad \vdash \Gamma_1, \hat{\beta}^\vee = A \vee B, \Gamma_2} \hat{\alpha}^\vee \text{R Auto}$$

Future Work

- Completeness of intersection/union system
- GADTs

Related Work

- [Cardelli '93]: Greedy method for System $F_{<}$.
- [Pierce & Turner '00]: Local type inference
- [Davies '05] ML-style \forall , finitely many refinements
- [Peyton Jones et al. '07]: Higher-rank \forall , mixed $:/\uparrow\downarrow$
- [Le Botlan & Rémy '03, Rémy & Yakobowski '08]: ML^F : impred. \forall
- [Leijen '07–'09]: HMF, HML: impred. \forall , relatively simple, annotations on impredicative uses only

Summary

- Bidirectional typing
- First-class polymorphism in a bidirectional setting
 - Ordered contexts
 - Complete for predicative polymorphism
- For subtyping: Intersection and union types

More Information

<http://www.cs.cmu.edu/~joshuad/papers/poly/>

- Acknowledgments:
 - Frank Pfenning, Sungwoo Park, Brigitte Pientka
 - ICFP and ML Workshop reviewers