Decision Problems for Restricted Variants of Two-Dimensional Automata

CIAA 2019

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July 22, 2019

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Two-Dimensional Automata

- A two-dimensional (2D) automaton is a generalization of a one-dimensional automaton.
- Two major differences:
    1. Different input word
    2. Different transition function

Two-Dimensional Automata

- A two-dimensional (2D) automaton is a generalization of a one-dimensional automaton.
- Two major differences:
  1. **Different input word**
  2. Different transition function

$$\begin{array}{cccccc}
\# & \# & \# & \cdots & \# & \# \\
\# & a_{1,1} & a_{1,2} & \cdots & a_{1,n} & \# \\
\# & a_{2,1} & a_{2,2} & \cdots & a_{2,n} & \# \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\# & a_{m,1} & a_{m,2} & \cdots & a_{m,n} & \# \\
\# & \# & \# & \cdots & \# & \#
\end{array}$$

Two-Dimensional Automata

- A two-dimensional (2D) automaton is a generalization of a one-dimensional automaton.
- Two major differences:
  1. Different input word
  2. **Different transition function**

$$\delta : (Q \setminus q_{\text{accept}}) \times (\Sigma \cup \{\#\}) \rightarrow Q \times \{U, D, L, R\}$$

Deterministic
four-way
(2DFA-4W)

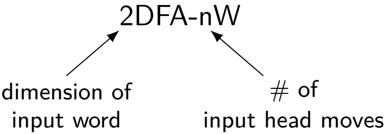
$$\delta : (Q \setminus q_{\text{accept}}) \times (\Sigma \cup \{\#\}) \rightarrow 2^{Q \times \{U,D,L,R\}}$$

Nondeterministic
four-way
(2NFA-4W)

Two-Dimensional Automata

Remark

A note on notation. . .



2DFA-nW

dimension of input word

# of input head moves

Notation like “4DFA” is found in literature discussing 2DFA-4W.

Restricted 2D Automata

- 2D automata do not have to be four-way automata.
- Restrict the transition function to get:
  - Three-way (3W) automata: $\{D, L, R\}$
  - Two-way (2W) automata: $\{D, R\}$
- Three-way automata cannot return to a row after moving downward, but they can read symbols multiple times in a row.
- Two-way automata are “read-once”.
  - Similar to a one-way one-dimensional automaton.

Decision Problems

	2DFA-4W	2NFA-4W	2DFA-3W	2NFA-3W	2DFA-2W	2NFA-2W
membership	✓	✓	✓	✓	✓	✓
emptiness	✗	✗	✓	✓	✓	✓
universality	✗	✗	✓	✗	✓	?
equivalence	✗	✗	?	✗	?	?

Decision Problems

	2DFA-4W	2NFA-4W	2DFA-3W	2NFA-3W	2DFA-2W	2NFA-2W
membership	✓	✓	✓	✓	✓	✓
emptiness	✗	✗	✓	✓†	✓∗	✓∗
universality	✗	✗	✓	✗	✓	?
equivalence	✗	✗	?	✗	✓○	?

○: new decidability result

∗: new complexity bound over general alphabets

†: new complexity bound over unary alphabets

Decision Problems

	2DFA-4W	2NFA-4W	2DFA-3W	2NFA-3W	2DFA-2W	2NFA-2W
membership	✓	✓	✓	✓	✓	✓
emptiness	✗	✗	✓	✓†	✓∗	✓∗
universality	✗	✗	✓	✗	✓	○✗
equivalence	✗	✗	?	✗	○✓	○✗

○: new decidability result

∗: new complexity bound over general alphabets

†: new complexity bound over unary alphabets

○: new decidability result, post-publication

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Language Emptiness

- Language emptiness asks: given an automaton $\mathcal{A}$, is $L(\mathcal{A}) = \emptyset$?
- Known to be decidable for three-way 2D automata.
  - Also PSPACE-hard for general alphabets.
  - See Inoue/Takanami (1980) and Petersen (1995).
- Easily seen to be decidable for two-way 2D automata.
  - Simply guess an accepting computation.

Language Emptiness: 2NFA-3W

- The emptiness problem for two-way 1D automata is NP-complete.
    - See Galil (1976).
- A unary two-way 1D automaton is a special case of a unary three-way 2D automaton.
    - The 2D problem is at least NP-hard.

Theorem

The emptiness problem for unary 2NFA-3W is NP-complete.

Proof Sketch

Given a unary 2NFA-3W automaton $\mathcal{A}$, replace all downward moves with "stay-in-place" moves. Call the new automaton $\mathcal{A}'$. Evidently, $L(\mathcal{A}) = \emptyset$ iff $L(\mathcal{A}') = \emptyset$. This is equivalent to deciding emptiness for a unary two-way 1D automaton.

Language Emptiness: 2NFA-2W

- The proof of decidability of emptiness for two-way nondeterministic 2D automata is also a proof that the problem is in NP.
- It is possible to improve this bound.

Theorem

The emptiness problem for 2NFA-2W is in P.

Proof Sketch

Via a reachability procedure.

If q_{accept} is reachable, then the language is not empty. Given a 2NFA-2W automaton $\mathcal{A}$ with n states, we can check reachability in polynomial time without nondeterminism.

Language Emptiness: 2NFA-2W

- The proof of decidability of emptiness for two-way nondeterministic 2D automata is also a proof that the problem is in NP.
- It is possible to improve this bound.

Corollary

The emptiness problem for 2DFA-2W is in P.

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Language Equivalence

- Language equivalence asks: given two automata $\mathcal{A}$ and $\mathcal{B}$, is $L(\mathcal{A}) = L(\mathcal{B})$?
- Known to be undecidable for four-way 2D automata.
  - See Blum/Hewitt (1967).
- Known to be undecidable for three-way nondeterministic 2D automata.
  - See Inoue/Takanami (1980).
- Unknown if decidable for three-way deterministic 2D automata.

Language Equivalence: 2DFA-2W

- What about two-way 2D automata?
  - Seems like it should be decidable.
  - The argument is not very straightforward, though.

Theorem

The equivalence problem for 2DFA-2W is decidable.

Proof Sketch

Via a technical lemma.

Language Equivalence: 2DFA-2W

Lemma

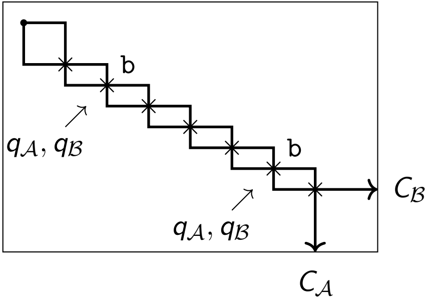
Let $\mathcal{A}$ and $\mathcal{B}$ be 2DFA-2W with m and n states, respectively.
Denote $z = m \cdot n \cdot |\Sigma|^2 + 1$ and $f(z) = z^2 \cdot (z^2 + z - 1)$.
If $L(\mathcal{A}) - L(\mathcal{B}) \neq \emptyset$, then $L(\mathcal{A}) - L(\mathcal{B})$ contains a 2D word with at most $f(z)$ rows and $f(z)$ columns.

- In what follows, assume we have a word $W \in L(\mathcal{A}) - L(\mathcal{B})$.
- Intuitively speaking:
  - Suppose $\mathcal{A}$ has an accepting computation $C_{\mathcal{A}}$ on W.
  - Suppose $\mathcal{B}$ has a rejecting computation $C_{\mathcal{B}}$ on W.
  - The lemma uses a pumping property to reduce the dimension of W by removing shared states of $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$.
  - This gives a witness word of bounded dimension separating $L(\mathcal{A})$ and $L(\mathcal{B})$.

Language Equivalence: 2DFA-2W

Proof Sketch

Case 1: $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ share at least z positions.
At least two of these shared positions must have been reached from the same states of $\mathcal{A}$ and $\mathcal{B}$ on the same symbol.
We can reduce dimension by identifying/removing these positions.



Language Equivalence: 2DFA-2W

Proof Sketch (cont'd)

Case 2: $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ share fewer than z positions.
There exists some subword Z consisting of $z \cdot (z^2 + z - 1)$ consecutive complete columns of the input word where $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ do not intersect.

Language Equivalence: 2DFA-2W

Proof Sketch (cont'd)

Case 2: $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ share fewer than z positions.
There exists some subword Z consisting of $z \cdot (z^2 + z - 1)$ consecutive complete columns of the input word where $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ do not intersect.

Case 2(a): One or both of $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ do not enter Z.
We can reduce dimension without affecting $C_{\mathcal{A}}$ or $C_{\mathcal{B}}$.

Language Equivalence: 2DFA-2W

Proof Sketch (cont'd)

Case 2: $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ share fewer than z positions.
There exists some subword Z consisting of $z \cdot (z^2 + z - 1)$ consecutive complete columns of the input word where $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ do not intersect.

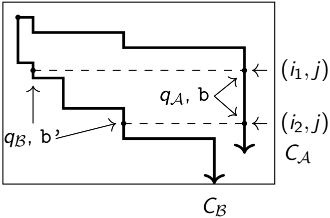
Case 2(b): Both $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ enter Z without visiting all columns.
Assume that $C_{\mathcal{A}}$ is above $C_{\mathcal{B}}$ and that $C_{\mathcal{A}}$ continues to the last row of Z.
We can reduce the columns without affecting $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$.

Language Equivalence: 2DFA-2W

Proof Sketch (cont'd)

Case 2: $C_\mathcal{A}$ and $C_\mathcal{B}$ share fewer than z positions.
There exists some subword Z consisting of $z \cdot (z^2 + z - 1)$ consecutive complete columns of the input word where $C_\mathcal{A}$ and $C_\mathcal{B}$ do not intersect.

Case 2(c): $C_\mathcal{A}$ has a vertical drop of z steps within Z, or $C_\mathcal{A}$ finishes computation at least z positions higher than $C_\mathcal{B}$.
We can reduce the rows without affecting $C_\mathcal{A}$ and $C_\mathcal{B}$.



Language Equivalence: 2DFA-2W

Proof Sketch (cont'd)

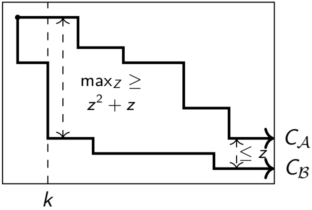
At this point, we know that:

- $C_{\mathcal{A}}$ is above $C_{\mathcal{B}}$ within Z; and
- the vertical distance between each computation at the end is at most z.

Let $\max_Z$ denote the maximal vertical difference of $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ at any fixed column in Z.

Proof Sketch (cont'd)

Case 2(c)(i): Assume $\max_Z \geq z^2 + z$.
Suppose the leftmost column of maximal vertical distance is k.
There must exist z columns between k and the last column of Z where vertical distance monotonically decreases or stays the same.
In two of these columns, the states of $C_\mathcal{A}$ and $C_\mathcal{B}$ and corresponding alphabet symbols coincide.
We can reduce the columns without affecting $C_\mathcal{A}$ or $C_\mathcal{B}$.



Language Equivalence: 2DFA-2W

Proof Sketch (cont'd)

Case 2(c)(ii): Assume $\max_Z \leq z^2 + z - 1$.
Since Z consists of $z \cdot (z^2 + z - 1)$ columns, there must exist z columns within Z where vertical distance stays the same.
In two of these columns, the states of $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ and corresponding alphabet symbols coincide.
We can reduce the columns without affecting $C_{\mathcal{A}}$ or $C_{\mathcal{B}}$.

Language Equivalence: 2DFA-2W

- The previous cases give a method to reduce the columns of the input word.
- The method to reduce rows is analogous.
- Altogether, the lemma gives a brute-force algorithm to decide equivalence by focusing only on input words up to a given dimension.

Language Equivalence: 2DFA-2W

Benefits of Approach

- The same argument also works for the inclusion problem.
    - The inclusion problem is therefore decidable for 2DFA-2W.

Detriments of Approach

- The algorithm given by the technical lemma is inefficient.
- The same argument cannot be used for 2DFA-3W.

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Row/Column Projection

- Row/column projections are operations on 2D words.
- The row (resp., column) projection of a 2D language L is the 1D language consisting of all first rows (resp., columns) of 2D words in L.

$$
\begin{array}{ccccc}
\# & \# & \# & \# & \# \\
\# & 1 & 0 & 1 & \# \\
\# & 0 & 1 & 0 & \# \\
\# & 1 & 1 & 1 & \# \\
\# & 0 & 0 & 0 & \# \\
\# & \# & \# & \# & \#
\end{array}
$$

$$\downarrow \text{(row proj.)}$$

$$
\begin{array}{ccccc}
\# & 1 & 0 & 1 & \#
\end{array}
$$

Row/Column Projection

- Row/column projections over four-way 2D automata do not always produce regular languages.
  - This model can recognize words of dimension $2^n \times 2^n$.
  - Doesn't work even for a unary alphabet.
- What about three-way/two-way 2D automata?
  - These models cannot recognize exponential-dimension words.

Row/Column Projection: 3W

Theorem

The row projection language $L(\mathcal{A})$ of a unary 2NFA-3W $\mathcal{A}$ is regular.

Proof Sketch

Replace downward moves with “stay-in-place” moves, as mentioned earlier.

Row/Column Projection: 3W

Theorem

The column projection language $L(\mathcal{A})$ of a unary 2DFA-3W $\mathcal{A}$ is not always regular.

Proof Sketch

Construct a unary 2DFA-3W whose language is equal to

$$L_{\text{composite}} = \{a^m \mid m > 1 \text{ and } m \text{ is not prime}\}.$$

Row/Column Projection: 2W

Theorem

The row projection language $L(\mathcal{A})$ of a general-alphabet 2NFA-2W $\mathcal{A}$ is regular.

Proof Sketch

Construct a one-way nondeterministic 1D automaton $\mathcal{B}$ with "stay-in-place" moves to simulate the computation of $\mathcal{A}$.
$\mathcal{B}$ nondeterministically guesses symbols that $\mathcal{A}$ reads when it moves downward and rightward.
$\mathcal{B}$ measures the number of rightward moves it made and the length of remaining input, and checks if there exists a matching accepting computation of $\mathcal{A}$.

Row/Column Projection: 2W

Corollary

The column projection language $L(\mathcal{A})$ of a general-alphabet 2NFA-2W $\mathcal{A}$ is regular.

Row/Column Projection

	2DFA-4W	2NFA-4W	2DFA-3W	2NFA-3W	2DFA-2W	2NFA-2W
row projection	✗	✗	✓∗	✓∗	✓	✓
column projection	✗	✗	✗	✗	✓	✓

✓: always regular
✗: sometimes non-regular
∗: unary alphabets only

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Conclusions

- 2D automata can be restricted to move in fewer than four directions.
- This restriction brings about interesting decidability properties.
- Language emptiness is NP-complete for unary 2NFA-3W.
- Language emptiness is in P for 2DFA-2W/2NFA-2W.
- Language equivalence is decidable for 2DFA-2W.
- Row projections are always regular over unary three-way 2D automata.
- Row/column projections are always regular over two-way 2D automata.

Future Work

- For general-alphabet three-way 2D automata, is the emptiness problem in PSPACE?
- Does there exist an efficient algorithm to decide equivalence for 2DFA-2W?
- Is the equivalence problem decidable for 2DFA-3W?
- What else can we study about restricted 2D automaton models?

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