Some Results on Words in Two Dimensions

Formal Languages & Automata Theory Seminar

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Background

- Combinatorics on words is a well-studied subfield of theoretical computer science, with its origins in the early 20th century.
- Many results in the one-dimensional case have appeared.
- However, the two-dimensional case is not as popular, even though many of the one-dimensional results seem naturally extendible to higher dimensions.
- In this presentation, we investigate various two-dimensional generalizations of some well-known properties of words.

- A **two-dimensional word**

$$A = \begin{bmatrix} a_{0,0} & \cdots & a_{0,n-1} \\ \vdots & \ddots & \vdots \\ a_{m-1,0} & \cdots & a_{m-1,n-1} \end{bmatrix}$$

  is a map from $\{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, n-1\}$ to an alphabet Σ.
  - Also called an **array**, a **picture**, and a **figure** in the literature.
- The **set of two-dimensional words** $\Sigma^{m \times n}$ contains all two-dimensional words of dimension $m \times n$ over Σ.
  - We also have the sets Σ^{**} (all two-dimensional words over Σ) and Σ^{++} (all *nonempty* two-dimensional words over Σ).

Preliminaries

- A pair of two-dimensional words A and B may be **concatenated** in
  - the horizontal direction, denoted $A \ominus B$; or
  - the vertical direction, denoted $A \varobar B$.

- A pair of two-dimensional words A and B may be **concatenated** in
  - the horizontal direction, denoted $A \ominus B$; or
  - the vertical direction, denoted $A \oplus B$.

Example

Given

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \; B = \begin{bmatrix} 7 & 8 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 3 \\ 6 \end{bmatrix},$$

we have that

$$A \ominus B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \text{ and } A \oplus C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Preliminaries

- Two-dimensional words may have **powers**, **prefixes**, and **suffixes**.
  - A prefix/suffix is **nontrivial** if it is nonempty.
  - A prefix/suffix is **proper** if it is not equal to the word itself.

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Example

Given $A = \begin{bmatrix} 4 & 6 \end{bmatrix}$, the 2×3 power of A is

$$A^{2\times 3} = \begin{bmatrix} 4 & 6 & 4 & 6 & 4 & 6 \\ 4 & 6 & 4 & 6 & 4 & 6 \end{bmatrix}.$$

$A^{2\times 3}$ has, among others, the prefix/suffix

$$B = \begin{bmatrix} 4 & 6 \\ 4 & 6 \end{bmatrix}.$$

- A two-dimensional word A is **primitive** if it cannot be written as a power; that is, $A \neq B^{p \times q}$ for some $B \in \Sigma^{++}$ with either $p \geq 2$ or $q \geq 2$.

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Example

The two-dimensional word $B = \begin{bmatrix} 2 & 4 \end{bmatrix}$ is primitive.
The two-dimensional word

$$A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$$

is not primitive, since we can write $A = B^{2 \times 1}$.

- A two-dimensional word A is **bordered** if we can write

$$A = (Q \varobar R \varobar Q) \ominus (S \varobar T \varobar S) \ominus (Q \varobar R \varobar Q)$$

for $Q \in \Sigma^{++}$ and $R, S, T \in \Sigma^{**}$.

- A two-dimensional word A is **bordered** if we can write

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Example

$$A = \begin{bmatrix} 7 & 4 & 1 & 7 & 4 \\ 6 & 8 & 0 & 6 & 8 \\ 3 & 2 & 9 & 3 & 2 \\ 7 & 4 & 1 & 7 & 4 \\ 6 & 8 & 0 & 6 & 8 \end{bmatrix}$$

We see immediately that A is bordered.

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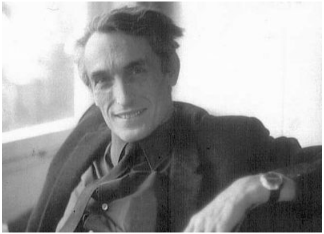
Background

- The **Lyndon-Schützenberger theorems** define a set of conditions for
  1. when a word has identical nontrivial proper prefixes and suffixes; and
  2. when the concatenation of two words x and y commutes; that is, when $xy = yx$.

R. C. Lyndon



M.-P. Schützenberger

1D First Lyndon-Schützenberger Theorem

Theorem

Let $y \in \Sigma^+$. Then the following are equivalent:

(1) *There exists $p \in \Sigma^+$ such that p is both a proper prefix and suffix of y;*

(2) *There exist $u \in \Sigma^+$, $v \in \Sigma^*$, and an integer $e \geq 1$ such that $y = (uv)^e u = u(vu)^e$.*

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(3) ***There exist $s \in \boldsymbol{\Sigma}^+$ and $t \in \boldsymbol{\Sigma}^*$ such that $y = sts$;***

(4) ***There exist $q \in \boldsymbol{\Sigma}^+$ and $r \in \boldsymbol{\Sigma}^*$ such that qr is a proper prefix of y and $qry = yrq$;***

(6) ***There exist a proper prefix $x \in \boldsymbol{\Sigma}^+$ of y, $w \in \boldsymbol{\Sigma}^*$, and an integer $i \geq 2$ such that $yw = x^i$.***

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Remark

There exist conditions (5) and (7) which are analogous to conditions (4) and (6) for suffixes.

1D Second Lyndon-Schützenberger Theorem

Theorem

Let $x, y \in \Sigma^+$. Then the following are equivalent:

(1) $xy = yx$;

(2) *There exist $z \in \Sigma^+$ and integers $k, l > 0$ such that $x = z^k$ and $y = z^l$;*

(3) *There exist integers $i, j > 0$ such that $x^i = y^j$.*

Theorem

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(3) *There exist integers* $i, j > 0$ *such that* $x^i = y^j$;

(4) ***There exist integers*** $\boldsymbol{r, s > 0}$ ***such that*** $\boldsymbol{x^r y^s = y^s x^r}$;

(5) $\boldsymbol{x\{x, y\}^* \cap y\{x, y\}^* \neq \emptyset}$.

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(5) $\boldsymbol{x\{x, y\}^* \cap y\{x, y\}^* \neq \emptyset}$.

Remark

Condition (5) is essentially the **defect theorem** from the field of coding theory.

2D First Lyndon-Schützenberger Theorem

- We can extend the first Lyndon-Schützenberger theorem to two dimensions by
  - considering two-dimensional overlapping words; or
  - considering two-dimensional bordered words.
- The overlapping extension is not very interesting.
  - Simply apply the 1D version of the theorem to each row/column of the pair of two-dimensional words.
- We will focus on the bordered extension.

2D First Lyndon-Schützenberger Theorem

Theorem

*Let $A \in \Sigma^{m \times n}$ be a nonempty two-dimensional **bordered** word. Then the following are equivalent:*

(1) *There exist $P_1, P_2 \in \Sigma^{++}$ such that P_1 is a proper prefix/suffix of A horizontally and P_2 is a proper prefix/suffix of A vertically;*

(2) *There exist $U_1, U_2 \in \Sigma^{++}$, $V_1, V_2 \in \Sigma^{**}$, and integers $e, f \geq 1$ such that $A = (U_1 \ominus V_1)^e \ominus U_1 = (U_2 \oplus V_2)^f \oplus U_2$;*

(3) *There exist $S_1, S_2 \in \Sigma^{++}$ and $T_1, T_2 \in \Sigma^{**}$ such that $A = S_1 \ominus T_1 \ominus S_1 = S_2 \oplus T_2 \oplus S_2$;*

Theorem (Cont.)

Let $A \in \Sigma^{m \times n}$ *be a nonempty two-dimensional* ***bordered*** *word. Then the following are equivalent:*

(4) *There exist* $U_1, U_2 \in \Sigma^{++}$ *and* $V_1, V_2 \in \Sigma^{**}$ *such that* $U_1 \ominus V_1 \ominus A = A \ominus V_1 \ominus U_1$ *and* $U_2 \oplus V_2 \oplus A = A \oplus V_2 \oplus U_2$*;*

(5) *There exist* $X_1, X_2 \in \Sigma^{++}$*, which are proper prefixes of* A *horizontally and vertically, respectively;* $Z_1, Z_2 \in \Sigma^{**}$*; and integers* $i_1, i_2 \geq 2$ *such that* $A \ominus Z_1 = X_1^{i_1 \times 1}$ *and* $A \oplus Z_2 = X_2^{1 \times i_2}$*;*

(6) *There exist* $R_1, R_2 \in \Sigma^{++}$*, which are proper suffixes of* A *horizontally and vertically, respectively;* $W_1, W_2 \in \Sigma^{**}$*; and integers* $j_1, j_2 \geq 2$ *such that* $W_1 \ominus A = R_1^{j_1 \times 1}$ *and* $W_2 \oplus A = R_2^{1 \times j_2}$*.*

Theorem

Let A and B be nonempty two-dimensional words. Then the following are equivalent:

(1) *There exist positive integers* p_1, p_2, q_1, q_2 *such that* $A^{p_1 \times q_1} = B^{p_2 \times q_2}$.

(2) *There exist* $C \in \Sigma^{++}$ *and positive integers* r_1, r_2, s_1, s_2 *such that* $A = C^{r_1 \times s_1}$ *and* $B = C^{r_2 \times s_2}$.

(3) *There exist positive integers* t_1, t_2, u_1, u_2 *such that* $A^{t_1 \times u_1} \circ B^{t_2 \times u_2} = B^{t_2 \times u_2} \circ A^{t_1 \times u_1}$ *where* $\circ \in \{\varobar, \ominus\}$.

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Primitive Enumeration

- Over an alphabet of size k, there are

$$\psi_k(n) = \sum_{d|n} \mu(d) k^{n/d}$$

1D primitive words of length n, where $\mu(d)$ is the **Möbius function**, defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n \text{ has an even number of prime divisors;} \\ -1, & \text{if } n \text{ has an odd number of prime divisors; and} \\ 0, & \text{if } n \text{ is divisible by a square} > 1. \end{cases}$$

Example

Enumerating all primitive words of length 4 over a binary alphabet:

$$\begin{aligned}\psi_2(4) &= \sum_{d|4} \mu(d) 2^{4/d} \\ &= \mu(1)2^{4/1} + \mu(2)2^{4/2} + \mu(4)2^{4/4} \\ &= (1)(2^4) + (-1)(2^2) + (0)(2^1) \\ &= 16 \text{ total words } - \underbrace{4 \text{ non-primitive words}}_{\text{copies of } 00,01,10,11}\end{aligned}$$

Indeed, the 12 primitive words are 0001, 0010, 0011, 0100, 0110, 0111, 1000, 1001, 1011, 1100, 1101, and 1110.

- We can produce an analogous 2D formula that enumerates all two-dimensional primitive words of size $m \times n$.
- Before we continue, we require the following corollary of the 2D second Lyndon-Schützenberger theorem.

Corollary

Given $A \in \Sigma^{++}$, there exist a unique primitive $C \in \Sigma^{++}$ and positive integers i and j such that $A = C^{i \times j}$.

Theorem

Let $\psi_k(m, n)$ denote the number of two-dimensional primitive words of dimension $m \times n$ over a k-letter alphabet. Then

$$\psi_k(m, n) = \sum_{d_1 | m} \sum_{d_2 | n} \mu(d_1)\mu(d_2) k^{mn/(d_1 d_2)}.$$

Primitive Verification

- The literature features a good deal of previous work on pattern matching in two-dimensional words.
- However, none of this work is directly related to the matters of primitivity or periodicity.
- It would be desirable to have an (efficient) algorithm to check the primitivity of a two-dimensional word.

Primitive Verification

- Could we take the elements of the two-dimensional word in row-major/column-major order, then check if this resulting word is primitive?
- No, since this method does not work in some cases.

Primitive Verification

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Example

The two-dimensional word $A = \begin{bmatrix} \mathtt{a} & \mathtt{a} \\ \mathtt{b} & \mathtt{b} \end{bmatrix}$ is not 2D primitive.

Its row-majorized word $A_{\mathrm{RM}} = [\mathtt{aa}][\mathtt{bb}]$ is 1D primitive.

Example

The two-dimensional word $A = \begin{bmatrix} \mathtt{a} & \mathtt{b} & \mathtt{a} \\ \mathtt{b} & \mathtt{a} & \mathtt{b} \end{bmatrix}$ is 2D primitive.

Its row-majorized word $A_{\mathrm{RM}} = [\mathtt{aba}][\mathtt{bab}]$ is not 1D primitive.

Primitive Verification

- Before we continue, we make the following observations.

Remark

- A word w is primitive if and only if w is not a subword of the word $w_F w_L$, where w_F is w with the first symbol removed and w_L is w with the last symbol removed.
- We can check this in linear time by using, for example, the Knuth-Morris-Pratt string-matching algorithm.
- There exists an algorithm $\textsc{1DPrimitiveRoot}(w)$ to obtain the primitive root of some word w.

- Before we continue, we require the following lemma.

Lemma

Let $A \in \Sigma^{m \times n}$. Let the primitive root of row i of A be r_i and the primitive root of column j of A be c_j. Then the primitive root of A has dimension $p \times q$, where

$$p = \mathrm{lcm}(|c_0|, |c_1|, \ldots, |c_{n-1}|)$$

and

$$q = \mathrm{lcm}(|r_0|, |r_1|, \ldots, |r_{m-1}|).$$

Primitive Verification

Theorem

It is possible to check whether a $m \times n$ two-dimensional word is primitive and to compute the primitive root in $O(mn)$ time, for fixed alphabet size.

Primitive Verification

Algorithm 1: Computing the primitive root of A

procedure $\textsc{2DPrimitiveRoot}(A)$
    for $0 \leq i < m$ **do**
        $r_i \leftarrow \textsc{1DPrimitiveRoot}(A[i, 0..n-1])$
    $q \leftarrow \mathsf{lcm}(|r_0|, |r_1|, \ldots, |r_{m-1}|)$
    for $0 \leq j < n$ **do**
        $c_j \leftarrow \textsc{1DPrimitiveRoot}(A[0..m-1, j])$
    $p \leftarrow \mathsf{lcm}(|c_0|, |c_1|, \ldots, |c_{n-1}|)$
    for $0 \leq i < p$ **do**
        for $0 \leq j < q$ **do**
            $C[i, j] \leftarrow A[i, j]$
    return (C, p, q)

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Bordered Enumeration

- The number of one-dimensional unbordered words of length n over an alphabet of size k satisfies

$$u_k(n) = \begin{cases} k, & \text{if } n = 1; \\ k(k-1), & \text{if } n = 2; \\ k \cdot u_k(n-1), & \text{if } n \geq 3 \text{ is odd}; \\ k \cdot u_k(n-1) - u_k(n/2), & \text{if } n \geq 4 \text{ is even}. \end{cases}$$

- The number of bordered words of length n is therefore $b_k(n) = k^n - u_k(n)$.
- How can we enumerate the number of two-dimensional unbordered words of size mn, $U_k(m, n)$?

- We say that a one-dimensional word w has period p if $w[i] = w[i+p]$ for all i.

Lemma

Let $1 \leq p < n$. A one-dimensional word w of length n has period p if and only if w has a border of length $n - p$.

Corollary

If a one-dimensional word has a border of length $> \lfloor n/2 \rfloor$, then it also has a shorter border.

Bordered Enumeration

Technique 1

- Use the inclusion-exclusion principle.
- Take a two-dimensional word A and consider each column of A to be a "symbol".
- If A is bordered, then each "symbol" is bordered.
- We use our lemma to determine the possible one-dimensional border lengths.

Example

Consider one-dimensional words of length 3. These words can only have period length 2. Given such a word, specifying 2 symbols in that word fixes the remaining symbol.

Removing this symbol from the word and considering each possible pair of remaining symbols as being members of an alphabet of 4 "symbols", we get

$$\begin{aligned} U_2(3, n) &= 2^{3n} - b_{2^2}(n) \\ &= 2^{3n} - b_4(n), \end{aligned}$$

where $m = 3$ and $n > 1$.

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Technique 2

- Use polynomials.
- Find the most general word w of length m having all periods from a set of periods P.
- Consider all nonempty subsets S of P.
- Starting with $P(x) = 0$, add the term $(-1)^{|S|}x^{d(w)}$, where $d(w)$ denotes the number of distinct symbols in w.
  - This is another application of the inclusion-exclusion principle, but with a different approach.

Example

Let $m = 5$. Then $P = \{3, 4\}$.

- For $S_1 = \{3\}$, the most general word of length 5 with period 3 is 12312.
- For $S_2 = \{4\}$, the most general word of length 5 with period 4 is 12341.
- For $S_3 = \{3, 4\}$, the most general word of length 5 with periods 3 and 4 is 11211.

This gives $P(x) = -x^3 - x^4 + x^2$, so

$$\begin{aligned} U_2(5, n) &= 2^{5n} - b_{2^3}(n) - b_{2^4}(n) + b_{2^2}(n) \\ &= 2^{5n} - b_8(n) - b_{16}(n) + b_4(n). \end{aligned}$$

Bordered Verification

- It would again be desirable to have an (efficient) algorithm to check whether a given two-dimensional word is bordered.
- Recall the following results:

Lemma

Let $1 \leq p < n$. A one-dimensional word w of length n has period p if and only if w has a border of length $n - p$.

Corollary

A one-dimensional word w of length n has no period shorter than n if and only if w is unbordered.

Bordered Verification

- Before we continue, we make the following observations.

Remark

- There exists an algorithm $\textsc{1DPeriod}(w)$ to obtain the periods of a one-dimensional word w.
- This algorithm returns the periods as a bit vector P where the ith bit of the vector is 1 if a period of length i exists in the word and 0 otherwise.
- By our observation, this algorithm need only search for periods p of length $\lceil n/2 \rceil \leq p \leq n - 1$.

Bordered Verification

Theorem

It is possible to check whether a $m \times n$ two-dimensional word is bordered and compute the dimension of the largest border in $O(mn)$ time, for fixed alphabet size.

Bordered Verification

Algorithm 2: Computing the primitive root of A

procedure $\text{2DBorder}(A, m, n)$

for $0 \le i < m$ **do**

$P_i \leftarrow \text{1DPeriod}(A[i,\ 0..n-1])$

$P \leftarrow P \cap P_i$

if $P = \emptyset$ **then**

return "unbordered"

$d \leftarrow$ smallest period common to all P_i

for $0 \le j < n$ **do**

$Q_j \leftarrow \text{1DPeriod}(A[0..m-1,\ j])$

$Q \leftarrow Q \cap Q_j$

if $Q = \emptyset$ **then**

return "unbordered"

$e \leftarrow$ smallest period common to all Q_j

return $(m - e, n - d)$

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Conclusions

- Properties of two-dimensional words is an area ripe for investigation.
- We saw generalizations of the one-dimensional Lyndon-Schützenberger theorems and extensions of the theorems to two-dimensions.
- We showed methods of enumerating and verifying primitive words and bordered words in two dimensions.
- The algorithms to perform this verification are very efficient. (Linear time!)

Future Work

- Can we generalize properties of words (e.g., overlaps, borders) to words of dimension greater than 2?
- Is there a better method for enumerating all two-dimensional unbordered words of dimension $m \times n$ over a k-letter alphabet?

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